# ON THE RELATION BETWEEN HARMONIC SUMMABILITY AND SUMMABILITY BY RIESZ MEANS OF CERTAIN TYPE

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(Recived September 19, 1958)

**1.** An infinite series $\sum_{n=0}^{\infty} u_n$ with partial sums $s_n = \sum_{0}^{n} u_k$ is said to be *summable by Harmonic means* [3], if the sequence $\{y_n\}$ tends to a limit as $n \to \infty$, where

$$(1.\ 1) \qquad y_n = \frac{b_n s_0 + b_{n-1} s_1 + \ldots\ldots + b_0 s_n}{b_0 + b_1 + \ldots\ldots + b_n}, \quad \left(b_n = \frac{1}{n+1}\right).$$

We write $B_n = b_0 + b_1 + \ldots\ldots + b_n$ so that $B_n \sim \log n$.

The main interest of the method lies in the Tauberian theorem associated with it.

THEOREM A [2]. *If $\Sigma u_n$ is summable by Harmonic means, and*

$$u_n = O(n^{-\delta}) \qquad 0 < \delta < 1,$$

*then $\Sigma u_n$ is convergent.*

If $\delta = 1$, Theorem A reduces to well known Tauber's first theorem, in view of the fact that Harmonic summability implies $(C, \delta)$ summability for every $\delta > 0$.

If $p_n \geqq 0$, $p_0 > 0$, $\Sigma p_n = \infty$, (so that $P_n = p_0 + p_1 + \ldots\ldots + p_n \to \infty$), and

$$(1.\ 2) \qquad t_n = \frac{p_0 s_0 + p_1 s_1 + \ldots\ldots + p_n s_n}{p_0 + p_1 + \ldots\ldots + p_n} \to s$$

as $n \to \infty$, then we say that $s_n \to s(R, p_n)$ [1, p. 57]. If we choose $P_n = \exp n^{\alpha}$ $(0 < \alpha < 1)$, then the Tauberian condition of Theorem A is also the Tauberian condition of $(R, p_n)$ summability.

The object of this note is to give an indirect proof of Theorem A by proving the following theorem:

THEOREM I. *If an infinite series $\Sigma u_n$ is summable by Harmonic means to the sum $s$, then it is also summable $(R, p_n)$ to the same sum, where $P_n = \exp n^{\alpha}$ $(0 < \alpha < 1)$.*

**2.** Let $a_n$ be defined by

$$(2.\ 1)\qquad \left(1-\sum_{r=1}^{\infty} a_r x^r\right)\left(\sum_{0}^{\infty} b_r x^r\right)=1.$$

We shall be using the following known relations [2].

$$(2.\ 2)\qquad b_n=\sum_{r=1}^{n} a_r\, b_{n-r}$$

$$(2.\ 3)\qquad B_n=1+\sum_{r=1}^{n} a_r\, B_{n-r},$$

$$(2.\ 4)\qquad a_n=O\left(\frac{1}{n(\log n)^2}\right)$$

$$(2.\ 5)\qquad a_n+a_{n+1}+\ldots\ldots=O\left(\frac{1}{\log n}\right).$$

In our case $a_n \geqq 0$ by Kaluza's theorem [1, p. 68].

We shall also need the following lemma:

LEMMA. *If* $P_n=\exp n^{\alpha}$ $(0<\alpha<1)$ *and* $m<n^{1-\alpha}$, *then*

$$(2.\ 6)\qquad \frac{p_{n-m}}{p_n}=1+O\left(\frac{m}{n^{1-\alpha}}\right).$$

PROOF.
$$\frac{p_{n-m}}{p_n}=\left(\frac{n}{n-m}\right)^{1-\alpha}\exp\{(n-m)^{\alpha}-n^{\alpha})\}$$

$$=\left\{1+O\left(\frac{m}{n-m}\right)\right\}\left\{1-O\left(\frac{m}{n^{1-\alpha}}\right)\right\}$$

$$=1+O\left(\frac{m}{n^{1-\alpha}}\right).$$

**3. Proof of Theorem I.** Without loss of generality we may assume that $s=0$. From (1. 1) and (1. 2), we obtain

$$(3.\ 1)\qquad s_n=B_n\, y_n-\sum_{r=1}^{n} a_r\, B_{n-r}\, y_{n-r},$$

and

$$t_n=\sum_{k=0}^{n} y_n\, c_{n,k},$$

where

$$c_{n,k}=\frac{B_k}{P_n}(p_k-a_1 p_{k+1}-\ldots\ldots-a_{n-k}\, p_n)\qquad \text{for } k=0,1,2,\ldots\ldots n.$$

For the proof of our theorem it is sufficient to prove [1, p. 43] that

$$\text{(3. 2)}\quad \begin{aligned} &\text{(i)} \quad \lim_{n\to\infty} c_{n,k} = 0 && \text{for each } k\,; \\ &\text{(ii)} \quad \sum_{k=0}^{n} c_{n,k} = \lambda_n \to 1 && \text{as } n \to \infty\,; \\ &\text{(iii)} \quad \sum_{k=0}^{n} |c_{n,k}| < H && \text{where } H \text{ is independent of } n. \end{aligned}$$

Since $\Sigma\, a_n$ is convergent and $\dfrac{p_n}{P_n} = O\left(\dfrac{1}{n^{1-\alpha}}\right)$ it is easy to prove (3. 2) (i). For proving (3. 2) (ii) we observe that

$$\begin{aligned} &\frac{1}{P_n}\sum_{k=0}^{n} B_k\,(p_k - a_1 p_{k+1} - \cdots\cdots - a_{n-k}\,p_n) \\ &= \frac{1}{P_n}\sum_{r=0}^{n} p_r\,[B_r - a_1\,B_{r-1}\cdots\cdots - a_r\,B_0] \\ &= \frac{1}{P_n}\sum_{r=0}^{n} p_r \\ &= 1, \end{aligned}$$

by using (2. 3).

For proving (3. 2) (iii) we assume that $n_0 = [n^{1-\alpha+\epsilon}]$ and $m_0 = [n^{1-\alpha-\epsilon}]$ where $\varepsilon$ is a fixed positive number.

Now

$$\begin{aligned} \sum_{k=0}^{n} |c_{n,k}| &= \frac{1}{P_n}\sum_{k=0}^{n} B_k\,|p_k - a_1\,p_{k+1} - \cdots\cdots - a_{n-k}\,p_n| \\ &\leqq \frac{1}{P_n}\sum_{k=0}^{n-n_0} B_k\,p_k + \frac{1}{P_n}\sum_{k=0}^{n-n_0} B_k\,(a_1 p_{k+1} + \cdots\cdots + a_{n-k}p_n) \\ &\qquad + \frac{1}{P_n}\sum_{k=n-n_0+1}^{n} B_k\,|p_k - a_1\,p_{k+1} - \cdots\cdots - a_{n-k}\,p_n| \\ &= \Sigma_1 + \Sigma_2 + \Sigma_3, \text{ say.} \end{aligned}$$

First we consider $\Sigma_1$.

$$\text{(3. 3)}\quad \Sigma_1 = O(B_n)\,\frac{P_{n-n_0}}{P_n} = O(B_n)\exp\left(-\alpha n_0/n^{1-\alpha}\right) = O(1).$$

Again using (2. 3) and (2. 5) we have

$$\Sigma_2 = \frac{1}{P_n}\sum_{k=0}^{n-n_0} B_k(a_1\,p_{k+1} + \cdots\cdots + a_{n-k}\,p_n)$$

$$= \frac{1}{P_n} \sum_{r=1}^{n-n_0+1} p_r(B_0 a_r + B_1 a_{r-1} + \ldots\ldots + B_{r-1} a_1)$$

$$+ \frac{1}{P_n} \sum_{r=n-n_0+2}^{n} p_r(B_0 a_r + \ldots\ldots + B_{n-n_0} a_{r-(n-n_0)})$$

$$= \frac{1}{P_n} \sum_{r=1}^{n-n_0+1} p_r(B_r - 1) + \frac{1}{P_n} \sum_{r=n-n_0+2}^{n-n_0+1+m_0} p_r (B_0 a_r + \ldots\ldots + B_{n-n_0} a_{r-(n-n_0)})$$

$$+ \frac{1}{P_n} \sum_{r=n-n_0+2+m_0}^{n} p_r(B_0 a_r + \ldots\ldots + B_{n-n_0} a_{r-(n-n_0)})$$

$$= O\left(\frac{B_n P_{n-n_0}}{P_n}\right) + O\left(\frac{P_{n-n_0+1+m_0}}{P_n}\right) B_n \max_{n-n_0+2 \leqq r \leqq n-n_0+1+m_0} (a_r + a_{r-1} + \ldots\ldots + a_{r-(n-n_0)})$$

$$+ O\left(\frac{P_n B_n}{P_n}\right) \max_{n-n_0+2+m_0 \leqq r \leqq n} (a_r + a_{r-1} + \ldots\ldots + a_{r-(n-n_0)})$$

$$= O(1) + O(\log n) \exp\{-\alpha(n_0 - m_0 - 1)/n^{1-\alpha}\} + O(B_n/\log m_0)$$

$$(3.\ 4) \qquad = O(1).$$

Finally

$$\Sigma_3 = O(B_n/P_n) \sum_{k=n-n_0+1}^{n} |p_k - a_1 p_{k+1} - \ldots\ldots - a_{n-k} p_n|$$

$$= O(B_n/P_n) \sum_{k=n-n_0+1}^{n-m_0-1} |p_k - a_1 p_{k+1} - \ldots\ldots - a_{m_0} p_{m_0+k}|$$

$$+ O(B_n/P_n) \sum_{k=n-n_0+1}^{n-m_0-1} (a_{m_0+1} p_{m_0+1+k} + \ldots\ldots + a_{n-k} p_n)$$

$$+ O(B_n/P_n) \sum_{k=n-m_0}^{n} |p_k - a_1 p_{k+1} - \ldots\ldots - a_{n-k} p_n|$$

$$= \Sigma_{31} + O(\log n/P_n) \sum_{r=n-n_0+m_0+2}^{n} p_r(a_{m_0+1} + a_{m_0+2} + \ldots\ldots + a_{r-n+n_0-1})$$

$$+ O(\log n/P_n) p_n \sum_{k=n-m_0}^{n} (1 + a_1 + a_2 + \ldots\ldots + a_{n-k})$$

$$= \Sigma_{31} + O(\log n\, P_n/P_n \log m_0) + O(B_n p_n m_0/P_n)$$

$$(3.\ 5) \quad = \Sigma_{31} + O(1)$$

Making use of (2. 6) we obtain

$$\Sigma_{31} = O(\log n/P_n) \sum_{k=n-n_0+1}^{n-m_0-1} |p_k - a_1 p_{k+1} - \ldots\ldots - a_{m_0} p_{k+m_0}|$$

$$= O(\log n/P_n) \sum_{k=n-n_0+1}^{n-m_0-1} p_{k+m_0} \left| \frac{p_k}{p_{k+m_0}} - a_1 \frac{p_{k+1}}{p_{k+m_0}} - \ldots\ldots - a_{m_0} \right|$$

$$= O(\log n/P_n) \sum_{n-n_0+1}^{n-m_0-1} p_{k+m_0}(1 - a_1 - \cdots\cdots - a_{m_0})$$

$$+ O(\log n/P_n) \sum_{n-n_0+1}^{n-m_0-1} \frac{p_{k+m_0}}{(k+m_0)^{1-\alpha}} [m_0 + (m_0 - 1)a_1 + \cdots\cdots + a_{m_0-1}]$$

$$= O(\log n/\log m_0) + O(m_0 \log n/P_n(n-n_0)^{1-\alpha}) \sum_{n-n_0+1}^{n-m_0-1} p_{k+m_0}$$

$$= O(1) + O\left(\frac{m_0}{(n-n_0)^{1-\alpha}} \cdot \frac{P_n \log n}{P_n}\right)$$

$$(3.\ 6) \qquad = O(1).$$

Collecting (3. 3), (3. 4), (3. 5) and (3. 6) we see that (3. 2) (iii) is also satisfied. This completes the proof of the theorem.

I am indebted to Prof. M. L. Misra for his kind interest and advice in the preparation of this paper.

## REFERENCES

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