ON THE OPERATORS WHICH GENERATE CONTINUOUS VON NEUMANN ALGEBRAS

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In the global theory of operators on a Hilbert space, it is a principal problem to fully classify all operators, especially non-normal operators. The character of an operator is closely related to the appearance of a von Neumann algebra generated by the operator. Let A be an arbitrary operator on a Hilbert space, then a von Neumann algebra generated by A means the smallest von Neumann algebra containing A, and it is denoted by $\mathbf{R}(A)$. An operator A on a Hilbert space is said to be of type I, II_1, II_∞, III if a von Neumann algebra $\mathbf{R}(A)$ is of type I, II_1 II_∞, III. Up to now it has been shown that there exist several non-normal operators which are of type I [1], [8], but we have not yet known whether the operators of type II, III actually exist. The object of the present note is to show that there exist the operators of type II, III on a separable Hilbert space. Throughout this note, we shall deal with only the operators on a separable Hilbert space and for the terminology of von Neumann algebra, we shall refer to [2].

1. We start with the case of type I. The normal operator is trivially the special case of type I. In fact, such an operator generates an abelian von Neumann algebra. Conversely the following result was obtained by von Neumann [5].

LEMMA 1. *Any abelian von Neumann algebra on a separable Hilbert space is generated by a single Hermitian operator.*

More generally, C.Pearcy [8] proved the following

LEMMA 2. *Any von Neumann algebra of type I on a separable Hilbert space is generated by a single operator.*

His method used to prove this fact is naturally applicable to the case of continuous von Neumann algebra. Indeed, we can prove, in general, the following lemmas.

LEMMA 3. *Let* $\mathbf{A}$,$\mathbf{B}$ *be abelian von Neumann algebras on a separable Hilbert space. Then the von Neumann algebra* $\mathbf{M}$ *generated by* $\mathbf{A}, \mathbf{B}$ *has a* *single generator.*

PROOF. By Lemma 1, there exist Hermitian operators $A \in \mathbf{A}$ and $B \in \mathbf{B}$ such that $\mathbf{A} = \mathbf{R}(A)$, $\mathbf{B} = \mathbf{R}(B)$. Now put $C = A + iB$ as seen in [8], then we can easily see that A, B belong to $\mathbf{R}(C)$. Therefore

$$\mathbf{R}(C) = \mathbf{R}(A, B) = \mathbf{R}(\mathbf{A}, \mathbf{B}) = \mathbf{M}.$$

LEMMA 4. *Let $\mathbf{M}_n (n = 1, 2, \cdots)$ be a sequence of von Neumann algebras on a separable Hilbert space with the following properties:*

(i) *each $\mathbf{M}_n$ has a single generator A_n,*

(ii) *$\{\mathbf{M}_n\}$ commute with each other.*

Then the von Neumann algebra $\mathbf{M}$ generated by $\{\mathbf{M}_n\}$ has a single generator.

PROOF. Each A_n is written in the form $A_n = S_{n1} + iS_{n2}$, where S_{n1}, S_{n2} are Hermitian operators in $\mathbf{M}_n$. Since $\{S_{n1}\}$, $\{S_{n2}\}$ are sequences of commuting Hermitian operators respectively, $\{S_{n1}\}$, $\{S_{n2}\}$ generate abelian von Neumann algebras $\mathbf{A}, \mathbf{B}$ respectively. It is obvious that $\mathbf{M}$ is generated by $\mathbf{A}, \mathbf{B}$. Therefore $\mathbf{M}$ has a single generator by Lemma 3.

2. We concentrate our attention to the operators of type II, and recall that a finite von Neumann algebra $\mathbf{M}$ is said to be hyperfinite if (i) $\mathbf{M}$ is of type I, or (ii) there exists a sequence of von Neumann subalgebras $\{\mathbf{M}_n\}$ of $\mathbf{M}$ with the following properties:

(1) each $\mathbf{M}_n$ is of type I and the center of $\mathbf{M}_n$ coincides with that of $\mathbf{M}$,

(2) $\{\mathbf{M}_n\}$ commute with each other,

(3) $\mathbf{M}$ is generated by $\{\mathbf{M}_n\}$.

Such a von Neumann factor is nothing but a hyperfinite factor defined in [2: p.290] (cf. [3: Corollary 2, 1]). In the latter case, $\mathbf{M}$ is of type II_1 and called a generalized approximately finite von Neumann algebra in [3].

Combining Lemma 2 and Lemma 4, we can now see the existence of the operators of type II_1, that is, we have

THEOREM 1. *A hyperfinite von Neumann algebra on a separable Hilbert space is generated by a single operator.*

REMARK. From Lemma 3, we can also see that there is a non-hyperfinite von Neumann algebra with a single generator as follows. Let G be the free group with two generators a, b. and let $g \to U_g$ $(g \in G)$ be the regular representation of G, then the factor $\mathbf{M}$ of type II_1 generated by the unitary operatrs U_g $(g \in G)$ has a single generator (see [4 : §5, 3 and §6, 2]). In fact, $\mathbf{A} = \mathbf{R}(U_a)$, $\mathbf{B} = \mathbf{R}(U_b)$ are both abelian von Neumann algebras and $\mathbf{M}$ is generated by $\mathbf{A}, \mathbf{B}$, and the conclusion follows from Lemma 3. 1

Here we shall point out that for a hyperfinite factor of type II_1 Theorem 1 was already shown by Y.Misonou in 1956, but his proof was somewhat complicated as compared with ours. Furthermore, Lemma 4 is applicable to show the fact that the operators of type II_∞, III exist. Let $\mathbf{A}$ be a von Neumann algebra of type II_1 with a single generator A on a separable Hilbert space $\mathbf{H}_1$ and let $\mathbf{B}$ be a von Neumann algebra of type I on a separable Hilbert space $\mathbf{H}_2$, then the tensor product $\mathbf{M} = \mathbf{A} \otimes \mathbf{B}$ is of type II_∞. By Lemma 2, $\mathbf{B}$ has a single generator B. Thus the ampliation $\mathbf{A} \otimes C_{H_2}$ (resp. $C_{H_1} \otimes \mathbf{B}$) of $\mathbf{A}$ (resp. $\mathbf{B}$) has a generator $A \otimes I_{H_2}$ (resp. $I_{H_1} \otimes \mathbf{B}$), and $\mathbf{M}$ is generated by $\{\mathbf{A} \otimes C_{H_2}, C_{H_1} \otimes \mathbf{B}\}$. Using Lemma 4, we obtain that $\mathbf{M}$ has a single generator, which is anoperator of type II_∞.

3. In the last stage, we shall construct an operator of type III. Indeed, it is possible by the following von Neumann's result [6],[7] (cf. [9]). Let $\mathbf{H}_{n1}$, $\mathbf{H}_{n2}$ be two dimensional Hilbert spaces and $\mathbf{B}_{n1}$ the algebra of all bounded operators on $\mathbf{H}_{n1}$. Then there exists an equivalence class $\mathfrak{C}$ of C_0-sequence for which the incomplete infinite direct product $\mathbf{M} = \Pi \overset{\mathfrak{C}}{\otimes} (\mathbf{B}_{n1} \otimes C_{H_{n_2}})$ is a factor of type III. Let $\widetilde{\mathbf{B}}_n$ be an extension of $\mathbf{B}_{n1} \otimes C_{H_{n_2}}$ to a separable Hilbert space $\Pi \overset{\mathfrak{C}}{\otimes} (\mathbf{H}_{n1} \otimes \mathbf{H}_{n2})$, then $\{\widetilde{\mathbf{B}}_n\}$ is a commuting sequence of factor of type I_2 and $\mathbf{M}$ is generated by $\{\widetilde{\mathbf{B}}_n\}$. By Lemma 2 and Lemma 4, we see that $\mathbf{M}$ has a single generator, which is an operator of type III.

Summing up the above results, we obtain

THEOREM 2. *There exist the operators of type* II_1, II_∞, III *on a separable Hilbert space.*

Added in Proof. Recently, we have known that C, Pearcy showed the existence of operators of type II_1 and II_∞. That is, in his paper "*On certain von Neumann algebras which are generated by partial isometries*" (to appear in the Proceedings of the A. M. S.), he proved that there exists a partial isometry V (resp. W) which generates a von Neumann algebra of type II_1 (resp. II_∞).

REFERENCES

[1] A. BROWN, The unitary equivalence of binormal operators, Amer. Journ. Math., 76 (1954), 414-439.

[2] J. DIXMIER, Les algèbres d'opérateurs dans l'espace hilbertien, Gauthier-Villars, Paris,

1957.
[3] Y. MISONOU, Generalized approximately finite W^*-algebras, Tôhoku Math. Journ., 7(1955), 192-205.
[4] F. J. MURRAY and J. VON NEUMANN, On rings of operators IV, Ann. of Math., 44(1943), 716-808.
[5] J. VON NEUMANN, Zur Algebra der Funktionaloperationen und Theorie der normalen Operatoren, Math. Ann., 102(1929), 370-427.
[6] J. VON NEUMANN, On infinite direct products, Compositio Math., 6(1938), 1-77.
[7] J. VON NEUMANN, On rings of operators III, Ann. of Math., 41(1940), 94-161.
[8] C. PEARCY, W^*-algebras with a single generator, Proc. Amer. Math. Soc., 13(1962), 831-832.
[9] T. SAITÔ, On incomplete infinite direct product of W^*-algebras, Tôhoku Math. Journ., 10 (1958), 165-171.

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