ON THE CLASS OF SATURATION IN THE THEORY OF APPROXIMATION II[1]

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1. Introduction. We have already determined the class of saturation for various methods of summation in the theory of Fourier series and Fourier integral (Sunouchi-Watari [4], Sunouchi [5]). In the present note, the author will determine the saturation class of local approximation of functions. In the preceding case, we have used Fourier transform or Fourier series essentially. But in the present case, we have to make another device. For the sake of simplicity, we will only study about the uniform approximation by the first arithmetic means. But it is evident that our method is applicable for another approximation norms and another summation methods.

Incidently, this method is applicable to determine the saturation class for such approximation process as Landau's singular integral. This approximation process has somewhat different feature from the periodic case.

2. Local saturation. Let $f(x)$ be integrable and periodic with period 2π and $\sigma_n(x, f) \equiv \sigma_n(x)$ be the n-th Fejér means of the Fourier series of $f(x)$. Moreover we suppose $[a, b]$ is a fixed closed subinterval situated in $[0, 2\pi]$. Then we have the following theorem.

THEOREM 1[2]. (1) *If $\sigma_n(x) - f(x) = o(1/n)$ uniformly in $[a, b]$, then $\tilde{f}(x)$ is a constant in $[a, b]$.*

(2) *If $\sigma_n(x) - f(x) = O(1/n)$ uniformly in $[a, b]$, then $\tilde{f}'(x)$ is essentially bounded in $[a, b]$.*

For the proof of Theorem 1, we need a lemma.

LEMMA. *If $h(x)$ is periodic with period 2π and $\tilde{h}''(x)$ is continuous in $[0, 2\pi]$, then*

$$\lim_{n\to\infty} n\{\sigma_n(x, h) - h(x)\} = \tilde{h}'(x)$$

bourdedly.

PROOF. Interchanging the roles of $h(x)$ and the conjugate of $h(x)$, we shall

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prove

$$\lim_{n\to\infty} n\{\widetilde{\sigma}_n(x, h) - \widetilde{h}(x)\} = h'(x)$$

bourdedly, under the assumption that $h''(x)$ is continuous.

The Taylor expansion yields

$$h(x + t) - h(x - t)$$
$$= 2th'(x) + 2t^2h''\{x + \theta(x - t)\}.$$

By the well known formula (Zygmund [6], p.91), we have

$$\widetilde{\sigma}_n(x, h) - \widetilde{h}(x)$$
$$= \frac{1}{\pi(n+1)}\int_0^\pi \{h(x + t) - h(x - t)\}\frac{\sin(n+1)t}{(2\sin t/2)^2}\,dt$$
$$= \frac{h'(x)}{\pi(n+1)}\int_0^\pi \frac{t\sin(n+1)t}{2(\sin t/2)^2}\,dt$$
$$+ \frac{1}{\pi(n+1)}\int_0^{2\pi} \frac{2t^2h''\{x+\theta(x-t)\}\sin(n+1)t}{2(\sin t/2)^2}\,dt$$
$$= I+J,$$

say. But

$$\lim_{n\to\infty} \frac{1}{\pi}\int_0^\pi \frac{t\sin(n+1)t}{2(\sin t/2)^2}\,dt = 1$$

and

$J = O\ (1/n)$ by the Riemann-Lebesgue theorem. Thus we complete the proof of Lemma.

PROOF OF THEOREM. (1) If we denote by C_0, the class of functions $g(x)$ such that $g(x) = 0$ outside of $[a, b]$ and $g'''(x)$ is continuous in $[0, 2\pi]$. Then evidently $\widetilde{g}''(x)$ is continuous in $[0, 2\pi]$.

Since

$$n\{\sigma_n(x, f) - f(x)\}$$

tends to zero bourdedly in $[a, b]$, we have

$$\lim_{n\to\infty}\int_0^{2\pi} n\{\sigma_n(x, f) - f(x)\}g(x)dx = 0$$

for all $g(x) \in C_0$.

Since $\sigma_n(x, f)$ has a symmetric kernel, we can interchange $f(x)$ and $g(x)$, that is,

$$\int_0^{2\pi} n\{\sigma_n(x, f) - f(x)\} g(x) dx$$

$$= \int_0^{2\pi} n\{\sigma_n(x, g) - g(x)\} f(x) dx.$$

On the other hand, Lemma gives

$$\lim_{n\to\infty} n\{\sigma_n(x, g) - g(x)\} = \tilde{g}'(x),$$

bourdedly. Thus we get

$$\int_0^{2\pi} f(x)\tilde{g}'(x) dx = 0.$$

By Parseval's relation, this is equivalent to

$$\int_0^{2\pi} \tilde{F}(x) g''(x) dx = 0$$

where $F(x)$ is an indefinite integral of $f(x)$. So, by the well known lemma (Courant and Hilbert [3], p.201), $\tilde{F}(x)$ is a linear function in $[a, b]$ and $\tilde{f}(x)$ is a constant in $[a, b]$.

(2) The proof of the second part is almost the same. Since

$$n\{\sigma_n(x, f) - f(x)\} = O\ (1)$$

uniformly in $[a, b]$, by the weak compactness of the space $L_\infty(a, b)$, we can take a subsequence n_k and a function $h(x) \in L_\infty(a, b)$ such that for all $g(x) \in C_0$,

$$\lim_{k\to\infty} \int_0^{2\pi} n_k\{\sigma_{n_k}(x, f) - f(x)\} g(x)\, dx$$

$$= \int_0^{2\pi} h(x) g(x) dx.$$

But the left-hand side is

$$\int_0^{2\pi} f(x)\tilde{g}'(x) dx = \int_0^{2\pi} \tilde{F}(x) g''(x) dx$$

and the right-hand side is

$$\int_0^{2\pi} H_2(x) g''(x) dx,$$

where $H_2(x)$ is the second integral of $h(x)$.

Thus we can conclude that $\tilde{f}(x)$ is an indefinite integral of a bounded function in $[a, b]$.

THEOREM 2. *We suppose that $f(x)$ is integrable in* $[0, 2\pi]$. *If $\tilde{f}(x)$ is constant in $[a, b]$, then $\sigma_n(x) - f(x) = o(1/n)$ uniformly in $[a + \delta, b - \delta]$ for any fixed $\delta > 0$. And if $\tilde{f}'(x)$ is bounded in $[a, b]$, then $\sigma_n(x) - f(x) = O(1/n)$ uniformly in $[a + \delta, b - \delta]$.*

PROOF. We set $D_n(x)$ the Dirichlet kernel and $A_k(x) = a_k \cos kx + b_k \sin kx$. Then by a theorem of localization [Zygmund 6, p. 367],

$$\sum_{k=1}^{n} kA_k(x) - \frac{1}{\pi}\int_a^b F(t)\lambda(t)\frac{d^4}{dt^4}[D_n(x-t)]dt$$

is uniformly summable $(C,1)$ to zero in $[a + \delta/2, b - \delta/2]$, where

$$F(t) = \sum_{k=0}^{3} c_k t^k + \sum_{k=1}^{\infty} \frac{kA_k(x)}{k^4},$$

$c_k =$ a constant, and

$$\lambda(t) = \begin{cases} 1, & \text{in } [a + \delta/2,\ b - \delta/2] \\ 0, & \text{outside } [a,\ b] \end{cases}$$

is continuous with derivatives of sufficiently high order. Integrating by parts successively,

$$\tau_n(x) - (C,1)\frac{1}{\pi}\int_a^b \frac{d^4}{dt^4}[F(t)\lambda(t)]D_n(x-t)dt = o(1)$$

uniformly in $[a + \delta/2, b - \delta/2]$, where $\tau_n(x)$ is the n-th $(C, 1)$-mean of trigonometric series $\sum kA_k(x)$. Since $\tilde{f}'(x)$ exists in $[a, b]$, by the Riemann-Lebesgue theorem we have

$$\tau_n(x) - (C,1)\frac{1}{\pi}\int_{a+\delta/2}^{b-\delta/2} \tilde{f}'(x)D_n(x-t)dt = o(1)$$

uniformly in $[a + \delta, b - \delta]$
Hence, in the case $\tilde{f}'(x) = 0$ in $[a, b]$

$$\|\tau_n(x)\|_C = o(1)$$

where C means C-norm with respect to the interval $[a + \delta, b - \delta]$. Since

$$\sigma_n(x) - \sigma_{n-1}(x) = S_{n-1}(x)/n(n-1)$$

where $S_n(x)$ is the n-th partial sum of $\sum kA_k(x)$,

$$\sigma_M(x) - \sigma_N(x) = \sum_{n=N+1}^{M} S_{n-1}(x)/n(n-1)$$

$$= \sum_{n=N+1}^{M-1} \tau_n(x)\left(\frac{1}{n-1} - \frac{1}{n+1}\right) + \frac{\tau_M(x)}{M-1} - \frac{\tau_N(x)}{N-1}.$$

Hence we get

$$\|\sigma_M(x) - \sigma_N(x)\|_C$$
$$= \sum_{n=N+1}^{M-1} o(1)\, O\left(\frac{1}{n^2}\right) + o\left(\frac{1}{M}\right) + o\left(\frac{1}{N}\right).$$

Letting $M \to \infty$,

$$\|f(x) - \sigma_N(x)\|_C = o\left(\frac{1}{N}\right).$$

Thus we get the first part of the Theorem. The proof of the remaining part is the same.

From the above two theorems, we can conclude Theorem 3.

THEOREM 3. *For the Fourier series of $f(x)$, which is integrable and periodic, the local saturation of the first arithmetic means for the uniform approximation over (a, b) is $[\{f(x) \mid \tilde{f}'(x)$ is bounded over $[a, b]\}$, $1/n$, $\tilde{f}(x)$ is constant over $(a, b)]$.*

3. Landau's singular integral. The singular integral of Landau is defined by

$$S_n(x,f) = \frac{1}{k_n}\int_0^1 f(t)\{1-(x-t)^2\}^n dt$$

where k_n is the normalization factor. Since $f(x)$ is not necessarily periodic, we subtract a linear function from $f(x)$ and suppose $f(x)$ is periodic. But the kernel is not periodic. A useful result of Arnold [1] is that

$$\lim_{n\to\infty} n\{S_n(x,f) - f(x)\} = \frac{1}{4}f''(x)$$

provided that $f''(x)$ exists and this converges boundedly provided that $f''(x)$ is bounded.

Our theorem is the following.

THEOREM 4. (1) *If $S_n(x,f) - f(x) = o(1/n)$ uniformly, then $f(x)$ is a linear function.*

(2) *If $S_n(x,f) - f(x) = O(1/n)$ uniformly, then $f'(x) \in$* Lip1.

(3) *If $f'(x) \in$ Lip 1, then $S_n(x,f) - f(x) = O(1/n)$ uniformly.*

From this, we can say: The saturationof Landau's singular integral is

$$\mathrm{Sat}\,[S_n(x)]_C = [\{f(x)\,|\,f'(x) \in \mathrm{Lip}\ 1\},\ 1/n,\ \text{linear function}].$$

This is the affirmative answer to a conjecture of Butzer ([2], p100).

PROOF. (1) From the hypothesis, we have

$$\lim_{n\to\infty}\int_0^1 n\{f(x) - S_n(x,f)\}e^{-2\pi ikx}dx = 0.$$

Noticing the fact that the kernel is symmetric and using the above mentioned Arnold's results,the left-hand side is equal to

$$\lim_{n\to\infty}\int_0^1 f(t)[n\{e^{-2\pi ikt} - S_n(t,e^{-2\pi ikx})\}]dt$$

$$= \int_0^1 f(t)\frac{1}{4}(2\pi)^2k^2e^{-2\pi ikt}dt$$

$$= \pi^2k^2a_k,$$

where a_k is the Fourier coefficient of $f(x)$. Consequently we get

$$a_k = 0 \quad (k = \pm 1,\ \pm 2, \dots\dots\dots).$$

(2) Since

$$\|f(x) - S_n(x,f)\|_C = O(1/n),$$

by the weak compactness L_∞-space, there is a function $g(x) \in L_\infty(0, 1)$ and a subsequence $\{n_p\}$ of natural numbers such that

$$\lim_{p\to\infty}\int_0^1 n_p\{f(x) - S_{n_p}(x,f)\}e^{-2\pi ikx}dx$$

$$= \int_0^1 g(x)e^{-2\pi ikr}dx.$$

By the same argument with the case (1), we have

$$\pi^2k^2a_k = \int_0^1 g(x)e^{-2\pi ikx}dx,\ (k = \pm 1,\ \pm 2, \dots\dots\dots),$$

that is $f''(x) \in L_\infty(0, 1)$. This is equivalent with the fact $f'(x) \in \mathrm{Lip}\ 1$.

(3) If $f'(x) \in \mathrm{Lip}\ 1$, then $f''(x)$ exists almost everywhere and is essentially bounded. So we have, from Arnold's result,

$$\|f(x) - S_n(x,f)\|L_\infty = O(1/n).$$

But the continuity of $f(x) - S_n(x, f)$ yields

$$f(x) - S_n(x, f) = O(1/n)$$

uniformly.

Thus the theorem is proved completely.

LITERATURES

[1] G. A. ARNOLD, On the order of approximation of continuous functions by singular integrals of certain special form. Math. Sborn. N. S. 29 (1951), 427-432 [Russ].

[2] P. L. BUTZER, Zur Frage der Saturationsklassen singulärer Integraloperatoren, Math. Zeitschr., 70 (1958), 93-112.

[3] R. COURANT AND D. HILBERT, Methods of Mathematical Physics, New York, 1953.

[4] G. SUNOUCHI AND C. WATARI, On determination of the class of saturation in the theory of approximation of functions I, Proc, Japan Acad. 34 (1958), 477-481. II, Tôhoku Math. Journ. 11 (1959), 480-488.

[5] G. SUNOUCHI, On the class of saturation in the theory of approximation I, Tôhoku Mah. Journ. 12 (1960), 339-344.

[6] A. ZYGMUND, Trigonometrical series I, Cambridge, 1959.

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