THE STRUCTURE OF A RIEMANNIAN MANIFOLD ADMITTING A PARALLEL FIELD OF ONE-DIMENSIONAL TANGENT VECTOR SUBSPACES

SHŌBIN KASHIWABARA

(Received February 4, 1959)

1. Introduction. Let M be an n-dimensional connected complete Riemannian manifold of class C^2, admitting a parallel field of one-dimensional tangent vector subspaces. M is also regarded as a Riemannian manifold whose homogeneous holonomy group fixes a one-dimensional tangent vector subspace. The purpose of this note is to discuss the global geometrical structure of M. Locally, the parallel field is generated from a parallel field of non-zero tangent vectors which is locally defined. The structure, that is the local decomposition of the Riemannian metric, has been well known to many geometricians. Starting from this local structure we proceed to determine the global structure of M. The main results are shown in Theorems $1-7$ Among them Theorems $1-6$ give structures in respective cases, and from the last Theorem 7 we can know a general structure of M.

From now on, the word "k-dimensional" is abbreviated "k-", say, like k-space (but "R-" etc have not such a meaning). Let us suppose that Latin indices a, b run from 1 to $n-1$ and Greek indices α, β, γ from 1 to n. Let E be a Euclidean 1-space with the coordinate system $\{t| -\infty < t < \infty\}$ and dt denotes the infinitesimal distance. Let E' be the part $\{t|0 \leqq t < \infty\}$ of E. Moreover for a constant $L > 0$, let $[L]$ be the part $\{t|0 \leqq t \leqq L\}$ of E.

The following conventions in a Riemannian manifold X are also applied to all of Riemannian manifolds: The *parallelism* in X means the one of Levi-Civita. A *neighborhood* in X is always an open set homeomorphic to Euclidean space. Take any $x, y \in X$. Let $[x, y]$ denote a geodesic arc joining x to y. And further, take a unit tangent vector v at x. Given a real number c, $g(x, v, c)$ is defined to be the geodesic arc issuing from x, whose length is $|c|$ and whose initial vector is v or $-v$ according as $c > 0$ or < 0. Let (x, v, c) denote its terminal point. Note that a geodesic arc is not necessarily simple and sometimes may be closed. Let a curve α: $x(t)$ (say, $a \leqq t \leqq b$) be given in X. At $x_0 = x(a)$ we take a unit vector v_0 tangent to X. Corresponding to each t, let $v(t)$ be the unit vector at $x(t)$ parallel to v_0 along α. Moreover if a geodesic arc $g(x_0, v_0, c)$ is given, each geodesic arc

$g(x(t), v(t), c)$ is said to be *parallel* to $g(x_0, v_0, c)$ along α. And as usual, to displace the latter arc parallelly along α is to obtain the former arcs. A *covering manifold* $C(X)$ of X is defined to be a connected covering manifold of X with the Riemannian metric naturally induced from X by the covering map p. $C(X)$ is of the same differentiability class as X. However, we sometimes allow local coordinate systems whose differentiability classes are minus 1 from that of X. Especially, if $p^{-1}(x)$ $(x \in X)$ consists of just two points, $C(X)$ is called a *double covering manifold* of X. Let us take the product $X \times E$. Into it, we introduce a Riemannian metric by $ds^2 = ds_X^2 + dt^2$ where ds_X denotes the Riemannian metric in X. We get thus a Riemannian manifold $X \times E$, which is usually called the *metric product* of X and E. Similarly, the metric products $X \times E'$, $X \times [L]$ are considered and they are Riemannian spaces. And a point of $X \times E$ etc. is denoted by (x, t) where $x \in X$, as usual The notation "$\times$" always means the operation of a metric product. Over X, a *field of vectors* (*vector spaces*) implies that to each point of X a vector (a vector space) is assigned. Let S be a field of vector 1-spaces and let V be a field of vectors. Then, the expression that S is *generated* from V means that at each point of X the vector 1-space of S is generated from the vector of V. Moreover, the expression "X *admits* (or is *admitting*) a field" implies always to admit the field throughout X.

2. Preliminaries Let M be a connected complete Riemannian n-manifold $(n > 1)$ of class C^2, admitting a parallel field of tangent vector 1-subspaces. (M is such one throughout the whole discussion.) The parallel field is called the *S-field* over M. Let us take the field of tangent vector $(n-1)$-subspaces, which is orthogonal to the S-field at each point of M. It is obvious that the field forms a parallel field over M, too. We call it the *R-field*. Such a manifold M will be called an *RS-manifold* of dimension n.

Take any $x_0 \in M$. Let U be an admissible coordinate neighborhood of x_0. Let (x^ν) be its coordinate system. Let $(g_{\alpha\beta})$ denote the fundamental tensor in U. U being simply-connected, we can find a parallel field $\{v(x) | x \in U\}$ in U of unit tangent vectors, from which the S-field restricted to U is generated. We denote its vector $v \equiv v(x)$ by (v^α). Put $v_\alpha = g_{\alpha\beta} v^\beta$. Then we have

$$\frac{\partial v_\beta}{\partial x^\gamma} - \left\{ {\alpha \atop \beta\gamma} \right\} v_\alpha = 0$$

where $\left\{ {\alpha \atop \beta\gamma} \right\}$ are Christoffel's symbols constructed from $g_{\alpha\beta}$. Hence

$$\frac{\partial v_\beta}{\partial x^\gamma} = \frac{\partial v_\gamma}{\partial x^\beta}.$$

This shows that the system of differential equations

$$\frac{\partial f}{\partial x^\alpha} = v_\alpha \tag{2.1}$$

has a solution of class C^3. We denote it by $f^n(x^1, \ldots\ldots x^n)$. Moreover we consider the differential equation

$$v^\alpha \frac{\partial f}{\partial x^\alpha} = 0. \tag{2.2}$$

Among the solutions there exist $n-1$ independent functions $f^a(x^1, \cdots\cdots, x^n)$ of class C^2. These functions $f^\alpha \equiv f^\alpha(x^1, \ldots\ldots, x^n)$ can be supposed to be defined in a neighborhood ($\subset U$) of x_0. We see easily

$$\frac{\partial(f^1, \ldots\ldots, f^n)}{\partial(x^1, \ldots\ldots, x^n)} \neq 0$$

at x_0. Put

$$x'^\alpha = f^\alpha(x^1, \cdots\cdots, x^n).$$

Using these, let us transform the coordinate system (x^α). We get thus a coordinate neighborhood $U'(\subset U)$ of x_0, which is covered by the new coordinate system (x'^α). Let $(g'_{\alpha\beta})$ be the fundamental tensor in U'. When $g^{\alpha\beta}$, $g'^{\alpha\beta}$ are defined by $g^{\alpha\gamma}g_{\gamma\beta} = \delta^\alpha_\beta$, $g'^{\alpha\gamma}g'_{\gamma\beta} = \delta^\alpha_\beta$, we have

$$g'^{an} = \frac{\partial f^a}{\partial x^\alpha}\frac{\partial f^n}{\partial x^\beta} g^{\alpha\beta}.$$

Since f^α satisfy (2.1) or (2.2), we get $g'^{an} = 0$, i.e. $g'_{an} = 0$. We see that in U' the vector v is represented by (δ^α_n). So, from the parallelism we have

$$\frac{\partial \delta^\alpha_n}{\partial x'^\beta} + \left\{ {\alpha \atop \beta\gamma} \right\}' \delta^\gamma_n = 0$$

where $\left\{ {\alpha \atop \beta\gamma} \right\}'$ are Christoffel's symbols constructed from $g'_{\alpha\beta}$. Hence

$$\left\{ {\alpha \atop \beta\gamma} \right\}' = 0.$$

From this and $g'_{an} = 0$, it follows that $g'_{nn} = \text{const.}$ and g'_{ab} are independent of x'^n. And further, $g'_{nn} = 1$, the vector v being a unit vector.

The results above are stated as follows: *At any $x_0 \in M$ there exists a coordinate neighborhood W such that the Riemannian metric in W is expressed by*

$$ds^2 = g_{ab}dx^a dx^b + (dx^n)^2$$

where (x^α) denotes the coordinate system in W and g_{ab} are the functions of $x^1, \ldots\ldots, x^{n-1}$ only. Moreover, in W we can see following facts: *If coordinates x^a are varied leaving x^n only fixed, we obtain an integral manifold of the R-field. If a coordinate x^n is varied leaving all of x^a fixed, we* *obtain an integral manifold of the S-field, that is a geodesic in which* x^n *plays the role of the arc-length.* Such a coordinate neighborhoods of x_0, whose coordinate system (x^α) consists of all of (x^α)'s such that $a^\alpha < x^\alpha < b^\alpha$ (where a^α, b^α are constants), is called a *reduced coordinate neighborhood* of x_0.

Let W, W' be two reduced coordinate neighborhoods of x_0. Let (x^α), (x'^α) be their coordinate systems respectively. Let W'' be the connected component of $W \cap W'$ containing x_0. *In* W'' *the coordinate systems* (x^α) *and* (x'^α) *are combined by the relations decomposed as follows*:

$$x'^a = f'^a(x^1, \dots\dots x^{n-1}),\ x'^n = \varepsilon x^n + \text{const.}\ (\varepsilon = +1 \text{ or } -1)$$

where f'^a *are the functions of class* C^2 *independent of* x^n.

Moreover we can see that through $x_0 \in M$ there passes a pair of the maximal, connected integral manifolds of the R-and S-fields. Let $R(x_0)$ and $S(x_0)$ denote the ones respectively. We give them the Riemannian metrics which are naturally induced from M, and call them R-and *S-submanifolds* of M respectively. They form Riemannian manifolds of class C^1 and the following fact is easily verified: *All of the R-and S-submanifolds are totally geodesic, and complete as Riemannian manifolds.* Indeed, each of the S-submanifolds is a geodesic. Accordingly it is also called an *S-geodesic.* Let $I(x_0)$ denote a subset $R(x_0) \cap S(x_0)$.

Let X be a connected complete Riemannian $(n-1)$-manifold of class C^1. That M is of one of the following types I—VI means that for suitable X etc. there is an isometric homeomorphism of class C^2, of M onto the corresponding Riemannian manifold, which maps each R-submanifold onto t=const.

Type I: The Riemannian manifold $X \times E$.

Type II: The Riemannian manifold constructed from $X \times [L]$ by identifying (x, L) with $(x, 0)$ for all $x \in X$.

Provided that there exists a non-trivial isometric homeomorphism ϕ of class C^2, of X onto itself, we define

Type III: The Riemannian manifold constructed from $X \times [L]$ by identifying (x, L) with $(\phi(x), 0)$ for all $x \in X$.

Next suppose that there exists an isometric involutive homeomorphism ψ of class C^2, having no fixed points, of X onto itself. (By the word "involutive" it is meant that $\psi\psi(x) = x$ for each $x \in X$.)

Type IV: The Riemannian manifold constructed from $X \times E'$ by identifying $(x, 0)$ with $(\psi(x), 0)$ for all $x \in X$.

Type V: The Riemannian manifold constructed from $X \times [L]$ by identifying $(x, 0)$ with $(\psi(x), 0)$, and (x, L) with $(\psi(x), L)$ for all $x \in X$.

Furthermore, provided that there exists another homeomorphism ψ' of

X onto itself, with the same property as ψ, we define

Type VI: The Riemannian manifold constructed from $X \times [L]$ by identifying $(x, 0)$ with $(\psi(x), 0)$, and (x, L) with $(\psi'(x), L)$ for all $x \in X$.

In M suppose that there exists a connected open submanifold M^0 which satisfies the following conditions 1) and 2), or 1) and 3):

1) M^0 is a union set of S-geodesics and the closure of M^0 is M.

2) M^0 is the maximal subset in which each point x is a limit point of $I(x)$ relative to each of submanifolds $R(x)$ and $S(x)$.

3) M^0 is a maximal subspace which becomes a fibre bundle where each fibre is an S-geodesic. (By the word "maximal" it is meant that there are no subspaces, $\supset M^0$, $\neq M^0$, which have the same property.)

When M^0 satisfies 1) and 2), M is said to be of *almost clustered type* with *kernel* M^0. In this case, if $M = M^0$, M is simply said to be of *clustered type*.

When M^0 satisfies 1) and 3), M is said to be of *almost fibred type* with *kernel* M^0. In this case if $M = M^0$, M is simply said to be of *fibred type*.

If M is not of almost fibred type but of type III (VI), M is said to be of *non-fibred type III* (*VI*). If M is not of one of types I—VI, M is said to be of *non-simple type*.

3. Fundamental lemmas. Take any $x_0 \in M$. An *R-neighborhood* of x_0 is a neighborhood in $R(x_0)$. A *normal vector* at x_0 is a unit tangent vector at x_0 orthogonal to $R(x_0)$. Let $n(x_0)$ always denote one of the normal vectors at x_0. Take an R-submanifold R_0 of M. At each point x of R_0 we plant a normal vector $n(x)$. If $n(x)$ becomes continuous over R_0, the set $\{n(x) \mid x \in R_0\}$ is said to be a *normal vector field* over R_0. Then R_0 admits just two normal vector fields and the normal vectors $n(x)$ are parallel to one another along any curves of class D^1 in R_0. Similarly over an R-neighborhood too, the notion of normal vector field is defined. In this case, there exist always just two normal vector fields, because it is simply-connected. For any two points x, y of an R-submanifold, let $d_R(x, y)$ denote the length of a minimizing geodesic in the R-submanifold joining x to y.

Again we take any $x_0 \in M$. Let n_0 be a normal vector at x_0. For a constant c, put $y_0 = (x_0, n_0, c)$. Then we have

LEMMA 3. 1. *There exists an R-neighborhood W_R at x_0 such that, if $\{n(x) \mid x \in W_R\}$ is the normal vector field over W_R where $n(x_0) = n_0$, $R(y_0)$ contains $(x, n(x), c)$ for all $x \in W_R$ and the map*

$$f:\ W_R \to R(y_0) \quad \text{defined by} \quad f(x) = (x, n(x), c)$$

is an isometric into-homeomorphism.

PROOF. 1) The case where the geodesic arc $g(x_0, n_0, c)$ is contained in a reduced coordinate neighborhood U. Let W_R be the connected component of $U \cap R(x_0)$ containing x_0. In U, let (x_0^α) denote x_0. Then, y_0 is denoted by $(x_0^a, x_0^n + \varepsilon c)$ where $\varepsilon = +1$ or -1. Moreover, if each $x \in W_R$ is denoted by (x^α), then $x^n = x_0^n$ and $(x, n(x), c)$ is denoted by $(x^a, x_0^n + \varepsilon c)$. Therefore by §2 we can see that W_R satisfies the condition in our lemma.

2) The other case. Take a finite system of reduced coordinate neighborhoods $U_\lambda (\lambda = 1, 2, \ldots\ldots, h)$ such that each U_λ contains a geodesic arc $[x_{\lambda-1}, x_\lambda]$ where the product curve $[x_0, x_1] \cdot [x_1, x_2] \cdot \ldots\ldots \cdot [x_{h-1}, x_h]$ becomes $g(x_0, n_0, c)$. To each U_λ and $[x_{\lambda-1}, x_\lambda]$, apply the result of 1). Thus we can easily find an R-neighborhood W_R at x_0 in our lemma.

Moreover let $x(t)\,(a \leqq t \leqq b)$, $x(a) = x_0$, be a curve of class D^1 in $R(x_0)$. Corresponding to each t, let $n(t)$ be the normal vector at $x(t)$ parallel to n_0 along the curve. Put $y(t) = (x(t), n(t), c)$ Let $n'(t)$ be the normal vector at $y(t)$ parallel to $n(t)$ along $g(x(t), n(t), c)\,(t:$ fixed). Then we have

LEMMA 3.2. 1) *$y(t)\,(a \leqq t \leqq b)$ is a curve of class D^1 in $R(y_0)$ and $\{n'(t) \mid a \leqq t \leqq b\}$ consists of normal vectors parallel to one another along the curve $y(t)$. 2) For any $y_1 \in R(y_0)$ there exist a point $x_1 \in R(x_0)$ and a normal vector n_1 at x_1 such that $y_1 = (x_1, n_1, c)$.*

PROOF. To prove 1), cover the curve $x(t)$ by a finite system of R-neighborhoods which have the same property as W_R in Lemma 3.1. Then 1) is easily verified. To prove 2), take a curve $z(t)$ $(0 \leqq t \leqq 1)$, $z(0) = y_0$, $z(1) = y_1$, of class D^1 in $R(y_0)$. Let n_0' be a normal vector at y_0 such that $(y_0, n_0', c) = x_0$. Let n_1' be the normal vector at y_1 parallel to n_0' along the curve $z(t)$. Now put $x_1 = (y_1, n_1', c)$. Then $x_1 \in R(x_0)$, and $y_1 = (x_1, \varepsilon n(x_1), c)$ for $\varepsilon = +1$ or -1. So 2) is proved.

Next, at $x_0 \in M$ we shall express $S(x_0)$ by $x(s)\,(-\infty < s < \infty)$ where $x(0) = x_0$ and s denotes the arc-length. If $S(x_0)$ is closed, it represents $S(x_0)$ many times. Let u_0 be a unit vector at x_0 tangent to $R(x_0)$ and let c be a positive constant. Now, displace u_0 parallelly along the curve $x(s)$. Then corresponding to each s, we get a vector $u(s)$ at $x(s)$ and it is tangent to $R(x(s))$. Hence, $g(x(s), u(s), c) \subset R(x(s))$. Put $z_0 = (x_0, u_0, c)$ and we have

LEMMA 3.3. *There exists a non-closed geodesic arc $x(s)\,(-\tau \leqq s \leqq \tau;$ $\tau > 0)$ such that $(x(s), u(s), c) \in S(z_0)$ for all s $(-\tau \leqq s \leqq \tau)$ and the map*

$$f:\ x(s)\,(-\tau \leqq s \leqq \tau) \rightarrow S(z_0) \text{ defined by } f(x(s)) = (x(s), u(s), c)$$

is an isometric into-homeomorphism.

PROOF. 1) The case where the geodesic arc $g(x_0, u_0, c)$ is contained in a reduced coordinate neighborhood U. Then there exists $\tau > 0$ such that $x(s)$ $(-\tau \leqq s \leqq \tau)$ is contained in U. In U let (x_0^α), (z_0^α), and (u_0^α) denote x_0, z_0, and u_0 respectively Then, $x_0^n = z_0^n$ and $u_0^n = 0$. And corresponding to each $s(-\tau \leqq s \leqq \tau)$ $x(s)$ is denoted by $(x_0^i, x_0^n + \varepsilon s)$ where $\varepsilon = +1$ or -1, and $u(s)$ by $(u^i, 0)$. Hence, $(x(s), u(s), c)$ is denoted by $(z_0^i, z_0^n + \varepsilon s)$. Therefore we can see that $x(s)(-\tau \leqq s \leqq \tau)$ satisfies the condition in our lemma.

2) The other case. Take a finite system of reduced coordinate neighborhoods U_λ $(\lambda = 1, 2, \ldots\ldots, h)$ such that each U_λ contains a geodesic arc $[x_{\lambda-1}, x_\lambda]$ where the product curve $[x_0, x_1]\cdot[x_1, x_2]\cdot\ldots\ldots\cdot[x_{h-1}x_h]$ becomes $g(x_0, u_0, c)$. To each U_λ and $[x_{\lambda-1}, x_\lambda]$ apply the result of 1). Thus we can easily find a geodesic arc in our lemma.

Under the same notations; put $z(s) = (x(s), \ u(s), c)$, then the following lemma is obvious:

LEMMA 3. 4. 1) *A curve* $z(s)(-\infty < s < \infty)$ *represents* $S(z_0)$ *(many times if it is closed) and the parameter* s *plays the role of the arc-length in* $S(z_0)$. 2) *If* $c = d_R(x_0, z_0)$, $d_R(x(s), \ z(s)) \leqq c$ *for any* s.

In M let x_0, y_0 be any two points. Then we have

LEMMA 3. 5. *A set* $R(x_0) \cap S(y_0)$ *is non-empty and at most countable.*

PROOF. 1) Let us prove $R(x_0) \cap S(y_0) \neq 0$. Take a geodesic arc $[y_0, x_0] = g(y_0, v_0, c)$ where $c > 0$. If v_0 is tangent to $R(y_0)$, $R(y_0)$ contains x_0 and obviously $R(x_0) \cap S(y_0) \neq 0$. Accordingly, let us consider the case where v_0 is not tangent to $R(x_0)$. Then at each $y \in [y_0, x_0]$ too, the tangent vector of $[y_0, x_0]$ is not tangent to $R(y)$. We can find a finite system of reduced coordinate neighborhoods U_λ $(\lambda = 1, 2, \ldots\ldots, h)$ such that each U_λ contains a geodesic arc $(y_0, v_0, s)(c_{\lambda-1} \leqq s \leqq c_\lambda)$ where $0 = c_0 < c_1 < \ldots\ldots < c_h = c$ Put $y_\lambda = (y_0, v_0, c_\lambda)$. In U_λ suppose that points $y_{\lambda-1}$ and y_λ are denoted by $(y_{\lambda-1,\lambda}^\alpha)$ and $(y_{\lambda,\lambda}^\alpha)$ respectively. Here, put $d_\lambda = |y_{\lambda-1,\lambda}^n - y_{\lambda,\lambda}^n|$. In U_0 let (v_0^α) denote v_0. Then $v_0^n \neq 0$, and let n_0 be a vector $(\varepsilon\delta_n^\alpha)$ (in U_0) at x_0 where $\varepsilon = +1$ or -1 according as $v_0^n > 0$ or < 0. Hence, $(y_0, n_0, d_1) \in R(y_1)$. By Lemma 3. 2 we can further verify $(y_0, n_0, d_1 + d_2) \in R(y_2), \ldots\ldots$, and finally $(y_0, n_0, d_1 + d_2 + \ldots\ldots + d_h) \in R(y_h)$. This implies $R(x_0) \cap S(y_0) \neq 0$

2) Let us prove that $R(x_0) \cap S(y_0)$ is at most countable. As a point set, $S(y_0)$ can be regarded as the union set of $\{\alpha_\lambda | \lambda = 1, 2, \ldots\ldots\}$ where α_λ is a subarc of $S(y_0)$ contained in a reduced coordinate neighborhood U_λ. (The index λ runs at most to ∞.) Here, $R(x_0) \cap \alpha_\lambda \subset R(x_0) \cap U_\lambda$. $R(x_0)$ satisfies the second countability axiom. So, $R(x_0) \cap U_\lambda$ consists of an at most countable system of non-intersecting R-neighborhoods in $R(x_0)$. Hence, $R(x_0) \cap \alpha_\lambda$ is at most countable. Therefore $R(x_0) \cap S(y_0)$ is at most countable.

4. Topology of R-submanifolds and structures. In M suppose that each of the R-submanifolds admits normal vector field. Then we have

LEMMA 4.1. *The S-field is generated from a parallel field of unit tangent vectors.*

PROOF. Take an R-submanifold R_0. At $x_0 \in R_0$ let us suppose that there exists $c > 0$ such that $(x_0, n(x_0), c) \in R_0$ for a suitable $n(x_0)$. Let $\{n(x) \mid x \in R_0\}$ be the normal vector field containing $n(x_0)$. Let n_c be the vector at $x_c \equiv (x_0, n(x_0), c)$ parallel to $n(x_0)$ along $g(x_0, n(x_0), c)$. Then, $n_c = n(x_c)$. If $n_c \neq n(x_c)$, we have $n(x_c) = -n_c$ and $x_0 \neq x_c$. Put $x_0' = (x_0, n(x_0), c/2)$. We displace $g(x_0, n(x_0), c/2)$ parallelly along a curve in $R(x_0)$ joining x_0 to x_c. Then the displacement shows that $n(x_0')$ is parallel to $-n(x_0')$ along the curve in $R(x_0')$. This implies that $R(x_0')$ does not admit normal vector field. It is contrary to the assumption. So, $n_c = n(x_c)$. From this fact and Lemmas 3.1, 3.5, our lemma is proved.

Conversely suppose that in M the S-field is generated from a parallel field $\{v(x) \mid x \in M\}$ of unit tangent vectors. Over each R-submanifold R, a subset $\{v(x) \mid x \in R\}$ becomes a normal vector field. That is, the converse of Lemma 4.1 holds good. Let R_0, R_1 be any two of the R-submanifolds. Take any $x_0 \in R_0$. By Lemma 3.5 there exists c such that $x_1 \equiv (x_0, v(x_0), c) \in R_1$. Then we have

LEMMA 4.2. *The map*

$$f\colon R_0 \to R_1 \quad \textit{defined by} \quad f(x) = (x, v(x), c)$$

where $x \in R_0$, is an isometric homeomorphism.

Such a map f is called the *R-map* with respect to a geodesic arc $g(x_0, v(x_0), c)$.

PROOF. Take $y_1 \in R_1$. By Lemma 3.2, $y_0 \equiv (y_1, -v(y_1), c) \in R_0$. So, $y_1 = (y_0, v(y_0), c)$. Hence, f is an onto-map. Next for $x_0', y_0' \in R_0$, if $(x_0', v(x_0'), c) = (y_0', v(y_0'), c)\ (\equiv x_1')$, then $(x_1', -v(x_1'), c) = x_0'$ and $= y_0'$. So, $x_0' = y_0'$. This implies that f is one-to-one. By Lemma 3.1 our lemma is proved.

In M, take an R-submanifold R_0. At $x_0 \in R_0$ let $N(x_0)$ denote the set of all positive numbers s such that at least one of two points $(x_0, \pm n(x_0), s)$ belongs to R_0. If $N(x_0)$ is non-empty, we denote the greatest lower bound of $N(x_0)$ by $\rho(x_0)$. If $N(x_0)$ is empty, we put $\rho(x_0) = \infty$. So, $0 \leqq \rho(x_0) \leqq \infty$. By Lemma 3.2 we have $\rho(x) = \rho(x_0)$ for any $x \in R_0$. Accordingly, we denote $\rho(x_0)$ by $\rho(R_0)$. We call $\rho(R_0)$ the *distance* of R_0. Let R_1 be another R-submanifold. At $x_0 \in R_0$ let $N(x_0, R_1)$ be the set of all positive numbers s such that at least one of two points $(x_0, \pm n(x_0), s)$ belongs to R_1. By Lemma

3. 5, $N(x_0, R_1)$ is non-empty. We denote the greatest lower bound of $N(x_0, R_1)$ by $\rho(x_0, R_1)$. So, $0 \leqq \rho(x_0, R_1) < \infty$. By Lemma 3. 2 we have $\rho(x_0, R_1) = \rho(x, R_1)$ for any $x \in R_0$. Accordingly, we denote $\rho(x_0, R_1)$ by $\rho(R_0, R_1)$. We call $\rho(R_0, R_1)$ the *distance* between R_0, R_1. Then we have

LEMMA 4. 3. 1) *If* $0 < \rho(R_0) < \infty$, *at least one of two points* $(x_0, \pm n(x_0), \rho(R_0))$ *belongs to* R_0. 2) *If* $\rho(R_0) > 0$, $\rho(R) > 0$ *for any R-submanifold R.* 3) *If* $\rho(R_0, R_1) > 0$, *at least one of two points* $(x_0, \pm n(x_0), \rho(R_0, R_1))$ *belongs to* R_1.

PROOF. The case where $0 < \rho(R_0) < \infty$. Suppose that two points $(x_0, \pm n(x_0), \rho(R_0))$ do not belong to R_0. Then for a suitable normal vector n_0 at x_0, we can find a sequence $\{s_\lambda \mid s_\lambda > s_{\lambda+1}; \lambda = 1, 2, \ldots\ldots\}$ such that lim $s_\lambda = \rho(R_0)$ and $x_\lambda \equiv (x_0, n_0, s_\lambda) \in R_0$ for all s_λ. Here, there exists an index m which satisfies $s_m - s_{m+1} < \rho(R_0)$. Let α be a curve of class D^1 in R_0 joining x_{m+1} to x_0. Take the arc $[x_0, x_0']$ parallel to a geodesic arc (x_0, n_0, s) $(s_{m+1} \leqq s \leqq s_m)$ along α. Indeed, it is one of two arcs $g(x_0, \pm n_0, s_m - s_{m+1})$. By Lemma 3. 2, $x_0' \in R_0$. Accordingly, $\rho(R_0) \leqq s_m - s_{m+1}$. This is obviously a contradiction. So, 1) holds good.

The case where $\rho(R_0) > 0$. Suppose that $\rho(R') = 0$ for an R-submanifold R'. By Lemma 3. 5 there exists a non-closed subarc $[x_0', x_0]$ of $S(x_0)$ where $x_0' \in R'$. For a suitable normal vector n_0' at x_0', we can find a sequence $\{s_\lambda | s_\lambda > s_{\lambda+1}; \lambda = 1, 2, \ldots\ldots\}$ such that lim $s_\lambda = 0$ and $x_\lambda' \equiv (x_0', n_0', s_\lambda) \in R'$ for all s_λ. Corresponding to each λ, let α_λ denote a curve of class D^1 in R' joining x_0' to x_λ'. Take the arc $[x_\lambda', x_\lambda]$ parallel to the arc $[x', x_0]$ along α_λ. The arc is a subarc of $S(x_0)$. By Lemma 3. 2, $x_\lambda \in R_0$. Moreover there exists a subarc $[x_\mu, x_\nu]$ of $S(x_0)$ (where $x_\mu, x_\nu \in \{x_\lambda\}$, $x_\mu \neq x_\nu$) whose length is smaller than $\rho(R_0)$. This is obviously a contradiction. So, 2) holds good.

The case where $\rho(R_0, R_1) > 0$. Suppose that two points $(x_0, \pm n(x_0), \rho(R_0, R_1))$ do not belong to R_1. Then, $\rho(R_1) = 0$ holds good. We can get thus a contradiction that the distance between R_0, R_1 is smaller than $\rho(R_0, R_1)$. So, 3) holds good.

In M, let R_0 be an R-submanifold. *The condition* $\rho(R_0) > 0$ *is equivalent to the condition that the topology of* R_0 *coincides with the relative one induced from M.* Lemma 4. 3 shows that, *if the topology of* R_0 *coincides with the relative one, this holds also good for other R-submanifold.*

THEOREM 1. *In M suppose that the topology of an R-submanifold coincides with the relative one induced from M. Then M is of one of types I—VI.*

PROOF. For any R-submanifold R, we have $\rho(R) > 0$ The following four cases are considered:

1) The case where each R-submanifold admits normal vector field. By Lemma 4 1, the S-field is generated from a parallel field of unit tangent vectors. We can see $\rho(R_0)=\rho(R_1)$ for any R-submanifolds R_0, R_1. By Lemma 4. 2 the following conclusion is now obvious: For any R-submanifold R, if $\rho(R)=\infty$, M is of type I, and if $\rho(R)<\infty$, M is of type II or III.

2) The case where an R-submanifold R_0 only does not admit normal vector field. Put $L=\rho(R_0)$. Of course $0<L\leqq\infty$. For each $c\,(0<c<L)$, let R_c be the R-submanifold passing through a point $(x_0, n(x_0), c)$ where $x_0 \in R_0$. In our case, $n(x)$ $(x\in R_0)$ is parallel to $-n(x)$ along a suitable curve in R_0. This and Lemma 3. 2 show that R_c consists of $(x, \pm n(x), c)$ for all $x\in R_0$. Here suppose $(x_1, n(x_1), c)=(x_1, -n(x_1), c)$ for $x_1\in R_0$. We can see that R_c does not admit normal vector field. This contradicts with our case. So, we have $(x, n(x), c)\neq(x, -n(x), c)$ for all $x\in R_0$. Next for $x_1, x_2\in R_0\,(x_1\neq x_2)$, suppose $(x_1, n(x_1), c)=(x_2, n(x_2), c)\,(\equiv y_1)$. $g(x_1, n(x_1), c)$ is parallel to $g(x_2, n(x_2), c)$ along a suitable curve in R_0. Hence by Lemma 3. 2 we can see that $n(y_1)$ is parallel to $-n(y_1)$ along the closed curve in R_c. Accordingly R_c does not admit normal vector field. This contradicts with our case, so we have $(x_1, n(x_1), c)\neq(x_2, n(x_2), c)$ for $x_1, x_2\in R_0$ $(x_1\neq x_2)$. These results and Lemma 3. 1 show that R_c is a double covering manifold of R_0. The covering map p satisfies $p(x_\varepsilon)=x$ where $x\in R_0$ and $x_\varepsilon\equiv(x, \varepsilon n(x), c)$ for $\varepsilon=+1$ or -1. It is now clear that R_c is isometrically homeomorphic to $R_{c'}(0<c'<L)$. Suppose $L<\infty$. By Lemmas 3. 2 and 4. 3, we have $x_0'\equiv(x_0, n(x_0), L)\in R_0$ where $x_0\in R_0$. So, there exists a normal vector $n(x_0')$ such that $(x_0', n(x_0'), L)=x_0$. Since $n(x_0')$ is parallel to $n(x_0)$along a suitable curve in R_0, the R-submanifold passing through a point $(x_0, n(x_0), L/2)$ does not admit normal vector field by Lemma 3. 2. This is contrary to our case. Therefore, $L=\infty$ must hold good. These results show that M is of type IV.

3) The case where two R-submanifolds R_0, R_1 only do not admit normal vector field. We get $\rho(R_0, R_1)>0$. Because, if $\rho(R_0, R_1)=0$, we have $\rho(R_0)=\rho(R_1)=0$ which contradicts with the assumption. Put $L=\rho(R_0, R_1)$ and take $x_0\in R_0$. By Lemmas 3. 2 and 4. 3, two points $(x_0, \pm n(x_0), L)$ belong to R_1. We get $\rho(R_0)=2L$. For each $c\,(0<c<L)$, let R_c be the R-submanifold passing through a point $(x_0, n(x_0), c)$. In the same way as in 2), R_c is a double covering manifold of R_0 and further of R_1. We get thus the conclusion that M is of type V or VI.

4) The case where three (or more) R-submanifolds do not admit normal vector field. Let R_0, R_1, and R_2 be such ones. As shown in 3) we have $\rho(R_0, R_1)>0$, and $\rho(R_0)=2\rho(R_0, R_1)$. Similarly, $\rho(R_0)=2\rho(R_0, R_2)$. Therefore,

$\rho(R_0, R_1) = \rho(R_0, R_2)$. By Lemmas 3.2 and 4.3, we get $R_1 = R_2$. This is a contradiction. So, such a case does not occur. This completes the proof of our theorem.

REMARK 1. 1) *The converse of Theorem* 1 *holds also true.*

2) *There exist RS-manifolds from type I to VI.*

3) *In M, if an R-submanifold R is compact, $\rho(R)$ is positive. Then the R-submanifolds are all compact* (by Theorem 1).

5. Fundamental groups and structures. In M, suppose that the S-field is generated from a parallel field $\{v(x) | x \in M\}$ of unit tangent vectors. Let R be an R-submanifold of M. Then we have

LEMMA 5.1. *The map*

$$p: R \times E \to M \quad \text{defined by} \quad p(y, t) = (y, v(y), t)$$

where $y \in R$ becomes a covering map. And $R \times E$ is regarded as a covering manifold of M.

Such a covering manifold is called the *natural covering manifold* of M.

PROOF. Put $N(M) = R \times E$. First we prove that the map p is an onto-map. Take any $x_0 \in M$. Let y_0 be a point of $R \cap S(x_0) (\neq 0$ by Lemma 3.5). There exists t_0 such that $x_0 = (y_0, v(y_0), t_0)$. Let $\tilde{x}_0$ denote a point (y_0, t_0) of $N(M)$. Then $p(x_0) = x_0$. So our assertion is true.

Next, we prove that p is locally an isometric homeomorphism. Take any $\tilde{x}_0 \equiv (y_0, t_0) \in N(M)$ and put $x_0 = p(\tilde{x}_0)$. Let $U(x_0)$ be a reduced coordinate neighborhood of x_0. It is represented by the product $U_R(x_0) \times g(t_1, t_2)$ where $U_R(x_0)$ is an R-neighborhood of x_0 and $g(t_1, t_2)$ is a geodesic arc $(y_0, v(y_0), t) (t_1 < t < t_2)$ not containing its end-points. Of course, $t_1 < t_0 < t_2$. Let $U_R(y_0)$ denote the R-neighborhood of y_0, isometrically homeomorphic to $U_R(x_0)$ under the R-map with respect to $g(x_0, -v(x_0), t_0)$. And further, let $I(t_1, t_2)$ denote the subspace $\{t | t_1 < t < t_2\}$ of E. Accordingly a product $U_R(y_0) \times I(t_1, t_2)$ is regarded as a neighborhood of $\tilde{x}_0$ in $N(M)$. We denote such a neighborhood in $N(M)$ by $U(\tilde{x}_0)$. It is now obvious that $U(\tilde{x}_0)$ is isometrically homeomorphic to $U(x_0)$ under the map p. So, p is locally an isometric homeomorphism.

Again take any $x_0 \in M$. We put $p^{-1}(x_0) = \{\tilde{x}_\lambda \mid \lambda \in J\}$ where J is a set of indices and at most countable by Lemma 3.5. Let $U(x_0)$ be a reduced coordinate neighborhood of x_0. As we have seen above, at each $\tilde{x}_\lambda$ there exists a neighborhood $U(\tilde{x}_\lambda)$ isometrically homeomorphic to $U(x_0)$ under p.

Then, $U(\tilde{x}_\mu) \cap U(\tilde{x}_\nu) = 0$ for $\tilde{x}_\mu, \tilde{x}_\nu \in p^{-1}(x_0)$ $(\tilde{x}_\mu \neq \tilde{x}_\nu)$. To prove this, suppose that it does not hold good. There exists a curve α contained in $U(\tilde{x}_\mu) \cup U(\tilde{x}_\nu)$, joining $\tilde{x}_\mu$ to $\tilde{x}_\nu$. Then, the curve $p(\alpha)$ becomes a closed curve with endpoint x_0, and we can easily find a contradiction. So, our assertion holds good. This completes the proof of our lemma.

Given any $x_0 \in M$, take a point $x_1 \in I(x_0)$ and put $x_1 = (x_0, n_0, c)$ where n_0 is a normal vector and c is a real number. Let α_1 be a curve in $R(x_0)$ joining x_0 to x_1. Let α_2 denote a geodesic arc $g(x_0, n_0, c)$. The product curve $\alpha_1 \cdot \alpha_2^{-1}$ is called an *RS-curve* with endpoint x_0, and according as $c \neq 0$ or $= 0$, is called *proper* or *improper*.

In M suppose that the S-field is generated from a parallel field of unit tangent vectors. Let R be an R-submanifold of M. Then we have

LEMMA 5.2. 1) *A closed curve α with endpoint $x_0 \in M$ is homotopic to an RS-curve leaving x_0 fixed. Moreover, a proper RS-curve is not homotopic to an improper RS-curve.* 2) *If $\rho(R) < \infty$, the fundamental group $\pi_1(M)$ has an infinite cyclic subgroup. If $\rho(R) = 0$ especially, $\pi_1(M)$ is not infinite cyclic.*

PROOF. First put $R_0 = R(x_0)$. By Lemma 5.1, we regard $R_0 \times E$ as the natural covering manifold of M. Let p be the covering map. Here we suppose that $p(x, 0) = x$ for all points $(x, 0)$ of the submanifold of $R_0 \times E$ defined by $t = 0$. This is possible. Let α_N be the curve in $R_0 \times E$ with initial point $(x_0, 0)$ such that $p(\alpha_N) = \alpha$. For convenience, let us represent α_N by a parametrized curve $(x(\tau), t(\tau))$ $(0 \leqq \tau \leqq 1)$. Of course, $(x(0), t(0)) = (x_0, 0)$. We denote a curve $(x(\tau), 0)$ $(0 \leqq \tau \leqq 1)$ by α_{1N} and a curve $(x(1), t(\tau))$ $(0 \leqq \tau \leqq 1)$ by α_{2N}. Then, α_N is homotopic to the product curve $\alpha_{1N} \cdot \alpha_{2N}$ leaving the endpoints fixed. Hence, α is homotopic to the curve $p(\alpha_{1N} \cdot \alpha_{2N})$ leaving x_0 fixed. The curve $p(\alpha_{1N} \cdot \alpha_{2N})$ being an RS-curve with endpoint x_0, the former part of 1) has been proved.

Suppose that the above curve α is a proper RS-curve. The curve α_N coincides with the product curve $\alpha_{1N} \cdot \alpha_{2N}$ where $t(1) \neq 0$. On the other hand, let α' be an improper RS-curve with endpoint x_0. Let α'_N be the curve in $R_0 \times E$ with initial point $(x_0, 0)$ such that $p(\alpha'_N) = \alpha'$. As $\alpha' \subset R_0$, α'_N is represented by $(\alpha', 0)$. Then the terminal points of α_N, α'_N are not the same point. This implies that α is not homotopic to α'. From this and Lemma 4.2, the latter part of 1) is easily proved.

Next we prove 2). In $R_0 = R(x_0)$, let us suppose that $\rho(R_0) < \infty$ and use the previous notations. Then there existsa proper RS-curve β of class D^1 with endpoint x_0. β is represented by the product curve $\beta_1 \cdot \beta_2^{-1}$ where β_1

$\subset R_0$ and $\beta_2 \subset S(x_0)$. β is not homotopic to zero. This is easily seen if we construct an inverse image of β by the map p. (Of course this is valid for any proper RS-curve.) We displace β_2 parallelly along β_1 and then denote the locus of terminal point of β_2 by β_1'. Let $g(x_0, n(x_0), c)$ $(c > 0)$ denote β_2. The terminal point of β_1' is expressed by $(x_0, n(x_0), 2c)$. Denote the geodesic arc $g(x_0, n(x_0), 2c)$ by β_2'. Then, a closed curve $\beta^2 (= \beta \cdot \beta)$ becomes homotopic to a proper RS-curve $\beta_1 \cdot \beta_1' \cdot \beta_2'^{-1}$ leaving their endpoints x_0 fixed. In fact, let β_N, β_N' be the curves in $R_0 \times E$ with the same initial point $(x_0, 0)$ such that $p\ (\beta_N) = \beta^2$ and $p(\beta_N') = \beta_1 \cdot \beta_1' \cdot \beta_2'^{-1}$. We can see that, the terminal points are the same point and they are homotopic leaving the endpoints fixed. From this our assertion is clear. I. e., β^2 is homotopic to a proper RS-curve with endpoint x_0. This is also valid for all of β^λ $(\lambda = 1, 2, \ldots\ldots)$, and they are not homotopic to zero as already mentioned. From this, the former part of 2) is proved. It is now easy to prove the latter part of 2).

In M we have

LEMMA 5. 3. *There exists at least one R-submanifold which admits normal vector field.*

PROOF. Suppose that all the R-submanifolds do not admit normal vector field. Take any two R_0, R_1 of them. At $x_0 \in R_0$ we can find $c \neq 0$ such that $(x_0, n(x_0), c) \in R_1$. Now let R' denote the R-submanifold passing through $x' = (x_0, n(x_0), c/2)$. By the assumption, a normal vector $n(x')$ is parallel to $-n(x')$ along a suitable curve in R'. Hence by Lemma 3. 2 we can see that the two points x_0 and $(x_0, n(x_0), c)$ are contained in the same R-submanifold. That is, R_0 coincides with R_1. This implies that M consists of an R-submanifold only, because R_0, R_1 are anyones. It contradicts with Lemma 3. 5. So our lemma is true.

In M, suppose that the S-field is not generated from a parallel field of unit tangent vectors, i. e., there is not such a parallel field which generates the S-field. Let $T(M)$ be the tangent bundle of M, so that each point of $T(M)$ is represented by a pair (x, v) of a point $x \in M$ and a tangent vector v at x. Let $\pi: T(M) \to M$ be the projection. We take the subspace $H(M)$ of $T(M)$ which consists of points $(x, \pm n(x))$ for all $x \in M$. $H(M)$ is an n-submanifold, and by the assumption connected. Put $h = \pi \mid H(M)$. Then *$H(M)$ is regarded as a double covering manifold of M under the Riemannian metric naturally induced from M by h. The map h is the covering map.* We call this covering manifold $H(M)$ the *holonomy covering manifold* of M. This has the following properties: *$H(M)$ admits a parallel* *field of unit tangent vectors which is induced from the S-field of M by h.* Accordingly, $H(M)$ is of course an RS-manifold of dimension n. Take an R-submanifold R_0 of M admitting normal vector field (Lemma 5.3). Let x_0 be any point of R_0. Then, *in $H(M)$ the R-submanifold passing through $(x_0, n(x_0))$ is distinct from the one passing through $(x_0, -n(x_0))$. Each of them is isometrically homeomorphic to R_0 under h.* Next, take an R-submanifold R_1 of M not admitting normal vector field (Lemma 4.1). Let x_1 be any point of R_1. Then *the R-submanifold R_{1H} of $H(M)$ passing through $(x_1, n(x_1))$ passes through $(x_1, -n(x_1))$ and is a double covering manifold of R_1 where $h|R_{1H}$ is the covering map.*

THEOREM 2. *In M if the fundamental group $\pi_1(M)$ is finite, M is of type I or IV.*

PROOF. 1) The case where the S-field is generated from a parallel field of unit tangent vectors. By Lemma 5.2, $\rho(R) = \infty$ where R is any R-submanifold of M. Accordingly by Theorem 1, M is of type I.

2) The other case. By Lemma 4.1, there exsts an R-submanifold R_0 not admitting normal vector field. Let $H(M)$ be the holonomy covering manifold of M. Let h be the covering map. Let R_{0H} be the R-submanifold of $H(M)$ such that $h(R_{0H}) = R_0$. In $H(M)$, the fundamental group is also finite. And further, $H(M)$ is an RS-manifold which satisfies the above case 1). So, $H(M)$ is of type I. That is, $H(M)$ is represented by the metric product $R_{0H} \times E$. We take any $x_0 \in R_0$. Let x_{0H} be a point of R_{0H} such that $p(x_{0H}) = x_0$. Let $S(x_{0H})$ denote the S-submanifold of $H(M)$ passing through x_{0H}. We have $h\cdot(S(x_{0H})) = S(x_0)$. Hence $S(x_0)$ is non-closed and $S(x_0) \cap R_0$ consists of the point x_0 only. So, $\rho(R_0) = \infty$. By Lemma 4.3, $\rho(R) > 0$ for any R-submanifold R. Accordingly by Theorem 1, M must be of type IV. This completes the proof of our theorem.

REMARK 2. 1) *In Theorem 2, types I, IV are characterized by the condition that $\pi_1(R)$, where R is an R-submanifold, is finite. If the order of $\pi_1(M)$ is odd, M is of type I and not of type IV.*

2) *There exist RS-manifolds of type I and ones of type IV, whose fundamental groups are all finite.*

THEOREM 3. *In M if the fundamental group $\pi_1(M)$ is infinite cyclic, M is one of types I—IV.*

PROOF. 1) The case where the S-field is generated from a parallel field of unit tangent vectors. Take an R-submanifold R of M. By Lemma 5.2, $\rho(R) > 0$. Accordingly by Theorem 1, M is of one of types I—III.

2) The other case. By Lemma 4.1, there exists an R-submanifold R_0

not admitting normal vector field. Take $x_0 \in R_0$. Let $H(M)$ be the holonomy covering manifold of M. Let h be the covering map. Let R_{0H} be the R-submanifold of $H(M)$ such that $h(R_{0H}) = R_0$. In $H(M)$ the fundamental group is infinite cyclic too. Accordingly by the above 1), $H(M)$ is of one of types I—III.

If $H(M)$ is of type I, M of type IV. This is verified by the same way as in Theorem 2. Note here that $\pi_1(R_0)$ is infinite cyclic.

If $H(M)$ is of type II or III, R_{0H} must be simply-connected by Lemma 5. 2. Hence $\pi_1(R_0)$ is cyclic of order 2. Accordingly, we get the conclusion that $\pi_1(M)$ contains a subgroup which is cyclic of order 2. This obviously contradicts with the assumption that $\pi_1(M)$ is infinite cyclic. So, $H(M)$ is not of type II or III. This completes the proof of our theorem.

REMARK 3. 1) *In Theorem* 3, *type I is characterized by the following condition a*) *and type IV by the following condition b*):

a) $\pi_1(R)$, *where R is an R-submanifold, is infinite cyclic.*

b) $\pi_1(R)$, *where R is the R-submanifold not admitting normal vector field, is infinite cyclic*

And, types II, III are characterized by the condition that an Rsubmanifold is simply-connected.

2) *There exist RS-manifolds from type I to IV, whose fundamental groups all are infinite cyclic*

6. Closedness of S-geodesics and structures. At $x_0 \in M$, let $T_R(x_0)$ denote the Euclidean vector $(n-1)$-space tangent to $R(x_0)$ at x_0. We denote the length of an S-geodesic S by $|S|$ So, S is closed or non-closed according as $|S| < \infty$ or $= \infty$. Again at $x_0 \in M$ take a subset $\{x | x \in R(x_0), d_R(x, x) < c\}$ where c is a positive constant. If the subset forms an R-neighborhood of x_0, we denote the R-neighborhood by $U_R(x_0; c)$. Especially if $U_R(x_0; c)$ can be covered by a normal coordinate system in $R(x_0)$ with center x_0, we call it a *normal R-neighborhood* of x_0. Then let $N_R(x_0; c)$ denote the R-neighborhood $U_R(x_0; c)$. The *exponential map* at x_0 is defined to be the map

$$\varphi: T_R(x_0) \rightarrow R(x_0)$$

such that $\varphi(v) = x_0$ for the zero vector $v \in T_R(x_0)$ and $\varphi(v) = (x_0, v/|v|, |v|)$ for any non-zero vector $v \in T_R(x_0)$ where $|v|$ is the length of v. Let $e(x_0)$ denote the greatest lower bound of $\{d_R(x_0, x) | x \in I(x_0) - x_0\}$ if $I(x_0) - x_0$ is non-empty. When $I(x_0) - x_0$ is empty, put $e(x_0) = +\infty$.

In M suppose that there exists $x_0 \in M$ such that $S(x_0)$ is non-closed and x_0 is not a limit point of $I(x_0)$ relative to $R(x_0)$. Then $e(x_0) > 0$. Take an R-neighborhood $U_R(x_0; a)$ where $0 < a < e(x_0)/2$. Let $\{n(x) | x \in U_R(x_0; a)\}$ be a normal vector field. Then we have

LEMMA 6.1. *The map*

$$f: U_R(x_0; a) \times E \to M \qquad \textit{defined by} \qquad f(x, t) = (x, n(x), t)$$

is an isometric into-homeomorphism.

Such a map f is called a *cylinder map* at x_0 and such an R-neighborhood $U_R(x_0; a)$ is called *proper.*

PROOF. First, suppose $(x_1, n(x_1), t_0) = (x_2, n(x_2), t_0)$ for $x_1, x_2 \in U_R(x_0; a)$ $(x_1 \neq x_2)$. Then we get $x_1 \neq x_0$ and $x_2 = (x_1, n(x_1), 2t_0)$. Take a minimizing geodesic $[x_1, x_0]$ in $R(x_0)$. Let $[x_2, x_0']$ be the geodesic arc parallel to $[x_1, x_0]$ along an arc $g(x_1, n(x_1), 2t_0)$. We have $[x_2, x_0'] \subset R(x_0)$ and $x_0' \in S(x_0)$ by Lemma 3.4. That is, $x_0' \in I(x_0)$. $S(x_0)$ being however non-closed, it follows that $x_0 \neq x_0'$. By Lemma 3.4, $d_R(x_2, x_0') \leqq d_R(x_1, x_0)$. Hence,

$$d_R(x_0, x_0') \leqq d_R(x_0, x_2) + d_R(x_2, x_0') < 2a < e(x_0).$$

This is contrary to the definition of $e(x_0)$. Next, suppose $(x_1, n(x_1), t_1) = (x_2, n(x_2), t_2)$ for $x_1, x_2 \in U_R(x_0; a)$ $(x_1 \neq x_2)$. We have $x_2 = (x_1, n(x_1), t')$ for $t' = t_1 + \varepsilon t_2 (\varepsilon = +1$ or $-1)$. By the same way, we get again the same contradiction. Accordingly, by Lemma 3.1 our lemma is proved.

THEOREM 4. *In M suppose that all S-geodesics are non-closed. If a point $x_0 \in M$ is a limit point of $I(x_0)$ relative to $R(x_0)$, each point $x \in M$ is also a limit point of $I(x)$ relative to $R(x)$ and then M is of non-fibred type III, non-fibred type VI, or clustered type. If a point $x_0 \in M$ is not a limit point of $I(x_0)$ relative to $R(x_0)$, then M is of fibred type.*

PROOF. We first prove that, if a point $x_0 \in M$ is not a limit point of $I(x_0)$ relative to $R(x_0)$, each $x \in M$ is not a limit point of $I(x)$ relative to $R(x)$.

Let R^0 be the maximal subset of $R(x_0)$, in which each x is not a limit point of $I(x)$ relative to $R(x)$ $(= R(x_0))$. Of course $R^0 \ni x_0$. By Lemma 6.1, at any $x \in R^0$ there exists a proper R-neighborhood U_R. Then $U_R \subset R^0$. Hence, R^0 is open in $R(x_0)$. Next let us verify that R^0 is closed in $R(x_0)$. Let $\overline{R^0}$ denote the closure of R^0 relative to $R(x_0)$. Suppose $\overline{R^0} \neq R^0$. At any $y_1 \in \overline{R^0} - R^0$ we take a normal R-neighborhood $N_R(y_1; c)$. A set $S(y_1) \cap N_R(y_1; c/2)$ is infinite, and countable by Lemma 3.5. We denote the set by $\{y_\lambda | \lambda = 1, 2, \ldots\ldots\}$. Let $\{n(y) | y \in N_R(y_1; c)\}$ be a normal vector field. For suitable t_λ, each y_λ is represented by $(y_1, n(y_1), t_\lambda)$. Take a point $z_1 \in R^0 \cap N_R(y_1; c/2)$. Let $[y_1, z_1]$ denote the geodesic arc in $N_R(y_1; c/2)$. And, displace it parallelly along $S(y_1)$. Then, at each y_λ a geodesic arc $[y_\lambda, z_\lambda]$ is obtained. It follows that $[y_\lambda, z_\lambda] \subset R(y_1)$. By Lemma 3.4, $z_\lambda \in S(z_1)$ and $d_R(y_\lambda, z_\lambda) \leqq d_R(y_1, z_1)$. Hence,

$$d_R(y_1, z_\lambda) \leqq d_R(y_1, y_\lambda) + d_R(y_\lambda, z_\lambda) < c/2 + c/2 = c.$$

So, $z_\lambda \in N_R(y_1; c)$. $S(z_1)$ being however non-closed, $z_\lambda\,(\lambda = 1, 2, \ldots\ldots)$ are distinct from one another. Accordingly, $N_R(y_1; c)$ contains an infinite set $\{z_\lambda | \lambda = 1, 2, \ldots\ldots\}$. On the other hand, the closure of $N_R(y_1; c)$ in $R(x_0)$ is compact. Consequent for any small $\delta > 0$ we can find z_μ, $z_\nu \in \{z_\lambda | \lambda = 1, 2, \ldots\ldots\}$ such that $d_R(z_\mu, z_\nu) < \delta$. This is contrary to the existence of a cylinder map at $z_1 \in R^0$. So, $\overline{R^0} = R^0$. That is, R^0 is closed in $R(x_0)$. Since R^0 is however open in $R(x_0)$, it follows that $R^0 = R(x_0)$. Therefore by Lemmas 3. 1 and 3. 5 our assertion is proved.

1) The case where there exists $x_0 \in M$, which is a limit point of $I(x_0)$ relative to $R(x_0)$. By the above assertion each point x of M is a limit point of $I(x)$ relative to $R(x)$. Hence if $\rho(R_0) > 0$, M is of non-fibred, type III or VI by Theorem 1. If $\rho(R_0) = 0$, M is of clustered type by Lemma 4. 3.

2) The other case. We take an R-submanifold R_0. If $x, y \in R_0$ belong to the same S-geodesic, we say that they are equivalent to each other. By this equivalence relation, we construct the quotient space of R_0 and denote it by B. Then by Lemma 6. 1 the space B is regarded as a connected complete Riemannian $(n-1)$-manifold of class C^1 under the Riemannian metric naturally induced from R_0. Next for any $y \in M$, let $[y]$ denote the point of B representing $R_0 \cap S(y)$. Then the map $\pi: M \to B$, defined by $\pi(y) = [y]$, is an onto-map by Lemma 3. 5. Thus it is now obvious that M is of fibred type. Here the base space is B and the projection is π and so on. This completes the proof of our theorem.

REMARK 4. 1) *If M is of clustered type, the S-geodesics are all non-closed.*

2) *There exist RS-manifolds of the respective types enumerated in Theorem 4, whose S-geodesics are all non-closed. In this case, an RS-manifold of fibred type is further of type I, III, IV, or VI, or non-simple type.* (See Appendix)

In M suppose that an S-submanifold S_0 is closed. We take a point $x_0 \in S_0$. Then $e(x_0) > 0$, because S_0 is closed. Let us put $L = |S_0|$. Now, take an R-neighborhood $U_R(x_0; a)$ where $0 < a < e(x_0)/2$. And if $\{n(x) | x \in U_R(x_0; a)\}$ is a normal vector field, by the similar way as in Lemma 6. 1 we can verify

LEMMA 6. 2. *The map*

$$f: U_R(x_0; a) \times [L] \to M \quad \textit{defined by} \quad f(x, t) = (x, n(x), t)$$

is an isometric into-homeomorphism provided that $U_R(x_0; a)$ is doubly treated in M as the images by f at $t = 0, L$.

Such a map is also called a *cylinder map* at x_0 and such an R-neighborhood $U_R(x_0; a)$ is called *proper*. Here we see that the map

$$x \to f(x, L)$$

for all $x \in U_R(x_0; a)$ is an isometric homeomorphism of $U_R(x_0; a)$ onto itself. Accordingly, this map induces a congruent transformation T in $T_R(x_0)$. T is called the congruent transformation in $T_R(x_0)$ *induced* from the cylinder map f. If we take a suitable orthonormal frame (e_a) in $T_R(x_0)$, then T, relative to (e_a), is represented by the following orthogonal matrix:

$$\begin{pmatrix} E_1 & & & & \\ & -E_2 & & 0 & \\ & & A_1 & & \\ & 0 & & \ddots & \\ & & & & A_k \end{pmatrix} \tag{6.1}$$

where E_1, E_2 denote the unit matrices of degrees r_1, r_2 respectively and

$$A_\lambda = \begin{pmatrix} \cos\theta_\lambda & -\sin\theta_\lambda \\ \sin\theta_\lambda & \cos\theta_\lambda \end{pmatrix}$$

for $0 < \theta_\lambda < \pi$ ($\lambda = 1, 2, \ldots\ldots k$; $r_1 + r_2 + 2k = n - 1$).

THEOREM 5. *In M suppose that among the S-geodesics there exist both closed one and non-closed one. Let M^0 be the subspace of M which is the union set of all non-closed S-geodesics. Then, M^0 is a connected open submanifold of M whose closure is M, and the maximal subset of M in which each point x is a limit point of $I(x)$ relative to $R(x)$. M is of non-fibred type III, non-fibred type VI, or almost clustered type with kernel M^0.*

PROOF. We take an R-submanifold R. Put $R^0 = \{x \mid x \in R, |S(x)| = \infty\}$. Then, two sets $R - R^0$ and R^0 are non-empty by the assumption and Lemma 3.5.

1) Take $x_0 \in R - R^0$ and $y_0 \in R^0$. Let $g(x_0, u_0, c)$ be a geodesic arc $[x_0, y_0]$ in R. $S(x_0)$ being closed, there exists a congruent transformation T in $T_R(x_0)$ induced from a cylinder map at x_0. However, $S(y_0)$ being non-closed, it follows that the vectors

$$u_0,\ Tu_0,\ \ldots\ldots,\ T^m u_0,\ \ldots\ldots$$

are distinct from one another. This implies that, if we represent T by a matrix (6.1), there exists at least one θ_λ such that π/θ_λ is an irrational number. On the other hand we take a vector $u \in T_R(x_0)$, for which there exists an integer $m > 0$ such that $T^m u = u$. u may be the zero vector. Here, such a vector u is said to be singular at x_0. All of singular vectors at x_0 from a vector subspace Z of $T_R(x_0)$. The existence of θ_λ implies that the dimension of Z is not greater than $n - 3$. Let φ denote the exponential map at x_0. Let $N_R(x_0; a)$ be a normal R-neighborhood. Then a set $\varphi(Z) \cap N_R(x_0; a)$ becomes a surface of dimension $\leqq n - 3$. This shows that a set $R^0 \cap N_R(x_0;$

a) is connected and open in R. Moreover we see that x_0 belongs to the closure $\overline{R^0}$ of R^0 relative to R. So, $\overline{R^0} = R$.

2) Take again any $y_1 \in R^0$. Suppose that every R-neighborhood of y_1 is not contained in R^0. Then at a suitable $x_1 \in R - R^0$, we can find a normal R-neighborhood $N_R(x_1; b)$ such that $y_1 \in N_R(x_1; b)$. By 1), $R^0 \cap N_R(x_1; b)$ $(\ni y_1)$ is a connected open set in R. This contradicts with our assumption. Accordingly, there exists an R-neighborhood of y_1, contained in R^0. This shows that R^0 is open in R.

3) Let z_1, z_2 be any points of R^0. Take a curve α in R joining z_1 to z_2. By 1) and 2), if we cover α by a finite number of suitable normal R-neighborhoods, we can see that z_1 and z_2 are joined by a curve in R^0. So, R^0 is connected.

By Lemma 3.5, M^0 is also regarded as the union set of all S-geodesics, each of which passes through a point of R^0. Accordingly 1) – 3) above show that M^0 is a connected open submanifold of M whose closure is M by Lemmas 3.1 and 3.5.

Next, take again any $x_0 \in R - R^0$. Let $N_R(x_0; a)$ be a normal R-neighborhood. Then, a point $y_0 \in R^0 \cap N_R(x_0; a)$ is a limit point of $I(y_0)$ relative to R. In fact, let $u_0 \in T_R(x_0)$ be the inverse image of y_0 by the exponential map at x_0. The vector u_0 is not singular at x_0. Hence, if we put $Y = \{y \mid y \in I(y_0),\ d_R(x_0, y) = |u_0|\}$ where $|u_0|$ is the length of u_0, Y is an infinite set. Take a normal R-neighborhood $N_R(y_0; c)$. For any $\delta\,(0 < \delta < c)$, we can find $y_\mu, y_\nu \in Y (y_\mu \neq y_\nu)$ such that $d_R(y_\mu, y_\nu) < \delta$. Here displace a minimizing geodesic $[y_\mu, y_\nu]$ parallelly along $S(y_0)$. At y_0 we get a geodesic arc $[y_0, y_0'] \subset N_R(y_0; c)$. By Lemmas 3.3 and 3.4, it follows that $y_0' \in I(y_0)$, $d_R(y_0, y_0') < \delta$, and $y_0 \neq y_0'$. Hence our assertion is easily seen. This fact shows that each y of R^0 is a limit point of $I(y)$ relative to R. For, if we express a geodesic arc $[x_0, y]$ by $g(x_0, u, d)$, the vector u is not singular. From this and the above fact, it is easily verified.

By Lemmas 3.1 and 3.5, we can now see that each $x \in M^0$ is a limit point of $I(x)$ relative to $R(x)$. This is not valid for any $x \notin M^0$, $S(x_0)$ being closed.

Accordingly, if $\rho(R) > 0$, M is of non-fibred type III or non-fibred type VI by Theorem 1. If $\rho(R) = 0$, M is of almost clustered type with kernel M^0 by Lemma 4.3. This completes the proof of our theorem.

REMARK 5. 1) *In Theorem* 5, *almost clustered type is not clustered type.*

2) *There exist RS-manifolds of the respective types enumerated in Theorem* 5 *each of which has both closed S-geodesic and non-closed one* (see Appendix).

THEOREM 6. *In M suppose that all the S-geodesics are closed. Then among them there exist S-geodesics with the longest length. In all of such ones let M^0 be the subspace of M which is their union set. Then M^0 is a connected open submanifold of M whose closure is M, and a maximal subspace which becomes a fibre bundle where each fibre is an S-geodesic. In other words, M is of almost fibred type with kernel M^0.*

PROOF. 1) First, take an R-submanifold R. At $x_0 \in R$ let $N_R(x_0; a)$ be a proper normal R-neighborhood. Let $\{n(x) | x \in N_R(x_0; a)\}$ be a normal vector feld. Here, put $L_0 = |S(x_0)|$. Let T be the congruent transformation in $T_R(x_0)$ induced from a cylinder map at x_0. Since all the S-geodesics are closed, we can find the least positive integer m such that T^m becomes the identity transformation. And there exists a unit vector $u_0 \in T_R(x_0)$ such that the vectors

$$u_0,\ Tu_0,\ \ldots\ldots, T^{m-1}u_0,$$

are distinct from one another where $T^m u_0 = u_0$. If $y \in N_R(x_0; a)$ is an interior point of a geodesic arc $g(x_0, u_0, a)$, we have $|S(y)| = mL_0$ because $N_R(x_0; a)$ is proper. Here we put $L = mL_0$. Take any $z \in R$. Let u be the vector at x_0 tangent to a geodesic arc $[x_0, z]$. Of course, $T^m u = u$. Hence, L is an integral multiple of $|S(z)|$ by Lemma 3. 4. So $|S(z)| \leqq L$. Consequently, the above $S(y)$ is an S-geodesic with the longest length.

2) We put $R^0 = \{x | x \in R,\ |S(x)| = L\}$. At $y_0 \in R^0$, let $N_R(y_0; b)$ be a proper normal R-neighborhood. Let $\{n(y) | y \in N_R(y_0; b)\}$ be a normal vector field. Define the cylinder map f: $N_R(y_0; b) \times [L] \to M$ by $f(y, t) = (y, n(y), t)$. Then for all $y \in N_R(y_0; b)$, we have $f(y, 0) = f(y, L)$. Hence $|S(y)| = L$, and $N_R(y_0; b) \subset R^0$. Accordingly, R^0 is open in R.

Next, provided that $R - R^0 \neq 0$, suppose that at $z_0 \in R - R^0$ there exists a normal R-neighborhood $N_R(z_0; c)$ which is contained in $R - R^0$. Hence, $|S(z)| < L$ for all $z \in N_R(z_0; c)$. And by Lemmas 3. 4 and 6. 2, $|S(x)| < L$ for all $x \in R$. This contradicts with 1). So, it follows that relative to R the closure of R^0 is R.

On the other hand, M^0 is also regarded as the union set of all S-geodesics whose lengths are all L. Accordingly, the above facts show that M^0 is an open submanifold of M, whose closure is M by Lemmas 3. 1 and 3. 5.

3) Let us prove that M^0 is connected. If $R = R^0$, we have $M = M^0$ and M^0 is obviously connected. So, suppose $R \neq R^0$. Take any $z_0 \in R - R^0$ and a proper normal R-neighborhood $N_R(z_0; c)$. In $N_R(z_0; c)$ we put $W_R = N_R(z_0; c) \cap R^0$. In W_R let W denote the union set of $S(y)$ for all $y \in W_R$. On the other hand, let T_0 be the congruent transformation in $T_R(z_0)$ induced

from a cylinder map at z_0. As mentioned in 1), there exists the least positive integer h such that T_0^h becomes the identity transformation. Then, $h \geqq 2$ and $L = h|S(z_0)|$. Here, let us call a vector $u \in T_R(z_0)$ singular at z_0 if $T_0^\mu u = u$ for an integer $\mu\,(0 < \mu < h)$. Of course, the zero vector of $T_R(z_0)$ is singular. Take a suitable frame (e_a) in $T_R(z_0)$, relative to which T_0 is represented by a matrix (6. 1). The following three cases are considered:

a) The case where $r_1 + r_2 = n - 1$, $r_2 = 1$. Then, $h = 2$. All of singular vectors at z_0 form an $(n-2)$ vector-subspace Z. If we map Z by the exponential map at z_0, we get in $N_R(z_0; c)$ an $(n-2)$-surface geodesic at z_0. So, W_R is not connected. However, W is connected. For, if $u \in T_R(z)$ is not singular, we have $T_0 u = -u$. From this it is obvious.

b) The case where $r_1 + r_2 = n - 1$, $r_2 \geqq 2$. Then, $h = 2$ too. All of singular vectors at z_0 form a vector subspace of dimension $\leqq n-3$. This implies that W_R is connected. So, W is connected.

c) The case where $r_1 + r_2 < n - 1$. Then $h > 2$, and for a suitable integer l,

$$h\theta_1 = \ldots\ldots = h\theta_k = 2\pi l.$$

Now we take a singular vector

$$u = u_1 e_1 + \ldots\ldots + u_{n-1} e_{n-1}.$$

From its components u_a, let us construct the following $k+1$ combinations:

$$(u_{r_1+1}, \ldots, u_{r_1+r_2}),\ (u_{r_1+r_2+1}, u_{r_1+r_2+2}),\ (u_{r_1+r_2+3}, u_{r_1+r_2+4}),$$
$$\ldots\ldots,\ (u_{r_1+r_2+2k-1}, u_{r_1+r_2+2k}).$$

Then, there exists at least one combination, all of whose elements are zero. So, we can see that all of singular vectors form the union set of a finite number of vector subspaces in $T_R(x_0)$. If $r_2 \neq 1$, all the singular vectors form the union set of some vector subspaces of respective dimensions $\leqq n-3$. This implies that W_R is connected. So, W is also connected. In the case $r_2 = 1$ too, it follows that W is connected even if W_R is not connected.

Consequently, it has been proved that W is connected. From this we can see that M^0 is connected. For, take any $y_1, y_2 \in R^0$ and a curve in R joining y_1 to y_2. Cover the curve by a finite number of proper normal R-neighborhoods $N_\lambda (\lambda = 1, 2, \ldots\ldots)$. Let W_λ denote the union set of $S(y)$ for all $y \in N_\lambda \cap R^0$. Then, W_λ are all connected. By 2), the union set of all W_λ is also connected. This implies that y_1, y_2 are joined by a curve in M^0. Therefore by Lemma 3. 5, M^0 is connected.

4) From the above results, we see that M is of almost fibred type with kernel M^0. In fact, at any $y_0 \in R^0$ let $N_R(y_0; b)$ be proper. By 2), we have $N_R(y_0; b) \subset R^0$. If we apply the cylinder map to $N_R(y_0; b) \times [L]$, the image is wholly contained in M^0. We can thus verify that M^0 is a fibre bundle

where each fibre is an S-geodesic (cf. Proof of Theorem 4). It is easy to see that, there is no subspace, $\supset M^0, \neq M^0$, which becomes such a fibre bundle. So our assertion is true. This completes the proof of our theorem.

REMARK 6. 1) *If M is of almost fibred type which is not fibred type, the S-geodesics are all closed* (by Theorems 4, 5).

2) *There exist RS-manifolds of almost fibred type which is not fibred type. Such an RS-manifold is further of type III or VI, or non-simple type. There exist RS-manifolds of fibred type whose S-geodesics are all closed. Such an RS-manifold is further of type II, III, V, or VI, or non-simple type.* (See Appendix.)

Finally, if we sum up Theorems 4—6, the following theorem is obtained:

THEOREM 7. *M is of almost fibred type, almost clustered type, non-fibred type III, or non-fibred type VI.*

The auther wishes to express here his sincere gratitude to Prof. S. Sasaki for his kind guidance during the preparation of the manuscript.

APPENDIX

In Remarks 1—6, we treated of the existence of RS-manifolds which satisfy some conditions. For the RS-manifolds there enumerated, we can all construct their models. Here let us show some of them, whose constructions seem comparatively to be difficult.

1. We take the torus D in Euclidean 4-space E^4, defined by

$$x_1 = \cos\sigma,\ x_2 = \sin\sigma,\ x_3 = \cos\tau,\ x_4 = \sin\tau$$
$$(-\infty < \sigma,\ \tau < \infty)$$

where $x_\lambda\,(\lambda = 1, 2, 3, 4)$ denote usual orthogonal coordinates in E^4. Let us regard D as a Euclidean 2-space form with the metric naturally induced from E^4. Construct the metric product $D \times [L]$. Let $D(0)$, $D(L)$ be the 2-submanifolds of $D \times [L]$ defined by $t = 0$, L respectively. Define

$$\phi_0 :\ D(0) \to D(0) \text{ by } \phi_0(\sigma, \tau, 0) = (\sigma + \pi,\ -\tau,\ 0).$$

Next, take a constant $\tau_0\,(0 < \tau_0 < \pi)$ and again define

$$\phi_L :\ D(L) \to D(L) \text{ by } \phi_L(\sigma, \tau, L) = (\sigma + \pi,\ -\tau + 2\tau_0,\ L).$$

Indeed, the maps ϕ_0, ϕ_L are isometric involutive homeomorphisms and have not fixed point. Accordingly in $D \times [L]$ if we identify x with $\phi_0(x)$ for all $x \in D(0)$ and y with $\phi_L(y)$ for all $y \in D(L)$, we get a Euclidean space form M^3. M^3 is also an RS-manifold of type VI. Especially, if π/τ_0 is an irrational number, M^3 becomes an *RS-manifold of non-fibred type VI whose S-geodesics are all non-closed* (Remark 4). If π/τ_0 is a rational number, M^3

becomes an *RS-manifold of fibred type and further type VI, whose S-geodesics are all closed* (Remark 6).

2. Instead of the torus D above, we take a cylinder in Euclidean 3-space E^3, defined by

$$x_1 = \cos\sigma,\ x_2 = \sin\sigma,\ x_3 = \tau\,(-\infty < \sigma, \tau < \infty),$$

where $x_\lambda(\lambda = 1, 2, 3)$ denote usual orthogonal coordinates in E^3. In the same way, we can also get an RS-manifold M^3. M^3 is an *RS-manifold of fibred type and further type VI, whose S-geodesics are all non-closed* (Remark 4).

3. In Euclidean 3-space E^3, let D be the subspace $\{(x_1, x_2, t) \mid |x_1| \leqq 1, 0 \leqq t \leqq 1\}$ where x_1, x_2, t denote usual orthogonal coordinates of E^3. Let $D(0)$, $D(1)$ be the subspaces of D defined by $t = 0$, 1 respectively. For a constant $c(\neq 0)$ we define

$$\phi : D(1) \to D(0) \text{ by } \phi(x_1, x_2, 1) = (x_1, x_2 + c, 0).$$

The map ϕ is an isometric homeomorphism. By this, identify $D(1)$ with $D(0)$. The space thus constructed from D we denote by D'. In D' let $D'(1)$, $D'(-1)$ denote the 2-submanifolds defined by $x_1 = 1, -1$ respectively. Take an irrational number $t_0 (0 < t_0 < 1)$ and again define

$$\psi : D'(1) \to D'(-1) \text{ by } \psi(1, x_2, t) = (-1, [x_2], [t + t_0])$$

where $[x_2] = x_2$ or $x_2 + c$ and $[t + t_0] = t + t_0$ or $t + t_0 - 1$ according as $t + t_0 < 1$ or $\geqq 1$. The map ψ is an isometric homeomorphism. By this, identify $D'(1)$ with $D'(-1)$. The space thus constructed from D' we denote by M^3. M^3 is a Euclidean 3-space form. M^3 is also an *RS-manifold of fibred type and further non-simple type, whose S-geodesics are all nonclosed* (Remark 4).

4. Let r, θ, z be cylindrical coordinates in Euclidean 3-space E^3. We take the closed domain D defined by $0 \leqq z \leqq 2$. In D, identify points $(r, \theta, 0)$ with points $(r, \theta, 2)$ foı all r, θ. We obtain thus a Euclidean 3-space form D'. In $D' \times [L]$ let $D'(0)$, $D'(L)$ denote the 3-submanifolds defined by $t = 0$, L respectively. Take a constant $\theta_0 (0 < \theta_0 < \pi)$ and define

$$\phi_0 : D'(0) \to D'(0) \quad \text{by } \phi_0(r, \theta, z, 0) = (r, -\theta, [z + 1], 0).$$

$$\phi_L : D'(L) \to D'(L) \quad \text{by } \phi_L(r, \theta, z, L) = (r, -\theta + 2\theta_0, [z + 1], L),$$

where $[z + 1] = z + 1$ or $z - 1$ according as $z < 1$ or $\geqq 1$. The maps ϕ_0, ϕ_L are isometric involutive homeomorphisms and have not fixed point. Accordingly in $D' \times [L]$, if we identify x with $\phi_0(x)$ for all $x \in D'(0)$ and further y with $\phi_L(y)$ for all $y \in D'(L)$, we get a Euclidean 4-space form M^4. M^4 is also an RS-manifold of type VI. If π/θ_0 is an irrational number, M^4 becomes an *RS-manifold of non-fibred type VI, among whose S-geodesics* *there exist both closed one and non-closed one* (Remark 5). If π/θ_0 is a rational number, M^4 becomes an *RS-manifold of almost fibred type* (*not fibred type*) *and further type VI* (Remark 6).

5. Let (e_λ) $(\lambda = 1, 2, 3, 4)$ be an orthonormal frame with origin O in Euclidean 4-space E^4. Take a constant α_0 $(0 < \alpha_0 < \pi)$. We consider the moving frames $(e_1, e_2(t), e_3(t), e_4)$ $(0 \leqq t \leqq 1)$ with origin O, where

$$(e_2(t), e_3(t)) = (e_2, e_3)\begin{pmatrix}\cos t\,\alpha_0 & -\sin t\,\alpha_0 \\ \sin t\,\alpha_0 & \cos t\,\alpha_0\end{pmatrix}.$$

In the following, let us represent points vectorially. Let D be the subspace of E^4, which consists of all points $x_1 e_1 + x_2 e_2 + x_3 e_3 + t\, e_4$ such that $|x_1| \leqq 1$, $0 \leqq t \leqq 1$. Let $D(0)$, $D(1)$ denote the subspaces of D defined by $t = 0, 1$ respectively. Define

$$\phi : D(1) \rightarrow D(0)$$

by $$\phi(x_1e_1 + x_2e_2(1) + x_3e_3(1) + e_4) = x_1e_1 + x_2e_2 + x_3e_3.$$

The map ϕ is an isometric homeomorphism. By this, identify $D(1)$ with $D(0)$. The space thus constructed from D we denote by D. In D' let $D'(1)$, $D'(-1)$ denote the 3-submanifolds defined by e_1-component $= 1, -1$ respectively. Take an irrational number t_0 $(0 < t_0 < 1)$ and again define

$$\psi : D'(1) \rightarrow D'(-1)$$

by $$\psi(e_1 + x_2e_2(t) + x_3\, e_3(t) + te_4) = -e_1 + x_2\, e_2([t + t_0]) + x_3\, e_3([t + t_0]) + [t + t_0]\, e_4$$

where $[t + t_0] = t + t_0$ or $t + t_0 - 1$ according as $t + t_0 < 1$ or $\geqq 1$. The map ψ is an isometric homeomorphism. By this, identify $D'(1)$ with $D'(-1)$. The space thus constructed from D' we denote by M^4. M^4 is a Euclidean 4-space form and also an RS-manifold. If π/α_0 is an irrational number, M^4 becomes an *RS-manifold of almost clustered type, among whose S-geodesics there exist both closed one and non-closed one* (Remark 5). If π/α_0 is a rational number, M^4 becomes an *RS-manifold of almost fibred type* (*not fibred type*) *and further non-simple type* (Remark 6).

YAMAGATA UNIVERSITY.