COEFFICIENTS OF COBORDISM DECOMPOSITION

YASURÔ TOMONAGA

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Introduction. In this paper we shall generalize the Hirzebruch polynomial ([1]), representing the index of a compact orientable and differentiable $4k$-manifold, by the simplest way and determine the coefficients of the cobordism decomposition by means of these polynomials. Moreover we shall compute the coefficients of cobordism decomposition for a submanifold. This provides us with an analogy of the Gauss-Codazzi equations in the differential geometry and hence has much to do with the problem of differentiable imbedding.

1. The Hirzebruch polynomial is defined as follows:

$$\sum_{i=0}^{\infty} L_i(p_1,\ldots\ldots,p_i)z^i = \prod_{i=1}^{m} \frac{\sqrt{\gamma_i z}}{\operatorname{tgh}\sqrt{\gamma_i z}}, \tag{1. 1}$$

where

$$\sum_{i=0}^{\infty} p_i z^i = \prod_{i=1}^{m} (1+\gamma_i z) \tag{1. 2}$$

and p_i denotes the $4i$-dimensional Pontryagin class. For example we have

$$L_1 = \frac{1}{3}p_1,\ L_2 = \frac{1}{45}(7p_2 - p_1^2),\ L_3 = \frac{1}{3^3\cdot 5\cdot 7}(62p_3 - 13p_1\cdot p_2 + 2p_1^3). \tag{1. 3}$$

Let X^{4k} be a compact orientable and differentiable $4k$-manifold. Then $L_k(p_1,\ldots\ldots,p_k)\,[X^{4k}]$ equals to the index of X^{4k}. In order to generalize $L_i(p_1,\ldots\ldots,p_i)$ we use a function

$$Q(z,y) = \frac{\sqrt{z}}{\operatorname{tgh}\sqrt{z}}(1 + y\operatorname{tgh}^2\sqrt{z}) \tag{1. 4}$$

instead of $\sqrt{z}/\operatorname{tgh}\sqrt{z}$ and define a new multiplicative series

$$\sum_{i=0}^{\infty} \Gamma_i(y,p_1,\ldots\ldots,p_i)\,z^i = \prod_{i=1}^{m} \frac{\sqrt{\gamma_i z}}{\operatorname{tgh}\sqrt{\gamma_i z}}(1 + y\operatorname{tgh}^2\sqrt{\gamma_i z}). \tag{1. 5}$$

The first three terms are given by

$$\Gamma_1(y,p_1) = \left(y + \frac{1}{3}\right)p_1,\ \Gamma_2(y,p_1,p_2) = p_2 y^2 + \frac{4p_2 - p_1^2}{3}y + \frac{7p_2 - p_1^2}{45}. \tag{1. 6}$$

$$\Gamma_3(y,p_1,p_2,p_3) = p_3y^3 + \frac{1}{3}(6p_3 - p_1p_2)y^2 + \frac{1}{15}(17p_3 - 8p_1p_2 + 2p_1^3)y$$
$$+ \frac{1}{3^3\cdot 5\cdot 7}(62p_3 - 13p_1p_2 + 2p_1^3).$$

It is clear that $\Gamma_i(y,p_1,\ldots\ldots,p_i)$ is a polynomial of degree i with respect to y and its coefficients are polynomials of $p_1,\ldots\ldots,p_i$ whose weight is i. Let M^{4i} or N^{4j} be a compact orientable and differentiable manifold of dimension $4i$ or $4j$ respectively. Since $\sum_i \Gamma_i(y,p_1,\ldots\ldots,p_i)$ is multiplicative ([3] II, p. 317) we have

(1. 7) $\Gamma_{i+j}(y,p_1,\ldots\ldots,p_{i+j})[M^{4i}\cdot N^{4j}] = \Gamma_i(y,p_1,\ldots\ldots,p_i)[M^{4i}]\Gamma_j(y,p_1,\ldots\ldots,p_j)[N^{4j}].$

It follows from the definition that

(1. 8) $$\Gamma_i(0,p_1,\ldots\ldots,p_i) = L_i(p_1,\ldots\ldots,p_i),$$

i. e.

(1. 9) $$\Gamma_k(0,p_1,\ldots\ldots,p_k)[X^{4n}] = \text{index of } X^{4k}.$$

Moreover it holds that

(1. 10) $$\Gamma_k(1,p_1,\ldots\ldots,p_k)[X^{4k}] = 2^{2k}(\text{index of } X^{4k}),$$

because

(1. 11) $$\frac{\sqrt{z}}{\mathrm{tgh}\sqrt{z}}(1 + \mathrm{tgh}^2\sqrt{z}) = \frac{2\sqrt{z}}{\mathrm{tgh}\,2\sqrt{z}}.$$

Furthermore we have

(1. 12) $$\Gamma_k(-1,p_1,\ldots\ldots,p_k)[X^{4k}] = A\text{-genus ([1], S. 14)}$$

because

(1. 13) $$\frac{\sqrt{z}}{\mathrm{tgh}\sqrt{z}}(1 - \mathrm{tgh}^2\sqrt{z}) = \frac{2\sqrt{z}}{\sinh 2\sqrt{z}}.$$

In some case we can easily prove the integrality of the coefficients of the polynomials $\Gamma_k(y,p_1,\ldots\ldots,p_k)[X^{4k}]$. Let X^{4k} be an almost complex split manifold. Then the Pontryagin class of X^{4k} takes the form

(1. 14) $$p = \prod(1 + a_i^2),\ a_i \in H^2(X^{4k}, Z).$$

We have

(1. 15) $$\prod_i \frac{\sqrt{\gamma_i}}{\mathrm{tgh}\sqrt{\gamma_i}}(1 + y\,\mathrm{tgh}^2\sqrt{\gamma_i}) = \prod_i \frac{\sqrt{\gamma_i}}{\mathrm{tgh}\sqrt{\gamma_i}}(1 + y\mathrm{tgh}^2\sqrt{\gamma_i})$$

$$= \sum_i L_i(p_1,\dots\dots,p_i) \sum_\alpha y^\alpha \sum \mathrm{tgh}^2\sqrt{\gamma_{i_1}}\dots\dots\mathrm{tgh}^2\sqrt{\gamma_{i\alpha}}$$

$$= \sum_i L_i(p_1,\dots\dots,p_i) \sum_\alpha y^\alpha \sum \mathrm{tgh}^2 a_{i_1}\dots\dots\mathrm{tgh}^2 a_{i\alpha}.$$

Meanwhile the index of a submanifold X^{4k-2r} determined by $v_1,\dots\dots v_r\ (\in H^2(X^{4k}, Z))$ ([1], S.87) is given by

$$(1.\ 16) \qquad \tau(X^{4k-2r}) = \kappa^{4k}[\mathrm{tgh} v_1\dots\dots\mathrm{tgh} v_r \sum L_i(p_1,\dots\dots,p_i)].$$

Comparing (1. 15) and (1. 16) we find that each coefficient of $\Gamma_k(y,p_1,\dots\dots,p_k)$ is a sum of many indices of submanifolds and hence is an integer. To prove the integrality of the coefficients of $\Gamma_k(y,p_1,\dots\dots,p_k)[X^{4k}]$ in general will be done in another chance.

2. Next we shall deal with the case where X^{4k} is the complex projective space $P_{2k}(C)$. The Pontryagin class of $P_{2k}(C)$ takes the form

$$(2.\ 1) \qquad (1+g^2)^{2k+1} = \sum_{i=0}^{k} p_i \mod g^{2k+2} \qquad ([1], \mathrm{S}.73),$$

where g denotes a generator of $H^2(P_{2k}(C), Z)$ and

$$(2.\ 2) \qquad g^{2k}[P_{2k}(C)] = 1,$$

i. e.

$$(2.\ 3) \qquad p_i = \binom{2k+1}{i} g^{2i} \ (i=1,\dots\dots,k).$$

Then we have from (1. 5) and (2. 1)

$$(2.\ 4) \qquad \Gamma_k(y,p_1,\dots\dots,p_k)[P_{2k}(C)]$$

$$= \frac{1}{2\pi i}\int \frac{1}{z^{k+1}}\left(\frac{\sqrt{z}}{\mathrm{tgh}\sqrt{z}}(1+y\,\mathrm{tgh}^2\sqrt{z})\right)^{2k+1} dz,$$

where the integral should be taken around $z=0$ in the positive direction. Changing variable to

$$(2.\ 5) \qquad u = \mathrm{tgh}\sqrt{z},$$

we have

$$(2.\ 6) \qquad \Gamma_k(y,p_1,\dots\dots,p_k)[P_{2k}(C)] = \frac{1}{2\pi i}\int \frac{(1+yu^2)^{2k+1}}{u^{2k+1}(1-u^2)} du,$$

where the integral should be taken around $u=0$ in positive direction. We have

$$(2.\ 7) \qquad \Gamma_k(y,p_1,\dots\dots,p_k)[P_{2k}(C)]$$

$$= \frac{1}{2\pi i}\int u^{-2k-1}(1+u^2+u^4+\cdots+u^{2k}+\cdots)(1+yu^2)^{2k+1}du$$

$$= 1+\binom{2k+1}{1}y+\binom{2k+1}{2}y^2+\cdots+\binom{2k+1}{k}y^k.$$

We put as follows:

$$(2.\ 8)\quad \begin{cases} \Gamma_1(y,p_1)[P_2(C)] = 3y+1 = Q_1(y), \\ \Gamma_2(y,p_1,p_2)[P_4(C)] = 10y^2+5y+1 = Q_2(y), \\ \Gamma_3(y,p_1,p_2,p_3)[P_6(C)] = 35y^3+21y^2+7y+1 = Q_3(y), \\ \vdots \\ \Gamma(y,p_1,p_1,\ldots\ldots,p_k)[P_{2k}(C)] = \binom{2k+1}{k}y^k+\cdots+\binom{2k+1}{1}y+1 = Q_k(y). \end{cases}$$

It should be noted that

$$(2.\ 9)\quad \Gamma_k(1,p_1,\ldots\ldots,p_k)[P_{2k}(C)] = 1+\binom{2k+1}{1}+\cdots\cdots+\binom{2k+1}{k} = 2^{2k},$$

which follows from (1. 10). It is clear that $\Gamma_k(y,p_1,\ldots\ldots p_k)[X^{4k}]$ is a cobordism invariant. Furthermore $\Gamma_k(y,p_1,\ldots\ldots,p_k)\ [X^{4k}]$ is multiplicative as we have seen in (1. 7). Meanwhile the classes of cobordism with respect to the rational coefficients are generated by the $P_{2i}(C)$'s ([2]). Hence we have from (2. 8)

$$(2.\ 11)\quad \Gamma_k(y,p_1,\ldots\ldots,p_k)\ [X^{4k}] = \sum_{i_1+\ldots+i_k=k} A_{i_1\ldots\ldots ik}Q_{i_1}(y)Q_{i_2}(y)\ldots\ldots Q_{i_k}(y)$$

according as

$$(2.\ 12)\quad X^{4k} \approx \sum_{i_1++i_k} A_{i_1\ldots\ldots i_k}P_{2i_1}(C)P_{2i_2}(C)\ldots\ldots P_{2i_k}(C) \text{ mod torsion,}$$

where $A_{i_1\ldots\ldots i_k}$'s denote some rational numbers and $\approx$ means "cobordantes" and $Q_0(y)=1$. For example we have

$$(2.\ 13)\quad \Gamma_3(y,p_1,p_2,p_3)[X^{12}] = A(35y^3+21y^2+7y+1)$$
$$+B(10y^2+5y+1)(3y+1)+C(3y+1)^3,$$

where A,B and C are some rational numbers.

3. Decomposition of X^8. Concerning the Thom algebra we shall make use of the following table ([4]):

k	1	2	3	4	5	6	7	8	9	10	11	12
Ω^k	0	0	0	Z	Z_2	0	0	$Z+Z$	Z_2+Z_2	Z_2	Z_2	$Z+Z+Z$

where Ω^k denotes the Thom algebra of dimension k with rational coefficients and the generators of Ω^4, Ω^8 and Ω^{12} are given by

$$\Omega^4 : P_2(C),$$
$$\Omega^8 : P_2(C)^2,\ P_4(C),$$
$$\Omega^{12} : P_2(C)^3,\ P_2(C)P_4(C),\ P_6(C).$$

First of all let us consider the cobordism decomposition of a X^8, i. e.

$$X^8 \approx AP_4(C) + BP_2(C)^2 \tag{3. 1}$$

where A and B denote some rational numbers. For this purpose we have to solve the equation

$$\Gamma_2(y, p_1, p_2)\,[X^8] = AQ_2(y) + BQ_1(y)^2, \tag{3. 2}$$

which follows from (2. 11) and (3. 1)

$$p_2[X^8]y^2 + \frac{1}{3}(4p_2 - p_1^2)[X^8]y + \frac{1}{45}(7p_2 - p_1^2)[X^8] \tag{3. 3}$$
$$= A(10y^2 + 5y + 1) + B(3y + 1)^2.$$

Comparing the coefficients of y_α's ($\alpha = 0, 1, 2$) we have

$$\left\{\begin{aligned} &10A + 9B = p_2[X^8], \\ &5A + 6B = \frac{1}{3}(4p_2 - p_1^2)[X^8], \\ &A + B = \frac{1}{45}(7p_2 - p_1^2)[X^8]. \end{aligned}\right. \tag{3. 4}$$

The first equation is linearly dependent of two others. Solving (3. 4) we have

$$\left\{\begin{aligned} &A = \frac{1}{5}(-2p_2 + p_1^2)[X^9], \\ &B = \frac{1}{9}(5p_2 - 2p_1^2)[X^8]. \\ &\text{index} = A + B = \frac{1}{45}(7p_2 - p_1^2)[X^8]. \end{aligned}\right. \tag{3. 5}$$

Hence we have

$$X^8 \approx \frac{1}{5}(-2p_2 + p_1^2)[X^8]P_4(C) + \frac{1}{9}(5p_2 - 2p_1^2)[X^8]P_2(C)^2 \tag{3. 6}$$

([2], p. 85).

For example we consider the quaternion projective space

$$(3.\ 7)\qquad P_{q-1}(K)=SP(q)/SP(1)\cdot Sp(q-1),\ q\geqq 2\qquad ([3], 1, \text{p. } 517).$$

The Pontryagin class of $P_{q-1}(K)$ is given by

$$(3.\ 8)\qquad p(P_{q-1}(K))=(1+u)^{2q}(1+4u)^{-1},\ u\in H^4(P_{q-1}(K), Z).$$

In the case $q=3$, (3. 8) becomes

$$(3.\ 9)\qquad p=1+2u+7u^2,$$

i. e.

$$(3.\ 10)\qquad p_1=2u,\quad p_2=7u^2.$$

We adopt an orientation

$$(3.\ 11)\qquad u^2[P_2(K)]=1\ \ ([3]\ \text{I, p. } 531).$$

Hence we have from (3. 6)

$$(3.\ 12)\qquad P_2(K)\approx -2P_4(C)+3P_2(C)^2.$$

It should be noted that

$$(3.\ 13)\qquad \text{index of } P_2(K)=\Gamma_2(0,p_1,p_2)[P_2(K)]=1$$

and hence

$$(3.\ 14)\qquad \Gamma_2(1,p_1,p_2)[P_2(K)]=2^4.$$

Another example is the complex quadric $Q_n=SO(n+2)/SO(2)\cdot SO(n)$ ([3] I, p. 525).

In this case the Pontryagin class is given by

$$(3.\ 15)\qquad p(Q_n)=(1+g^2)^{n+2}\cdot(1+4g^2),^{-1}\ (n>2),$$
$$g\in H^2(Q_n, Z).$$

In particular we have

$$(3.\ 16)\qquad \begin{cases} p(Q_4)=1+2g^2+7g^4, \\ p(Q_6)=1+4g^2+12g^4+8g^6. \end{cases}$$

Hence we have from (3. 6)

$$(3.\ 17)\qquad \begin{cases} Q_4\approx -2\tau P_4(C)+3\tau P_2(C)^2, \\ \tau=g^4[Q_4]=\text{index of } Q_4. \end{cases}$$

4. Cobordism decomposition of X^{12}. Next let us consider the cobordism decomposition of X^{12}, i. e.

$$X^{12}\approx AP_6(C)+BP_4(C)P_2(C)+CP_2(C)^3.$$

For this purpose we must solve the equation follows from (2. 11):

$$\Gamma_3(y, p_1, p_2, p_3)[X^{12}] = AQ_3(Y) + BQ_2(y)Q_1(y) + CQ_1(y)^3, \tag{4. 1}$$

i. e.

$$p_3[X^{12}]y^3 + \frac{1}{3}(6p_3 - p_1p_2)[X^{12}]y^2 + \frac{1}{15}(17p_3 - 8p_1p_2 + 2p_1^3)[X^{12}]$$

$$+ \frac{1}{3^3 \cdot 5 \cdot 7}(62p_3 - 13p_1p_2 + 2p_1^3)[X^{12}] = A(35y^3 + 21y^2$$

$$+ 7y + 1) + B(10y^2 + 5y + 1)(3y + 1) + C(3y + 1)^3,$$

where A, B and C denote some rational numbers. Comparing the coefficients of y_α's ($\alpha = 0, 1, 2, 3$) we have

$$\left\{\begin{aligned} &35A + 30B + 27C = p_3[X^{12}], \\ &21A + 25B + 27C = \frac{1}{3}(6p_3 - p_1p_2)[X^{12}], \\ &7A + 8B + 9C = \frac{1}{15}(17p_3 - 8p_1p_2 + 2p_1^3)[X^{12}], \\ &A + B + C = \frac{1}{3^3 \cdot 5 \cdot 7}(62p_3 - 13p_1p_2 + 2p_1^3)[X^{12}]. \end{aligned}\right. \tag{4. 2}$$

The first equation is linearly dependent of three others. Solving (4. 2) we have

$$\left\{\begin{aligned} &A = \frac{1}{7}(3p_3 - 3p_1p_2 + p_1^3)[X^{12}], \\ &B = \frac{1}{15}(-21p_3 + 19p_1p_2 - 6p_1^3)[X^{12}], \\ &C = \frac{1}{27}(28p_3 - 23p_1p_2 + 7p_1^3)[X^{12}]. \\ &\text{index} = A + B + C = \frac{1}{3^3 \cdot 5 \cdot 7}(62p_3 - 13p_1p_2 + 2p_1^3)[X^{12}]. \end{aligned}\right. \tag{4. 3}$$

Hence we have

$$X^{12} \approx \frac{1}{7}(3p_3 - 3p_1p_2 + p_1^3)[X^{12}]P_6(C) + \frac{1}{15}(-21p_3 \tag{4. 4}$$

$$+ 19p_1p_2 - 6p_1^3)[X^{12}]P_4(C)P_2(C) + \frac{1}{27}(28p_3 - 23p_1p_2$$

$$+ 7p_1^3)[X^{12}]P_2(C)^3.$$

For example we consider $P_3(K)$. In this case the Pontrygin class takes the form

$$p = 1 + 4u + 12u^2 + 8u^3,\ u \in H^4(P_3(K), Z), \tag{4. 5}$$

i. e.

$$p_1 = 4u,\ p_2 = 12u^2,\ p_3 = 8u^3. \tag{4. 6}$$

Hence we have from (3. 18)

$$P_3(K) \approx -8\lambda\{P_6(C) - 3P_4(C)P_2(C) + 2P_2(C)^3\}, \tag{4. 7}$$

where

$$\lambda = u^3[P_3(K)].$$

In this case it should be noted that

$$\text{Index of } P_3(K) = \Gamma_3(0, p_1, p_2, p_3)[P_3(K)] = 0. \tag{4. 8}$$

Another example is given by $Q_6 = SO(8)/SO(2)\cdot SO(6)$ ((3. 15)). From (3 .16) and (4. 3) we have

$$\begin{cases} Q_6 \approx -8\mu P_6(C) + 24\mu P_4(C)P_2(C) - 16\mu P_2(C)^3, \\ \text{index of } Q_6 = 0,\ \mu = g^6[Q_6]. \end{cases} \tag{4. 9}$$

5. Cobordism decomposition of X^{16}. Our multiplicative series $\sum\limits_i \Gamma_i(y, p_1, \ldots\ldots, p_i)$ is not available for the cobordism decomposition of X^{16}, because in this case the number of independent equations such as (3. 16) is less than 5. For this reason we introduce a new multiplicative series such that

$$\prod_{i=1}^{m} \frac{\sqrt{\gamma_i z}}{\mathrm{tgh}\sqrt{\gamma_i z}}(1 + y\,\mathrm{tgh}^2\sqrt{\gamma_i z})^{-1} = \sum_{i=0}^{\infty} \Lambda_i(y, p_1, \ldots\ldots, p_i) z^i, \tag{5. 1}$$

where

$$\prod_{i=1}^{m}(1 + \gamma_i z) = \sum_{i=0}^{\infty} p_i z^i. \tag{5. 2}$$

The first three terms of (5. 1) are given by

$$\begin{cases} \Lambda_0 = 1, \\ \Lambda_1 = \left(\frac{1}{3} - y\right)p_1, \\ \Lambda_2 = (p_1^2 - p_2)y^2 + \frac{1}{3}(p_1^2 - 4p_2)y + \frac{1}{45}(7p_2 - p_1^2), \\ \Lambda_3 = -(p_3 - 2p_1p_2 + p_1^3)y^3 + (-2p_3 + 3p_1p_2 - p_1^3)y^2 \end{cases} \tag{5. 3}$$

$$+\frac{1}{15}(-17p_3+8p_1p_2-2p_1^3)y+\frac{1}{3^3\cdot5\cdot7}(62p_3-13p_1p_2+2p_1^3),$$

$$\Lambda_4=(-p_4+2p_1p_3+p_2^2-3p_1^2p_2+p_1^4)y^4$$

$$+\frac{1}{3}(-8p_4+13p_1p_3+8p_2^2-18p_1^2p_2+5p_1^4)y^3$$

$$+\frac{1}{15}(-36p_4+43p_1p_3+29p_2^2-47p_1^2p_2+11p_1^4)y^2$$

$$+\frac{1}{3^3\cdot5\cdot7}(-744p_4+325p_1p_3+176p_2^2-248p_1^2p_2+51p_1^4)y$$

$$+\frac{1}{3^4\cdot5\cdot7}(381p_4-71p_1p_3-19p_2^2+22p_1^2p_2-3p_1^4).$$

It is clear that $\Lambda_i(y,p_1,\ldots\ldots,p_i)$ is a polynomial of degree i of y and each coefficient has weight i with regard to $(p_1,\ldots\ldots,p_i)$. Of course $\Lambda_i(0,p_1,\ldots\ldots,p_i)$ coincides with the Hirzebruch polynomial $L_i(p_1,\ldots\ldots,p_i)$, i.e. $\Lambda_i(0,p_1,\ldots\ldots,p_i)\ [X^{4i}]$ equals the index of X^{4i}. It holds that

$$\frac{1}{2\pi i}\int\frac{1}{z^{k+1}}\left\{\frac{\sqrt{z}}{\mathrm{tgh}\sqrt{z}}(1+y\,\mathrm{tgh}^2\sqrt{z})^{-1}\right\}^{2k+1}dz \tag{5, 4}$$

$$=\Lambda_k(y,p_1,\ldots\ldots,p_k)\ [P_{2k}(C)],$$

where the integral should be taken around $z=0$ in positive direction. Changing variable to

$$u=\mathrm{tgh}\sqrt{z}, \tag{5. 5}$$

we have

$$\Lambda_k(y,p_1,\ldots\ldots,p_k)\ [P_{2k}(C)]=\frac{1}{2\pi i}\int\frac{(1+yu^2)^{-2k-1}}{u^{2k+1}(1-u^2)}du, \tag{5. 6}$$

where the integral should be taken around $u=0$ in positive direction. We have from (4. 6)

$$R_k(y)\equiv\Lambda_k(y,p_1,\ldots\ldots,p_k)\ [P_{2k}(C)]=1-(2k+1)y \tag{5. 7}$$

$$+\frac{(2k+1)(2k+2)}{2}y^2-\cdots+(-1)^k\frac{(2k+1)(2k+2)\cdots(3k)}{k!}y^k.$$

Especially we have

$$(5.\ 8)\quad \begin{cases} R_1(y) \equiv \Lambda_1(y, p_1)[P_2(C)] = 1 - 3y, \\ R_2(y) \equiv \Lambda_2(y, p_1, p_2)[P_4(C)] = 1 - 5y + 15y^2, \\ R_3(y) \equiv \Lambda_3(y, p_1, p_2, p_3)[P_6(C)] = 1 - 7y + 28y^2 - 84y^3, \\ R_4(y) \equiv \Lambda_4(y, p_1, p_2, p_3, p_4)[P_8(C)] = 1 - 9y + 45y^2 - 165y^3 + 495y^4. \end{cases}$$

Suppose that

$$(5.\ 9)\quad X^{16} \approx AP_8(C) + BP_6(C)P_2(C) + CP_4(C)^2 + DP_2(C)^2P_4(C) + EP_2(C)^4$$

mod torsion, where A, B, C, D and E denote some rational numbers. Since $\sum_i \Lambda_i$ is multiplicative we have

$$(5.\ 10)\quad \Lambda_4(y, p_1, \ldots\ldots, p_4)[X^{16}] = AR_4(y) + BR_3(y)R_1(y) + CR_2(y)^2 + DR_2(y)R_1(y)^2 + ER_1(y)^4.$$

Comparing the coefficients of y^α's ($\alpha = 0, 1, 2, 3, 4$) we have from (5. 8) and (5. 3)

$$(5.\ 11)\quad \begin{cases} \text{(i)}\quad 495A + 252B + 225C + 135D + 81E \\ \qquad = (-p_4 + 2p_3p_1 + p_2^2 - 3p_1^2p_2 + p_1^4)[X^{16}], \\ \text{(ii)}\quad -165A - 168B - 150C - 135D - 108E \\ \qquad = \dfrac{1}{3}(-8p_4 + 13p_3p_1 + 8p_2^2 - 18p_1^2p_2 + 5p_1^4)[X^{16}], \\ \text{(iii)}\quad 45A + 49B + 55C + 54D + 54E \\ \qquad = \dfrac{1}{15}(-36p_4 + 43p_3p_1 + 29p_2^2 - 47p_1^2p_2 + 11p_1^4)[X^{16}], \\ \text{(iv)}\quad -9A - 10B - 10C - 11D - 12E \\ \qquad = \dfrac{1}{3^3\cdot 5\cdot 7}(-744p_4 + 325p_3p_1 + 176p_2^2 - 248p_1^2p_2 + 51p_1^4)[X^{16}], \\ \text{(v)}\quad A + B + C + D + E \\ \qquad = \dfrac{1}{3^4\cdot 5^2\cdot 7}(381p_4 - 71p_3p_1 - 19p_2^2 + 22p_1^2p_2 - 3p_1^4)[X^{16}]. \end{cases}$$

Solving (5. 11) we have

$$(5.\ 12)\quad \begin{cases} A = \dfrac{1}{9}(-4p_4 + 4p_3p_1 + 2p_2^2 - 4p_1^2p_2 + p_1^4)[X^{16}], \\ B = \dfrac{1}{21}(36p_4 - 33p_3p_1 - 18p_2^2 + 33p_1^2p_2 - 8p_1^4)[X^{16}], \\ C = \dfrac{1}{25}(18p_4 - 18p_3p_1 - 7p_2^2 + 16p_1^2p^2 - 4p_1^4)[X^{16}], \end{cases}$$

$$\begin{cases} D = \dfrac{1}{45}(-180p_4 + 159p_3p_1 + 80p_2^2 - 150p_1^2p_2 + 36p_1^4)[X^{16}], \\ E = \dfrac{1}{81}(165p_4 - 137p_3p_1 - 70p_2^2 + 127p_1^2p_2 - 30p_1^4)[X^{16}]. \end{cases}$$

In the case of the quaternion projective space $P_4(K)$ it is known from (3. 8) that

(5. 13) $\quad p_1 = 6u,\ p_2 = 21u^2,\ p_3 = 36u^3,\ p_4 = 66u^4,\ u \in H^4(P_4(K), Z).$

We put

(5. 14) $\quad u^4[P_4(K)] = \lambda.$

Then we have from (5. 12)

(5. 15) $\quad A = -\dfrac{82}{3}\lambda,\ B = 90\lambda,\ C = 45\lambda,\ D = -200\lambda,\ E = \dfrac{280}{3}\lambda.$

Hence we have

(5. 16) $\quad 3P_4(K) \approx -82\lambda P_8(C) + 270\lambda P_6(C)P_2(C) + 135\lambda P_4(C)^2$
$\qquad - 600\lambda P_4(C)P_2(C)^2 + 280\lambda P_2(C)^4 \quad$ mod torsion.

In this case λ equals to the index of $P_4(K)$, i. e.

(5. 17) $\quad \lambda = \Lambda_4(0, p_1, p_2, p_3, p_4)[P_4(K)]$

by virtue of (5. 3).

Another example is found in the manifold $W = F_4/Spin(9)$ ([3] I, p. 534).

The Pontryagin class of W is given by

(5. 18) $\quad p_1 = p_3 = 0,\ p_2 = 6u,\ p_4 = 39u^2,\ u^2[W] = 1,\ u \in H^8(W, Z).$

We have from (5. 12)

(5. 19) $\quad A = -\dfrac{28}{3},\ B = 36,\ C = 18,\ D = -92,\ E = \dfrac{145}{3}.$

Hence we have

(5. 20) $\quad 3W \approx -28P_8(C) + 108P_6(C)P_2(C) + 54P_4(C)^2$
$\qquad - 276P_4(C)P_2(C)^2 + 145P(C) \quad$ mod torsion.

The index of W equals to 1.

Of course our multiplicative series $\sum_i \Lambda_i$ is applicable for the cobordism decomposition of X^8 and X^{12}. The results coincide with those of § 3.

6. Genus of submanifold. Let V^{4k} be any submanifold of V^{4k+2}. We assume that both manifolds are compact orientable and differentiable. Let $v \in H^2(V^{4k+2}, Z)$ be the cohomology class representing V^{4k}. The Pontryagin class of the normal bundle of V^{4k} is given by $j^*(1+v^2)$, where j denotes the injection $j: V^{4k} \to V^{4k+2}$. Then the Pontryagin class of V^{4k} is given by

$$(6.1) \quad 1 + p_1(V^{4k}) + p_2(V^{4k}) + \cdots\cdots = j^*[(1 + p_1(V^{4k+2}) + \cdots\cdots)(1+v^2)^{-1}]$$

([1], S.86). Since $\sum_{i=0}^{\infty} \Gamma_i(y, p_1, \ldots\ldots, p_i) z^i$ is generated by $\frac{\sqrt{z}}{\mathrm{tgh}\sqrt{z}}(1 + y\,\mathrm{tgh}^2\sqrt{z})$ we have

$$(6.2) \quad \sum_{i=0}^{\infty} \Gamma_i(y, p_1, \ldots\ldots, p_i) = j^*\left[\frac{\mathrm{tgh}\, v}{v(1+y\,\mathrm{tgh}^2 v)} \sum_{i=0}^{\infty} \Gamma_i(y, p_1, \ldots\ldots, p_i)\right].$$

In general we have

$$(6.3) \quad j^*(X)[V^{4k}] = vX[V^{4k+2}],$$

where X denotes any $4k$-cohomology class, i. e. $X \in H^{4k}(V^{4k+2})$.[1] Hence we have from (3.2)

$$(6.4) \quad \Gamma_k(y, p_1, \ldots\ldots, p_k)[V^{4k}]^{2)} = \left[\kappa^{4k+2}\left\{\frac{\mathrm{tgh}\, v}{1+y\,\mathrm{tgh}^2 v} \sum_{i=0}^{\infty} \Gamma_i(y, p_1, \ldots\ldots, p_i)\right\}\right] \cdot [V^{4k+2}]$$

$$= \Bigg[\kappa^{4k+2}\Bigg\{v - \left(y + \frac{1}{3}\right)v^3 + \left(y^2 + y + \frac{2}{15}\right)v^5$$

$$- \left(\frac{17}{3^2 \cdot 5 \cdot 7} + \frac{2}{15}y + \frac{3y^2+2y}{9} + \frac{45y^3+60y^2+17y}{45}\right)v^7$$

$$+ \cdots\cdots\Bigg\} \sum_{i=0}^{\infty} \Gamma_i(y, p_1, \ldots\ldots, p_i)\Bigg][V^{4k+2}].$$

For example we have

$$(6.5) \quad \Gamma_1(y, p_1)[V^4] = \left\{(vp_2 - v^3)y + \frac{1}{3}(vp_1 - v^3)\right\}[V^6],$$

$$(6.6) \quad \Gamma_2(y, p_1, p_2)[V^8] = \Big[(v^5 - v^3 p_1 + vp_2)y^2 + \frac{1}{3}\{3v^5 - 2v^3 p_1$$

$$+ v(4p_2 - p_1^2)\}y + \frac{1}{45}\{6v^5 - 5v^3 p_1 + v(7p_2 - p_1^2)\}\Big][V^{10}],$$

$$(6.7) \quad \Gamma_3(y, p_1, p_2, p_3)[V^{12}] = [(-v^7 + v^5 p_1 - v^3 p_2 + vp_3)y^3 + \frac{1}{3}\{-5v^7 + 4v^5 p_1$$

1) Precisely speaking $x \in H^{4k}(V^{4k+2r}, A) \otimes B$ (A, B additive groups).

2) κ^n denotes the n-component of a cohomology ring.

$$+ (p_1^2 - 5p_2)v^3 + (6p_3 - p_1p_2)v\}y^2 + \frac{1}{15}\{-11v^7 + 7v^5p_1$$

$$+ (2p_1^2 - 9p_2)v^3 + (17p_3 - 8p_1p_2 + 2p_1^3)v\}y + \frac{1}{3^3\cdot 5\cdot 7}\{-51v^7 + 42v^5p_1$$

$$+ 7(p_1^2 - 7p_2)v^3 + (62p_3 - 13p_1p_2 + 2p_1^3)v\}][V^{14}].$$

Next we consider the case where V^{4k} is a submanifold of a V^{4k+2r} and both manifolds are compact orientable and differentiable. We assume that V^{4k} is determined by a sequence of cohomology classes $v_1,\ldots\ldots,v_r \in H^2(V^{4k+2r},Z)$ ([1] S. 87). In this case we have

$$j^*(X)[V_{4k}] = v_1\ldots\ldots v_r X[V^{4k+2r}], \tag{6. 8}$$

where $X \in H^{4k}(V^{4k+2r})$ and j denotes the injection $j: V^{4k} \to V^{4k+2r}$. Applying (6. 4) many times we have

$$\Gamma_k(y,p_1,\ldots\ldots,p_k)[V^{4k}] = \Big[\kappa^{4k+2r}\Big\{\Big(\frac{\operatorname{tgh} v_1}{1+y\operatorname{tgh}^2 v_1}\Big)\cdots\cdots\Big(\frac{\operatorname{tgh} v_r}{1+y\operatorname{tgh}^2 v_r}\Big)$$
$$\sum \Gamma_i(y,p_1,\ldots\ldots,p_i)\Big\}\Big][V^{4k+2r}]. \tag{6. 9}$$

For example we have

$$\Gamma_1(y,p_1)[V^4] = \Big[\Big(y + \frac{1}{3}\Big)\{v_1v_2p_1 - (v_1v_2^3 + v_2v_1^3)\}\Big][V^8], \tag{6. 10}$$

$$\Gamma_2(y,p_1,p_2)[V^8] = \Big[\{v_1^5v_2 + v_2^5v_1 + v_1^3v_2^3 - (v_1v_2^3 + v_2v_1^3)p_1 + v_1v_2p_2\}y^2 \tag{6. 11}$$
$$+\Big\{v_1^5v_2 + v_2^5v_1 + \frac{2}{3}v_1^3v_2^3 - \frac{2}{3}(v_1v_2^3 + v_2v_1^3)p_1$$
$$+ \frac{1}{3}v_1v_2(4p_2 - p_1^2)\Big\}y + \Big\{\frac{2}{15}(v_1^5v_2 + v_2^5v_1)$$
$$+ \frac{1}{9}v_1^3v_2^3 - \frac{1}{9}(v_1v_2^3 + v_2v_1^3)p_1 + \frac{1}{45}v_1v_2(7p_2 - p_1^2)\Big\}\Big][V^{12}].$$

7. Cobordism decomposition of $X^{4k} \subset X^{4k+2}$. In this paragraph we shall study the cobordism decomposition of a $4k$-manifold which is a submanifold of a $(4k+2)$-manifold. First of all let us consider a X^8 imbedded in a X^{10} where we assume that both manifolds are compact orientable and differentiable. Let $v \in H^2(X^{10}, Z)$ be a cohomology class corresponding to X^8 ([1] S. 87). Suppose that

$$X^8 \approx AP_4(C) + BP_2(C)^2. \tag{7. 1}$$

Then we have from (6. 6) and (3. 2)

$$(7.2)\qquad A(10y^2+5y+1)+B(3y+1)^2=\Big[(v^5-v^3p_1+vp_2)y^2$$
$$+\frac{1}{3}\{3v^5-2v^3p_1+v(4p_2-p_1^2)\}y$$
$$+\frac{1}{45}\{6v^5-5v^3p_1+v(7p_2-p_1^2)\}\Big][X^{10}].$$

Comparing the coefficients of y^{α}'s ($\alpha=0,1,2$) we have

$$(7.3)\qquad \begin{cases} 10A+9B=(v^5-v^3p_1+vp_2)[X^{10}], \\ 5A+6B=\dfrac{1}{3}\{3v^5-2v^3p_1+v(4p_2-p_1^2)\}[X^{10}], \\ A+B=\dfrac{1}{45}\{6v^5-5v^3p_1+v(7p_2-p_1^2)\}[X^{10}]. \end{cases}$$

The first equation follows from two others. Solving (7. 3) we have

$$(7.4)\qquad \begin{cases} A=\dfrac{1}{5}(-v^5-2vp_2+vp_1^2)[X^{10}], \\ B=\dfrac{1}{9}(3v^5-v^3p_1+5vp_2-2vp_1^2)[X^{10}], \\ \text{index}=A+B=\dfrac{1}{45}\{(6v^5-5v^3p_1+v(7p_2-p_1^2)\}[X^{10}]. \end{cases}$$

Thus *the cobordism components of a* X^8 *are vniquely determined by* v *and the Pontryagin classes of* X^{10}.

Let us consider the case where $X^{10}=P_5(C)$. In this case the Pontryagin class takes the form

$$(7.5)\qquad p=(1+g^2)^6=1+6g^2+15g^4+20g^6,\ g\in H^2(P_5(C),Z).$$

Hence we have

$$(7.6)\qquad \begin{cases} A=\dfrac{1}{5}(-v^5+6vg^4)[P_5(C)], \\ B=\dfrac{1}{3}(v^5-2v^3g+vg^4)[P_5(C)], \\ \text{index}=A+B=\dfrac{1}{15}(2v^5-10v^3g+23vg^4)[P_5(C)]. \end{cases}$$

Putting $v=\lambda g$ (λ : integer) we have

$$(7.7)\qquad A=\frac{1}{5}(6\lambda-\lambda^5),\ B=\frac{1}{3}(\lambda^5-2\lambda^3+\lambda)$$

$$\text{and index} = \frac{1}{15}(2\lambda^5 - 10\lambda^3 + 23\lambda).$$

It is easy to show that A and B are integers for every λ and they change sign with λ. The actual values A and B in the simple cases are given by the following table:

	$v=g$	$v=2g$	$v=3g$
A	1	-4	-45
B	0	6	64
index $=A+B$	1	2	19

(1)

The rule (7. 7) restricts the kind of differentiable 8-dimensional orientable submanifold of $P_5(C)$.

The relation

(3. 12) $\quad P_2(K) \approx -2P_4(C) + 3P_2(C)^2$

shows that $P_2(K)$ *cannot be "cobordantes" with any differentiable 8-dimensional submanifold of* $P_5(C)$. Concerning $Q_4 = SO(6)/SO(2)\cdot SO(4)$ we see from (3. 17) and (7. 7) that Q_4 *cannot be "cobordantes" with any differentiable 8-dimensional submanifold of* $P_5(C)$ *other than those which are determined by* $v = \pm 2g$.

The case $X^{12} \subset X^{14}$. Let X^{12} and X^{14} be compact orientable and differentiable manifolds and the former be a submanifold of the latter. We denote the cohomology class corresponds to X^{12} by $v \in H^2(X^{14}, Z)$. Suppose that

$$X^{12} \approx AP_6(C) + BP_4(C)P_2(C) + CP_2(C)^3.$$

From (6. 7) and (4. 1) we have

$$\begin{aligned}(7.\ 10)\quad & A(35y^3+21y^2+7y+1)+B(10y^2+5y+1)(3y+1)+C(3y+1)^3 \\ &= \Big[(-v^7+v^5p_1-v^3p_2+vp_3)y^3+\frac{1}{3}\{-5v^7+4v^5p_1 \\ &+v^3(p_1^2-5p_2)+v(6p_3-p_1p_2)\}y^2+\frac{1}{15}\{-11v^7+7v^5p_1 \\ &+v^3(2p_1^2-9p_2)+v(17p_3-8p_1p_2+2p_1^3)\}y+\frac{1}{3^3\cdot5\cdot7}\{-51v^7 \\ &+42v^5p_1+7v^3(p_1^2-7p_2)+v(62p_3-13p_1p_2+2p_1^3)\}\Big][X^{14}].\end{aligned}$$

Comparing the coefficients of y^α's ($\alpha=0,1,2,3$) we have

$$(7.\ 11)\quad \left\{\begin{aligned} &35A+30B+27C=(-v^7+v^5p_1-v^3p_2+vp_3)[X^{14}], \\ &21A+25B+27C=\frac{1}{3}\{-5v^7+4v^5p_1+v^3(p_1^2-5p_2) \\ &\qquad\qquad +v(6p_3-p_1p_2)\}[X^{14}]\end{aligned}\right.$$

$$\begin{cases} 7A + 8B + 9C = \dfrac{1}{15}\{-11v^7 + 7v^5p_1 + v^3(2p_1^2 - 9p_2) \\ \qquad\qquad + v(17p_3 - 8p_1p_2 + 2p_1^3)\}[X^{14}], \\ A + B + C = \dfrac{1}{3^3\cdot 5\cdot 7}\{-51v^7 + 42v^5p_1 + 7(p_1^2 - 7p_2)v^3 \\ \qquad\qquad + (62p_3 - 13p_1p_2 + 2p_1^3)v\}[X^{14}]. \end{cases}$$

The first equation follows from three others. Solving (7. 11) we have

$$\text{(7. 12)} \quad \begin{cases} A = \dfrac{1}{7}\{-v^7 + (3p_3 - 3p_1p_2 + p_1^3)v\}[X^{14}], \\ B = \dfrac{1}{15}\{8v^7 - v^5p_1 + (2p_2 - p_1^2)v^3 + (-21p_3 + 19p_1p_2 - 6p_1^3)v\}[X^{14}], \\ C = \dfrac{1}{27}\{-12v^7 + 3v^5p_1 + (2p_1^2 - 5p_2)v^3 \\ \qquad + (28p_3 - 23p_1p_2 + 7p_1^3)v\}[X^{14}], \\ \text{index} = A + B + C = \dfrac{1}{3^3\cdot 5\cdot 7}\{-51v^7 + 42v^5p_1 + 7(p_1^2 - 7p_2)v^3 \\ \qquad + (62p_3 - 13p_1p_2 + 2p_1^3)v\}[X^{14}]. \end{cases}$$

Thus *the cobordism components of* X^{12} *are uniquely determined by* v *and the Pontryagin classes of* X^{14}. When X^{14} is $P_7(C)$ the Pontryagin class taks the form

$$\text{(7. 13)} \quad p = (1 + g^2)^8, \qquad g \in H^2(P_7(C),Z),$$
$$p_1 = 8g^2,\ p_2 = 28g^4,\ p_3 = 56g^6, \qquad g^7[P_7(C)] = 1.$$

Hence we have from (6. 12)

$$\text{(7. 14)} \quad \begin{cases} A = \dfrac{1}{7}(-v^7 + 8vg^6)[P_7(C)], \\ B = \dfrac{8}{15}(v^7 - v^5g^2 - v^3g^4 + vg^6)[P_7(C)], \\ C = \dfrac{1}{9}(-4v^7 + 8v^5g^2 - 4v^3g^4)[P_7(C)]. \end{cases}$$

Putting $v = \lambda g$ (λ : integer) we have

$$\text{(7. 15)} \quad A = \frac{1}{7}(8\lambda - \lambda^7),\ B = \frac{8}{15}(\lambda^7 - \lambda^5 - \lambda^3 + \lambda),$$
$$C = \frac{1}{9}(-4\lambda^7 + 8\lambda^5 - 4\lambda^3),$$

$$\text{index} = A + B + C = \frac{1}{3^2.5.7}(-17\lambda^7 + 112\lambda^5 - 308\lambda^3 + 528\lambda).$$

Changing v over g, $2g$, and $3g$ we have the following table:

	$v=g$	$v=2g$	$v=3g$
A	1	-16	-309
B	0	48	1024
C	0	-32	-768
index= $A+B+C$	1	0	-53

(2)

It is easy to show that A, B and C are integers for every λ. We have seen

(4. 7) $P_3(K) \approx -8\lambda\{P_6(C) - 3P_4(C)P_2(C) + 2P_2(C)^3\}$,

$\lambda = u^3[P_3(K)]$,

$u \in H^4(P_3(K), Z)$.

From (7.15) we see that $P_3(K)$ *cannot be "ocbordantes" with any* 12-*dimensional differentiable submanifold of* $P_7(C)$ *other than those which are determined by* $v = \pm 2g$. *The same thing holds for* $Q_6((4.9))$.

8. Next we consider the case $X^8 \subset X^{12}$. We impose the same conditions upon X^8 and X^{12} as before. Let X^8 be determined by $v_1, v_2 \in H^2(X^{12}, Z)$ ([1] S.87). We have proved that

$$(6.\ 11)\quad \Gamma_2(y, p_1, p_2)[X^8] = \Big[\Big\{v_1^5v_2 + v_2^5v_1 + v_1^3v_2^3 - (v_1v_2^3 + v_2v_1^3)p_1 + v_1v_2p_2\Big\}y^2$$

$$+ \Big\{v_1^5v_2 + v_2^5v_1 + \frac{2}{3}v_1^3v_2^3 - \frac{2}{3}(v_1v_2^3 + v_2v_1^3)p_1$$

$$+ \frac{1}{3}v_1v_2(4p_2 - p_1^2)\Big\}y + \Big\{\frac{2}{15}(v_1^5v_2 + v_2^5v_1) + \frac{1}{9}v_1^3v_2^3$$

$$- \frac{1}{9}(v_1v_2^3 + v_2v_1^3)p_1 + \frac{1}{45}v_1v_2(7p_2 - p_1^2)\Big\}\Big][X^{12}].$$

Suppose that

$$(8.\ 2)\qquad X^8 \approx AP_4(C) + BP_2(C)^2.$$

Then we have from (6. 11)

$$(8.\ 3)\qquad A(10y^2 + 5y + 1) + B(3y + 1)^2 = [\{v_1^5v_2 + v_2^5v_1 + \cdots\cdots\}][X^{12}].$$

Comparing the coefficients of y^α's ($\alpha = 0,1,2$) we have

$$\left\{\begin{aligned} 10A + 9B &= \Big\{v_1^5v_2 + v_2^5v_1 + v_1^3v_2^3 - (v_1v_2^3 + v_2v_1^3)p_1 + v_1v_2p_2\Big\}[X^{12}], \\ 5A + 6B &= \Big\{v_1^5v_2 + v_2^5v_1 + \frac{2}{3}v_1^3v_2^3 - \frac{2}{3}(v_1v_2^3 + v_2v_1^3)p_1 \end{aligned}\right.$$

$$(8.\ 4)\quad \begin{cases} \qquad + \dfrac{1}{3}\, v_1 v_2 (4p_2 - p_1^2)\Big\}[X^{12}], \\ A + B = \Big\{\dfrac{2}{15}(v_1^5 v_2 + v_2^5 v_1) + \dfrac{1}{9} v_1^3 v_2^3 - \dfrac{1}{9}(v_1 v_2^3 + v_2 v_1^3) p_1 \\ \qquad + \dfrac{1}{45}\, v_1 v_2 (7p_2 - p_1^2)\Big\}[X^{12}]. \end{cases}$$

The first equation follows from two others. Solving (8. 4) we have

$$(8.\ 5)\quad \begin{cases} A = \Big\{-\dfrac{1}{5}(v_1^5 v_2 + v_2^5 v_1) + \dfrac{1}{5}(P_1^2 - 2P_2) v_1 v_2\Big\}[X^{12}], \\ B = \Big\{\dfrac{1}{3}(v_1^5 v_2 + v_2^5 v_1) + \dfrac{1}{9} v_1^3 v_2^3 - \dfrac{1}{9}(v_1 v_2^3 + v_2 v_1^3) p_1 \\ \qquad + \dfrac{1}{9}(5p_2 - 2p_1^2) v_1 v_2\Big\}[X^{12}], \\ \text{index} = A + B = \Big\{\dfrac{2}{15}(v_1^5 v_2 + v_2^5 v_1) + \dfrac{1}{9} v_1^3 v_2^3 - \dfrac{1}{9}(v_1 v_2^3 \\ \qquad + v_2 v_1^3) p_1 + \dfrac{1}{45}(7p_2 - p_1^2)\, v_1 v_2\Big\}[X^{12}]. \end{cases}$$

Thus *the coefficients of cobordism decomposition of* X^8 *are uniquely determined by* v_1, v_2 *and the Pontryagin class of* X^{12}.

When $X^{12} = P_6(C)$, the Pontryagin class takes the form

$$(8.\ 6)\quad \begin{cases} p = (1 + g^2)^7 = 1 + 7g^2 + 21g^4 + 35g^6,\ g \in H^2(P_6(C), Z), \\ g^6[P_6(C)] = 1. \end{cases}$$

Hence we have

$$(8.\ 7)\quad \begin{cases} A = \Big\{-\dfrac{1}{5}(v_1^5 v_2 + v_2^5 v_1) + \dfrac{7}{5}\, v_1 v_2 g^4\Big\}[P_6(C)] \\ B = \Big\{\dfrac{1}{3}(v_1^5 v_2 + v_2^5 v_1) + \dfrac{1}{9} v_1^3 v_2^3 - \dfrac{7}{9}(v_1 v_2^3 + v_2 v_1^3) g^2 \\ \qquad + \dfrac{7}{9}\, v_1 v_2 g^4\Big\}[P_6(C)]. \end{cases}$$

Putting

$$(8.\ 8)\qquad v_1 = \lambda g,\ v_2 = \mu g\ (\lambda, \mu : \text{integers}),$$

we have

$$(8.\ 9)\quad \begin{cases} A = -\dfrac{1}{5}(\lambda^5\mu + \mu^5\lambda) + \dfrac{7}{5}\lambda\mu, \\ B = \dfrac{1}{3}(\lambda^5\mu + \mu^5\lambda) + \dfrac{1}{9}\lambda^3\mu^3 - \dfrac{7}{9}(\lambda\mu^3 + \mu^3\lambda) + \dfrac{7}{9}\lambda\mu, \\ \text{index} = A + B = \dfrac{2}{15}(\lambda^5\mu + \mu^5\lambda) + \dfrac{1}{9}\lambda^3\mu^3 \\ \qquad\qquad - \dfrac{7}{9}(\lambda\mu^3 + \mu\lambda^3) + \dfrac{98}{45}\lambda\mu. \end{cases}$$

For example A and B take the following values :

	$\lambda=\mu=1$	$\lambda=2$, $\mu=1$	$\lambda=\mu=2$
A	1	-4	-20
B	0	6	28
index $=A+B$	1	2	8

(3)

It is easy to show that A and B are integers for every λ and μ. $P_2(K)$ *cannot be cobordantes with any differentiable 8-dimensional submanifold of $P_6(C)$ determined by any two cohomology classes $v_1,v_2 \in H^2(P_6(C),Z)$.* The Q_4 has some possibility because of the existence of the submanifold determined by $\lambda = 2, \mu = 1$.

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UTSUNOMIYA UNIVERSITY, JAPAN.