AN APPROXIMATION PROBLEM PROPOSED BY K. ITÔ

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1. Itô proposed the following problem: *Given a sequence f satisfying*

$$\sum_{n\geqq 0}\frac{\lambda^n}{n\,!}|f(n)|^2<+\infty \tag{1. 1}$$

and a number $\varepsilon>0$, does there exist a polynomial P such that

$$\sum_{n\geqq 0}\frac{\lambda^n}{n\,!}|f(n)-P(n)|^2\leqq\varepsilon\ ? \tag{1. 2}$$

Izumi has given [1] an affirmative answer by *constructing* such polynomials in the case in which (1. 1) is strengthened to

$$\sum_{n\geqq 0}|f(n)|^2/w^n<+\infty \tag{1. 1$'$}$$

for some w. This entails the *existence* of the approximating polynomials under the weaker hypothesis (1. 1) because the Dirac sequences satisfy (1. 1$'$) and are already total amongst those sequences which satisfy (1. 1). Izumi also deals in analogous fashion with the problem in which means with index 1 replace those with index 2 appearing above.

In this paper we give a rapid existential proof, based upon the Hahn-Banach Theorem, of a more general assertion.

2. The set of positive integers is replaced by an arbitrary locally compact space T, summation being replaced by integration with respect to a chosen positive Radon measure μ on T. Assume that $u_1,\ldots\ldots,u_n$ are n real-valued functions on T such that the mapping

$$u:\ t\to(u_1(t),\ldots\ldots,u_n(t)) \tag{2. 1}$$

is a homeomorphism of T into R^n (n dimensional real number space), and such that, if $\|u(t)\|=\sum_{k=1}^{n}|u_k(t)|$, then

$$\int_T \exp\,(a\|u(t)\|)d\mu(t)<+\infty \tag{2. 2}$$

for a suitable number $a>0$. Finally let p be an exponent satisfying $1\leqq p<+\infty$, and let p' be the conjugate exponent.

THEOREM. *If the mapping* (2. 1) *is a homeomorphism of T into R^n,* *and if* (2. 2) *holds for some* $a > 0$, *then the polynomials* $P(u_1, \ldots\ldots, u_n)$ *are dense in* $L^p(T, \mu)$.

PROOF. We use the Hahn-Banach Theorem. As is well known, if L is any continuous linear form on $\mathfrak{L}^p(T, \mu)$, there is a function $g \in \mathfrak{L}^{p'}(T, \mu)$ such that

$$L(f) = \int_T fg d\mu \tag{2. 3}$$

for all $f \in \mathfrak{L}^p(T, \mu)$. It suffices to show that L, if orthogonal to all polynomials $P(u_1, \ldots\ldots, u_n)$, vanishes identically on $\mathfrak{L}^p(T, \mu)$. Now (2. 2) implies that μ is bounded, so that g is certainly μ-integrable.

Consider the function

$$G(z_1, \ldots\ldots, z_n) = \int_T \exp\ (z_1 u_1(t) + \ldots\ldots + z_n u_n(t))\, g(t)\, d\mu(t)$$

for complex $z_k = x_k + iy_k$. The integral here converges absolutely for $|x_k| < a/p$ $(1 \leqq k \leqq n)$ and so G is holomorphic on this polystrip. Moreover

$$\frac{\partial^{r_1 + \cdots + r_n} G}{\partial z_1^{r_1} \ldots \partial z_n^{r_n}} (0, \ldots\ldots, 0) = \int_T u_1^{r_1} u_2^{r_2} \ldots\ldots u_n^{r_n} g d\mu$$

for arbitrary integers $r_k \geqq 0$. So, if L is orthogonal to all polynomials $P(u_1, \ldots\ldots, u_n)$, G is identically vanishing in the polystrip. This implies in particular that

$$\int_T \exp\ (iy_1 u_1(\mathrm{t}) + \ldots\ldots + iy_n u_n(t))\, g(t)\, d\mu(t) = 0 \tag{2. 4}$$

for all real y_k.

If we introduce a measure ν on R^n defined by

$$\int_{R^n} h\, d\nu = \int_T h(u) g\, d\mu \tag{2. 5}$$

for h a continuous function on R^n with compact support, we see at once that ν is bounded, that (2. 5) continues to hold for any bounded and continuous h, and that therefore (2. 4) entails that

$$\int_{R^n} \exp\ (iy_1 s_1 + \ldots\ldots + iy_n s_n)\, d\nu(s_1, \ldots\ldots, s_n) = 0 \tag{2. 6}$$

for all real y_k. By the uniqueness theorem for Fourier transforms of bounded measures, (2. 6) implies that $\nu = 0$. Then, since u is a homeomorphism of T into R^n, (2. 5) shows that $g = 0$ a. e. (μ), so that (2. 3) gives $L(f) = 0$ for all $f \in \mathfrak{L}^p(T, \mu)$. This completes the proof.

REMARKS. It is clear that this theorem yields on specialisation the solution of Itô's problem, as well as Izumi's theorems. It can also be adapted easily to give a form of Itô's conjecture in which uniformity with respect to λ is contemplated.

REFERENCE

[1] S. IZUMI, On an approximation theorem in the theory of probability, Tôhoku Math. Journ., 5 (1953), 22–28.

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