ON THE EXISTENCE OF HARMONIC FUNCTIONS ON A RIEMANN SURFACE

AKIRA SAGAWA

(Received April 16, 1963)

1. Let R be an open Riemann surface and G be a non-compact region on R whose relative boundary C consists of a finite or infinite number of compact or non-compact analytic curves clustering nowhere on R.

Let HP, HB and HD denote the classes of single valued harmonic functions, which are respectively positive, bounded or have a finite Dirichlet integral.

We denote by $O_{HX}(X = P,\ B$ or $D)$ the class of Riemann surfaces R such that every function $u(p) \in HX$ on R reduces to a constant. Further, denote by SO_{HX} the class of non-compact regions G with relative boundary C, such that every function $u(p) \in HX$ on G which vanishes continuously at every point on C, vanishes throughout G.

R.Nevanlinna [3], R.Bader-M. Parreau [1] and A.Mori [2] proved the following theorem.

THEOREM. *A Riemann surface R does not belong to $O_{HX}(X = B, D)$, if and only if there exist two non-compact regions G_1 and G_2 on R which are disjoint from each other and do not belong to $SO_{HX}(X = B, D)$.*

In this paper, by modifying Nevanlinna's method [3], we shall give a necessary and sufficient condition in order that a Riemann surface belongs to $O_{HB} - O_{HP}$.

Further, we shall give some criteria for a Riemann surface to belong to $O_{HX}(X = B, D)$ which is slightly different from Theorem mentioned above.

2. Let R be an open Riemann surface and C_1 be a system of at most an enumerable number of compact or non-compact analytic curves clustering nowhere on R.

We suppose that C_1 separates R into two non-compact regions G_1 and $R - \overline{G}_1$ $(\overline{G}_1 = G_1 \cup C_1)$.

Let G_2 be a non-compact region on R which contains $R - \overline{G}_1$ and whose relative boundary C_2 is contained in G_1.

We have the following theorem.

THEOREM 1. *Suppose that there exist two non-compact regions G_1 and*

G_2 on R satisfying the following conditions:

(i) $G_1 \in SO_{HB}$ *and* $G_2 \overline{\in} SO_{HB}$,

(ii) $\sup_{C_1} \omega(p, C_2, G_2) = \lambda < 1$, *where* $\omega(p, C_2, G_2)$ *is the harmonic measure of* C_2 *with respect to* G_2,

(iii) *there exists a non-constant single-valued positive harmonic function* $V(p)$ *in* G_1 *which is bounded* $(< M)$ *on* $G_1 \cap G_2$ *and equals zero on* C_1.

Then $R \overline{\in} O_{HP}$. *Conversely, if* $R \in O_{HB} - O_{HP}$, *we can find non-compact regions* G_1 *and* G_2 *satisfying* (i), (ii) *and* (iii).

PROOF. Suppose that there exist two non-compact regions G_1 and G_2 satisfying the conditions (i), (ii) and (iii).

Let $\{R_n\}$ $(n = 0, 1, \cdots)$ be an exhaustion of R such that R_n is compact with respect to R and the boundary Γ_n of R_n consists of a finite number of analytic closed curves on R and $R_n \cup \Gamma_n \subset R_{n+1}$.

Let $f(q)$ be an arbitrary continuous real function on C_2 such that $0 < f(q) \leqq M$ on C_2.

We construct a harmonic function $f_n(p)$ in $G_2 \cap R_n$ such that $f_n(p) = 0$ on $\Gamma_n \cap G_2$ and $f_n(p) = f(p)$ on $C_2 \cap R_n$.

By the maximum principle, we get $f_n(p) < f_{n+1}(p) < M$ for each n, whence the sequence $\{f_n(p)\}$ converges to a harmonic function $u(p)$ in G_2 which is uniquely determined by $f(q)$.

For simplicity, we shall call $u(p)$ the lower function in G_2 with the boundary value $f(q)$ on C_2.

Further, by the maximum principle, we have

$$u(p) \leqq M \cdot \omega(p, C_2, G_2)$$

for any $p \in G_2$, so that from (iii),

$$(1) \qquad \sup_{C_1} u(p) \leqq M \cdot \lambda.$$

Now we construct two sequences $\{u_n(p)\}$ and $\{v_n(p)\}$ $(n = 0, 1, \cdots)$ as follows.

First we put

$$v_0(p) = V(p)$$

in G_1. Let $u_n(p)$ $(n = 0, 1, \cdots)$ be the lower function in G_2 with boundary value $v_n(p)$ on C_2 and $v_{n+1}(p)$ be the harmonic function in G_1 such that $v_{n+1}(p) - v_0(p)$ is bounded in G_1 and $v_{n+1}(p) = u_n(p)$ on C_1.

From the formula (1), we see easily that for $n \geqq 0$

$$u_n(p) < M(\lambda + \lambda^2 + \cdots + \lambda^{n+1}) < M\frac{\lambda}{1-\lambda}$$

on C_1 and

$$v_{n+1}(p) < M(1 + \lambda + \cdots + \lambda^{n+1}) < M\frac{1}{1-\lambda}$$

on C_2. Since by the maximum principle, two sequences $\{u_n(p)\}$ and $\{v_n(p)\}$ increase monotonically with n, these sequences converge to harmonic functions $u(p)$ and $v(p)$ in G_2 *and* G_1, respectively.

Since $u_n(p) - v_n(p) = 0$ on C_2, $u_n(p) - v_n(p) = u_n(p) - u_{n-1}(p)$ on C_1 and $|u_n(p) - v_n(p)| < \dfrac{M}{1-\lambda}$ in $G_1 \cap G_2$, we obtain by the maximum principle applied in $G_1 \cap G_2 \cap R_m$

$$|u_n(p) - v_n(p)| < u_n(p) - v_n(p) + \frac{M}{1-\lambda}\,\omega(p, \Gamma_m \cap G_1 \cap G_2, R_m \cap G_1 \cap G_2),$$

where $\omega(p, \Gamma_m \cap G_1 \cap G_2, R_m \cap G_1 \cap G_2)$ is the harmonic measure of $\Gamma_m \cap G_1 \cap G_2$ with respect to $R_m \cap G_1 \cap G_2$.

Making $m \to \infty$ and next $n \to \infty$, we have from $G_1 \in SO_{HB}$

$$u(p) \equiv v(p)$$

in $G_1 \cap G_2$. Hence if we put $U(p) = v(p)$ in G_1 and $U(p) = u(p)$ in G_2, then $U(p)$ is non-negative and harmonic on R.

It is easy to see that

$$U(p) = u(p) \leqq \frac{M}{1-\lambda}\,\omega(p, C_2, G_2)$$

in G_2.

Then, since $G_2 \overline{\in} SO_{HB}$, it follows that

$$\inf_{G_2} U(p) \leqq \frac{M}{1-\lambda} \inf_{G_2} \omega(p, C_2, G_2) = 0.$$

On the other hand, since $G_1 \in SO_{HB}$, we have

$$\sup_{G_1} U(p) = \sup_{G_1} v(p) \geqq \sup_{G_1} V(p) = \infty,$$

hence $U(p)$ is non-constant, so that $R \overline{\in} O_{HP}$.

Next, we suppose that $R \in O_{HB} - O_{HP}$. Then there exists a non-constant single-valued positive harmonic function $U(p)$ on R. We choose a point p_1 on R and consider the open set on R, where $U(p) < U(p_1)$.

Then it is obvious that each connected component of this open set is a non-compact region not belonging to SO_{HB}.

Since $R \in O_{HB}$, it follows by Theorem stated in **1** that this open set consists of only one non-compact region G which does not belong to the class SO_{HB}.

We denote by G_1 any connected component of the complememtary set $R-\overline{G}$ of G with respect to R and by C_1 its boundary.

It is evident that G_1 is non-compact region. We choose a point p_2 in G_1.

The part of the niveau curve $U(p)=U(p_2)$ contained in G_1 divides R into two or more parts and denote by G_2 that part, which contains G. On the relative boundary C_2 of G_2 with respect to R, $U(p)$ equals $U(p_2)$.

Since $G \overline{\in} SO_{HB}$ and $G \subset G_2$, we have $G_2 \overline{\in} SO_{HB}$.

Further, since $G \overline{\in} SO_{HB}$ and $R \in O_{HB}$, we have $G_1 \in SO_{HB}$ by Theorem stated in **1**.

Next, putting $k=\inf_R U(p)$, we see easily by the maximum principle that

$$\omega(p, C_2 \cap R_m,\ G_2 \cap R_m) \leqq \frac{U(p)-k}{U(p_2)-k}$$

in $G_2 \cap R_m$, so that

$$\omega(p, C_2, G_2) \leqq \frac{U(p)-k}{U(p_2)-k}$$

in G_2. Hence we obtain

$$\sup_{C_1} \omega\ (p, C_2, G_2) \leqq \frac{U(p_1)-k}{U(p_2)-k} < 1.$$

If we put $U(p)-U(p_1)=V(p)$, then $V(p)$ satisfies the condition (iii), so that two non-compact regions G_1 and G_2 satisfy the conditions (i), (ii) and (iii).

3. We can prove the following theorem.

THEOREM 2. *A Riemann surface R does not belong to the class O_{HB}, if and only if there exists a non-compact region G on R satisfying the following conditions*:

(i) $$G \overline{\in} SO_{HB},$$

(ii) $$\int_C \frac{\partial g(p,q)}{\partial \nu}\, ds > 0,$$

where $g(p,q)$ is the Green function of R with its pole at $q \in G$ and C is relative boundary of G and ν is the inner normal with respect to G.

PROOF. First we suppose that every ideal boundary component of R is regular for the Green function of R.

Let $\{\varepsilon_n\}$ $(n=0,1,\cdots)$ be a monotonically decreasing sequence which tends to zero. We denote by $R_n (n=0,1,\cdots,)$ the set of all points p on R satisfying the inequality $g(p,q)>\varepsilon_n$ and by Γ_n the boundary of R_n. By the assumption, R_n is compact with respect to R.

By putting $g(p,q)-\varepsilon_n=g_n(p,q)$, it is obvious that $g_n(p,q)$ is the Green function of R_n.

We consider the bounded harmonic function $\omega_n(p) = \omega(p, \Gamma_n \cap G, R_n)$ in R_n with boundary values $\omega_n(p) = 1$ on $\Gamma_n \cap G$ and $\omega_n(p) = 0$ on $\Gamma_n - \Gamma_n \cap G$.

Since $\omega_n(p)\,(n = 0, 1, \cdots)$ are uniformly bounded, by taking suitable subsequence, we may assume that $\omega_n(p)$ converges to a bounded harmonic function $\omega(p)$ uniformly on any compact region on R.

By Green's formula, we obtain that

$$\omega_n(q) = \frac{1}{2\pi}\int_{\Gamma_n \cap G} \frac{\partial g_n(p,q)}{\partial \nu}\, ds = \frac{1}{2\pi}\int_{\Gamma_n \cap G} \frac{\partial g(p,q)}{\partial \nu}\, ds$$
$$= 1 - \frac{1}{2\pi}\int_{C \cap R_n} \frac{\partial g(p,q)}{\partial \nu}\, ds.$$

Hence

$$\omega(q) = 1 - \frac{1}{2\pi}\int_C \frac{\partial g(p,q)}{\partial \nu}\, ds. \tag{2}$$

SUFFICIENCY. Suppose that there exists a region G satisfying the conditions (i) and (ii).

Then, from (2), it follows that

$$\omega(q) < 1.$$

On the other hand, we have

$$\omega(p, \Gamma_n \cap G, G \cap R_n) < \omega_n(p)$$

in $G \cap R_n$, so that

$$\lim_{n\to\infty} \omega(p, \Gamma_n \cap G,\ G \cap R_n) \leqq \omega(p)$$

in G.

Hence from (i), we have

$$1 = \sup_G \lim_{n\to\infty} \omega(p, \Gamma_n \cap G, G \cap R_n) \leqq \sup_G \omega(p).$$

Thus the function $\omega(p)$ is non-constant.

NECESSITY. If $R \overline{\in} O_{HB}$, there exists a non-constant single-valued bounded harmonic function $u(p)$ on R.

We take a point p_0 on R arbitrarily and denote by G a connected component of the open set on R, where $u(p) < u(p_0)$.

If we consider the function $u(p_0) - u(p)$ in G, then it is easy to see that $G \overline{\in} SO_{HB}$. Similarly we can prove that each connected component of $R - \overline{G}$ is non-compact region and does not belong to SO_{HB}.

If $\int_C \frac{\partial g(p,q)}{\partial \nu}\, ds = 0$, then from (2) and maximum principle, the function

$1-\omega(p)$ reduces to the constant zero, hence each component of $R-\bar{G}$ belongs to the class SO_{HB}, which is a contradiction.

Thus we have our assertion, when any ideal boundary component of R is regular for the Green function of R.

Next we suppose that at least an ideal boundary component of R is irregular for the Green function of R.

We use the same notations as above. Then we may assume that R_n is non-compact region.

As is easily seen from the above proof, it is sufficient to prove that the equality (2) holds in this case.

Let $\{F_m\}$ $(m=0,1,\cdots)$ be an exhaustion of R such that, for each m, F_m contains q and an outer point of R_n.

Let $R_n^{(m)}$ be the connected component of $R_n \cap F_m$ which contains q. Denote by $\Gamma_n^{(m)}$ the boundary of $R_n^{(m)}$ and by $\gamma_n^{(m)}$ the part of $\Gamma_n^{(m)}$ contained in R_n. Denote by $\omega_n^{(m)}(p)$ the harmonic measure of $(\Gamma_n^{(m)}-\gamma_n^{(m)})\cap G$ with respect to $R_n^{(m)}$ and by $g_n^{(m)}(p,q)$ the Green function of $R_n^{(m)}$ with its pole at q.

By Green's formula, we have

$$\omega_n^{(m)}(q)=\frac{1}{2\pi}\int_{(\Gamma_n^{(m)}\cap\gamma_n^{(m)})\cap G}\frac{\partial g_n^{(m)}(p,q)}{\partial\nu}ds.$$

Since R_n belongs to SO_{HB}, it is easy to see that

$$\lim_{m\to\infty}\omega_n^{(m)}(p)=\omega_n(p).$$

It is evident that

$$\int_{(\Gamma_n^{(m_0)}-\gamma_n^{(m_0)})\cap G}\frac{\partial g_n^{(m)}(p,q)}{\partial\nu}ds<\int_{(\Gamma_n^{(m)}-\gamma_n^{(m)})\cap G}\frac{\partial g_n^{(m)}(p,q)}{\partial\nu}ds$$

for $m>m_0$.

Letting first $m\to\infty$ and next $m_0\to\infty$, we obtain

$$\int_{\Gamma_n\cap G}\frac{\partial g_n(p,q)}{\partial\nu}ds\leqq\lim_{m\to\infty}\int_{(\Gamma_n^{(m)}-\gamma_n^{(m)})\cap G}\frac{\partial g_n^{(m)}(p,q)}{\partial\nu}ds.$$

On the other hand, since $g_n^{(m)}(p,q)=g_n(p,q)$ in $p\in R_n^{(m)}$, it follows that

$$\lim_{m\to\infty}\int_{(\Gamma_n^{(m)}-\gamma_n^{(m)})\cap G}\frac{\partial g_n^{(m)}(p,q)}{\partial\nu}ds\leqq\int_{\Gamma_n\cap G}\frac{\partial g_n(p,q)}{\partial\nu}ds.$$

Hence we obtain

$$\omega_n(q)=\frac{1}{2\pi}\int_{\Gamma_n\cap G}\frac{\partial g_n(p,q)}{\partial\nu}ds.$$

Therefore, it holds that

$$\omega(q) = 1 - \frac{1}{2\pi}\int_C \frac{\partial g(p,q)}{\partial \nu}\,ds,$$

as was required.

THEOREM 3. *A Riemann surface does not belong to the class O_{HD}, if and only if there exists a non-compact region G on R satisfying the following conditions*:

(i) $$G \overline{\in} SO_{HD},$$

(ii) $$\int_C \frac{\partial g(p,q)}{\partial \nu}\,ds > 0,$$

where $g(p,q)$ is the Green function of R with its pole at $q \in G$ and C is relative boundary of G and ν is the inner normal with respect to G.

PROOF. We consider the case when every ideal boundary component of R is regular for the Green function of R.

We use the same notations as in the proof of Theorem 2.

First, suppose that there exists a non-compact region G satisfying the conditions (i) and (ii).

Since $G \overline{\in} SO_{HD} = SO_{HBD}$, there exists a non-constant single-valued positive harmonic function $V(p)$ in G such that $V(p)=0$ on C, $\sup_G V(p)=1$ and its Dirichlet integral $D(V(p))$ taken over G is finite.

Let $U(p)$ be the function on R such that $U(p)=V(p)$ in G and $U(p)=0$ in $R-G$ and on C, and let $u_n(p)$ be the harmonic function in R_n whose boundary value on Γ_n equals to $U(p)$.

Since $0 \leqq u_n(p) \leqq 1$, we may suppose that sequence $\{u_n(p)\}$ converges to a harmonic function $u(p)$ on any compact set in R.

For $m<n$, we obtain

$$D_{R_m}(u_n) \leqq D_{R_m}(U) = D_{G\cap R_m}(V).$$

Letting $n\to\infty$ and next $m\to\infty$, we have

$$D_R(u) \leqq D_G(V) < \infty.$$

Since $V(p) \leqq u(p)$, $1=\sup_G V(p) \leqq \sup_G u(p)$ and since $0\leqq u_n(p)\leqq 1$, we have

$$\sup_G u(p) = 1.$$

On the other hand, since

$$u_n(q) = \frac{1}{2\pi}\int_{\Gamma_n\cap G} V(p)\frac{\partial g_n(p,q)}{\partial \nu}\,ds \leqq \frac{1}{2\pi}\int_{\Gamma_n\cap G}\frac{\partial g(p,q)}{\partial \nu}\,ds$$

$$= 1 - \frac{1}{2\pi} \int_{C \cap R_n} \frac{\partial g(p, q)}{\partial \nu} ds,$$

we have

$$u(q) \leqq 1 - \frac{1}{2\pi} \int_C \frac{\partial g(p, q)}{\partial \nu} ds. \tag{3}$$

Therefore, $u(p)$ is non-constant.

To prove the converse, we suppose that $u(p)$ is a non-constant harmonic function whose Dirichlet integral over R is finite.

We choose a point p_0 on R and denote by G a connected component of the open set on R, where $u(p) > u(p_0)$.

Since $D_G(u(p) - u(p_0)) < D_R(u(p))$, we have $G \overline{\in} SO_{HD}$, hence it follows that $G \overline{\in} SO_{HB}$.

Therefore, by the same argument as in the proof of Theorem 2, we get

$$\int_C \frac{\partial g(p, q)}{\partial \nu} ds > 0.$$

Thus our assertion is proved.

In the case when at least an ideal boundary component of R is irregular for the Green function of R, since the inequality (3) holds, we can prove the assertion similarly.

REFERENCES

[1] R. BADER-M. PARREAU, Domaines non-compacts et classification des surfaces de Riemann, C. R. Acad. Sci. Paris, 232(1951), 138-139.

[2] A. MORI, On the existence of harmonic functions on a Riemann surface, Journ. Fac. Sci. Univ. Tokyo. S. I., 6(1951), 247-257.

[3] R. NEVANLINNA, Über die Existenz von beschränkten Potentialfunktionen auf Flächen von unendlichen Geschlecht, Math. Zeits., 52(1950), 559-604.

COLLEGE OF ARTS AND SCIENCE, FACULTY OF EDUCATION,
TÔHOKU UNIVERCITY.