CERTAIN ALMOST CONTACT HYPERSURFACES IN KAEHLERIAN MANIFOLDS OF CONSTANT HOLOMORPHIC SECTIONAL CURVATURES

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Introduction. An odd-dimensional differentiable manifold is said to have an almost contact structure or to be an almost contact manifold if the structural group of its tangent bundle is reducible to the product of a unitary group with the 1-dimensional identity group. The study of almost contact manifolds, at the first time, has been developed by W. M. Boothby and H. C. Wang [1][1] and J. W. Gray [2] using a topological method. Recently, S. Sasaki [7] found a differential geometric method of investigation into the almost contact manifold and using this method Y. Tashiro [12] proved that in any orientable differentiable hypersurface in an almost complex manifold we can naturally define an almost contact structure. Hereafter, the almost contact structure of the hypersurface is studied by M. Kurita [4], Y. Tashiro and S. Tachibana [13] and the present author [5].

The purpose of the paper is to discuss normal almost contact hypersurfaces in a Kaehlerian manifold of constant holomorphic sectional curvature and to prove some fundamental properties of the hypersurfaces.

In §1, we give first of all some preliminaries of almost contact manifold and prove a certain condition for a Riemannian manifold to be a normal contact manifold for the later use. In §2, we consider hypersurfaces in a Kaehlerian manifold and give a condition for the induced almost contact structure of a hypersurface in a Kaehlerian manifold to be normal. After proving a lemma in §3, we show in §4 that, in a normal almost contact hypersurface of a Kaehlerian manifold of constant holomorphic sectional curvature, the second fundamental tensor can admit at most three distinct characteristic roots and that they are all constants. The distributions corresponding these characteristic roots are studied in §5 and integrability of these distributions is discussed.

In §6, the integral submanifolds of certain distributions are considered and using the theorem in §1, we prove that the integral submanifolds admit normal contact metric structures.

1. Almost contact structure and contact metric structure. On a $(2n-1)$-dimensional real differentiable manifold M^{2n-1} with local coordinate systems

1) The numbers in the brackets refer to the bibliography at the end of the paper.

$\{x^i\}$, if there exist a tensor field $\phi_j{}^i$, contravariant and covariant vector fields ξ^i and η_i satisfying the relations

$$\xi^i \eta_i = 1, \tag{1. 1}$$

$$\mathrm{rank}\ (\phi_j{}^i) = 2n-2, \tag{1. 2}$$

$$\phi_j{}^i \xi^j = 0, \quad \phi_j{}^i \eta_i = 0, \tag{1. 3}$$

$$\phi_j{}^i \phi_k{}^j = -\delta_k^i + \xi^i \eta_k, \tag{1. 4}$$

then the set $(\phi_j{}^i, \xi^i, \eta_j)$ is called an almost contact structure and the manifold with such a structure is called an almost contact manifold. It has been proved by S. Sasaki [7] that this definition of the almost contact manifold is equivalent to that used in J. W. Gray's paper [2]. It is known[2)] that an almost contact manifold always admits a positive definite Riemannian metric tensor g_{ji} satisfying

$$g_{ji}\xi^j = \eta_i, \tag{1. 5}$$

$$g_{ji}\phi_k{}^j \phi_h{}^i = g_{hk} - \eta_h \eta_k. \tag{1. 6}$$

The metric with above properties is called an associated metric to the almost contact structure and the almost contact manifold with such a Riemannian metric is called an almost contact metric manifold. In this paper, we always treat such a Riemannian metric tensor, so we use a notation η^i in stead of ξ^i.

The tensor $N_{ji}{}^h$ defined by the following is fundamental:

$$N_{ji}{}^h = \phi_j{}^r(\nabla_r \phi_i{}^h - \nabla_i \phi_r{}^h) - \phi_i{}^r(\nabla_r \phi_j{}^h - \nabla_j \phi_r{}^h) + \nabla_i \eta^h \eta_j - \nabla_j \eta^h \eta_i. \tag{1. 7}$$

Where and throughout the paper ∇_j denotes the operator of covariant differentiation with respect to the Christoffel symbols formed from the associated metric and put $\nabla^j = g^{jr}\nabla_r$.

An almost contact structure with vanishing $N_{ji}{}^h$ is called a normal almost contact structure. Totally geodesic hypersurfaces in a Kaehlerian manifold are examples of normal almost contact manifolds[3)].

A differentiable manifold M^{2n-1} is said to have a contact structure or to be a contact manifold if there exists a 1-form η over the manifold such that

$$\eta \wedge \overbrace{d\eta \wedge \cdots\cdot \wedge d\eta}^{n-1} \neq 0,$$

where operator $\wedge$ in the last equation means exterior multiplication. In an

2) Sasaki, S. [7], Hatakeyama, Y. [3].
3) Okumura, M. [5].

almost contact metric manifold if there is a relation for a constant c,

$$c^2\phi_{ji} = \partial_j\eta_i - \partial_i\eta_j, \tag{1. 8}$$

then the rank of the matrix (ϕ_{ji}) being $2n-2$, the structure is regarded as the one formed from a contact structure. So, we call such an almost contact structure a contact metric structure. A contact metric structure is called a normal contact metric structure if in the structure the tensor $N_{ji}{}^h$ vanishes identically.

The following theorem on normal contact structure is necessary for the later section.

THEOREM 1. 1.[4] *Let M^{2n-1} be a Riemannian manifold. If M^{2n-1} admits a Killing vector v_i of constant length satisfying*

$$c^2\nabla_j\nabla_i v_h = v_i g_{jh} - v_h g_{ji}, \tag{1. 9}$$

then, M^{2n-1} is a normal contact metric manifold such that the given Riemannian metric g_{ji} is the associated one.

PROOF. Let c_1 be the length of v_i and put $\eta_i = \dfrac{1}{c_1}v_i$. Then, we have

$$c^2\nabla_j\nabla_i\eta_h = \eta_i g_{jh} - \eta_h g_{ji}, \tag{1. 10}$$

and $\eta^i\eta_i=1$. Transvecting (1. 10) with η^h, we get

$$c^2\nabla_j\eta^h\nabla_i\eta_h = g_{ji} - \eta_i\eta_j,$$

because of $\eta^r\nabla_j\eta_r=0$. If we put $\phi_{ji}=c\nabla_j\eta_i$, the above equation changes its form as (1. 4). By the construction we easily see that (1. 1), (1. 3) are satisfied and that the existence of the solutions of (1. 3) and (1. 4) shows that the rank of $(\phi_j{}^i)$ is $2n-2$. Furthermore by definition

$$\phi_{ji} = \frac{c}{2}(\partial_j\eta_i - \partial_i\eta_j) = c\nabla_j\eta_i, \tag{1. 11}$$

which implies that the structure is the one induced from a contact structure. Substituting (1. 11) into (1. 10), we have

$$c\nabla_j\phi_{ih} = \eta_i g_{jh} - \eta_h g_{ji}, \tag{1. 12}$$

from which, together with (1. 7), we get $N_{ji}{}^h=0$. This completes the proof.

4) Okumura, M. and Y. Ogawa [6].

2. Induced almost contact structure of a hypersurface in a Kaehlerian manifold. Let us consider a real analytic $2n$-dimensional almost Hermitian manifold M^{2n} with local coordinate systems $\{X^\kappa\}$ and $(F_\mu{}^\lambda, G_{\mu\lambda})$ be the almost Hermitian structure, that is, $F_\mu{}^\lambda$ be the almost complex structure defined on M^{2n} and $G_{\mu\lambda}$ be the Riemannian metric tensor satisfying $G_{\kappa\lambda} = G_{\mu\nu}F_\kappa{}^\mu F_\lambda{}^\nu$. A hypersurface M^{2n-1} of M^{2n} may be represented parametrically by the equation $X^\kappa = X^\kappa(x^i)$. In this paper, we assume that the function $X^\kappa(x^i)$ be real analytic, because we discuss a complete integrability making use of the Frobenius' existence theorem for analytic differential equations. Furthermore, in the following, we assume that the hypersurface be orientable.

Let $B_i{}^\kappa = \partial_i X^\kappa$, $(\partial_i = \partial/\partial x^i)$, then they span the tangent plane of M^{2n-1} at each point and induced Riemannian metric g_{ji} in M^{2n-1} is given by

$$g_{ji} = G_{\mu\lambda} B_j{}^\mu B_i{}^\lambda . \tag{2. 1}$$

Choosing the unit normal vector C^κ to the hypersurface, we put

$$\phi_j{}^i = B_j{}^\lambda F_\lambda{}^\kappa B^i{}_\kappa , \tag{2. 2}$$

$$\eta_j = B_j{}^\mu F_\mu{}^\lambda C_\lambda = B_j{}^\mu F_{\mu\lambda} C^\lambda , \tag{2. 3}$$

where we have put $B^j{}_\kappa = G_{\lambda\kappa} B_i{}^\lambda g^{ji}$, $C_\lambda = G_{\lambda\kappa} C^\kappa$ and $F_{\lambda\mu} = G_{\kappa\mu} F_\lambda{}^\kappa$. Then the aggregate $(\phi_j{}^i, g^{ir}\eta_r, \eta_j, g_{ji})$ defines an almost contact metric structure in the hypersurface[5]. In the following we call an orientable hypersurface with the induced almost contact structure an almost contact hypersurface and if the structure is normal a normal almost contact hypersurface.

Assuming that M^{2n} be a Kaehlerian manifold we consider an almost contact hypersurface in M^{2n}.

Making use of Gauss and Weingarten equations

$$\nabla_j B_i{}^\kappa = H_{ji} C^\kappa , \tag{2. 4}$$

$$\nabla_j C_\kappa = - H_{ji} B^i{}_\kappa , \tag{2. 5}$$

where H_{ji} is the second fundamental tensor of the hypersurface, we see that the following identities are always valid.

$$\nabla_j \eta_i = - \phi_i{}^r H_{jr} , \tag{2. 6}$$

$$\nabla_j \phi_{ih} = \eta_i H_{jh} - \eta_h H_{ji} . \tag{2. 7}$$

Consequently the tensor $N_{ji}{}^h$ can be rewritten as

$$N_{kj}{}^i = \eta_j (\phi_k{}^r H_r{}^i + \phi^{ir} H_{rk}) - \eta_k (\phi_j{}^r H_r{}^i + \phi^{ir} H_{rj}) . \tag{2. 8}$$

5) Tashiro, Y. [12].

Contraction with respect to i and k in (2. 8) gives

$$N_{rj}{}^{r} = -\eta^{s} H_{rs} \phi_{j}{}^{r}, \tag{2. 9}$$

because of $H_{ji} = H_{ij}$ and $\phi_{ji} = -\phi_{ij}$.

Let M^{2n-1} be a normal almost contact hypersurface, then, transvecting (2. 9) with $\phi_i{}^j$, we have

$$H_{ij}\eta^{j} = \alpha\eta_{i}, \quad (\alpha = H_{ji}\eta^{j}\eta^{i}), \tag{2. 10}$$

that is, α is a characteristic root of the second fundamental tensor H_j^i and η^j is a corresponding eigenvector of α. Furthermore, transvecting (2. 8) with η^j and making use of (2. 10), we get

$$\phi_k{}^r H_r{}^i + \phi^{ir} H_{rk} = 0\,. \tag{2. 11}$$

This implies, together with (2. 6), that

$$\nabla_j \eta_i + \nabla_i \eta_j = 0\,, \tag{2. 12}$$

which means that the vector η_i is a Killing vector. Since η_i is a unit vector we have from the above equation

$$\nabla_j \eta_i \eta^i = 0, \quad \nabla_i \eta_j \eta^i = 0\,. \tag{2. 13}$$

Now, we prove the following

THEOREM 2. 1. *Let M^{2n} be a Kaehlerian manifold. In order that the induced almost contact structure of a hypersurface in M^{2n-1} be normal, it is necessary and sufficient that the vector η_i is a Killing vector.*

PROOF. We have only to prove the sufficiency of the condition. By means of (2. 6) if η_i is a Killing vector we have the relation (2. 11). Substituting (2. 11) into (2. 8), we get $N_{kj}{}^{i} = 0$. This proves the sufficiency of the condition. Q.E.D.

3. Normal almost contact hypersurfaces in a Kaehlerian manifold of constant holomorphic sectional curvature. A Kaehlerian manifold M^{2n} is called a manifold of constant holomorphic sectional curvature if the holomorphic sectional curvature at every point is independent of two dimensional directions at the point, and its curvature tensor is given by

$$R_{\nu\mu\lambda\kappa} = k(G_{\nu\kappa}G_{\mu\lambda} - G_{\mu\kappa}G_{\nu\lambda} + F_{\nu\kappa}F_{\mu\lambda} - F_{\mu\kappa}F_{\nu\lambda} - 2F_{\nu\mu}F_{\lambda\kappa}), \tag{3. 1}$$

k being a constant.

In this section we consider an orientable hypersurface in a Kaehlerian manifold of constant holomorphic sectional curvature.

Substituting (3. 1) into the Gauss and Codazzi equations[6)]

$$R_{kjih} = B_k^{\nu} B_j^{\mu} B_i^{\lambda} B_h^{\kappa} R_{\nu\mu\lambda\kappa} + H_{kh} H_{ji} - H_{jh} H_{ki}, \tag{3. 2}$$

$$\nabla_k H_{ji} - \nabla_j H_{ki} = B_k^{\nu} B_j^{\mu} B_i^{\lambda} C^{\kappa} R_{\nu\mu\lambda\kappa}, \tag{3. 3}$$

we have

$$R_{kjih} = k(g_{ji} g_{kh} - g_{ki} g_{jh} + \phi_{kh}\phi_{ji} - \phi_{jh}\phi_{ki} - 2\phi_{kj}\phi_{ih}) + H_{kh} H_{ji} - H_{jh} H_{ki} \tag{3. 4}$$

and

$$\nabla_k H_{ji} - \nabla_j H_{ki} = k(\eta_k \phi_{ji} - \eta_j \phi_{ki} - 2\phi_{kj}\eta_i), \tag{3. 5}$$

from which we get

$$(\nabla_k H_{ji} - \nabla_j H_{ki})\eta^j = -k\phi_{ki}, \tag{3. 6}$$

and

$$(\nabla_k H_{ji} - \nabla_j H_{ki})\eta^i = -2k\phi_{kj}. \tag{3. 7}$$

For a normal almost contact hypersurface in a Kaehlerian manifold of constant holomorphic sectional curvature it follows that

$$R_{kjih}\eta^k = \eta_h(kg_{ji} + \alpha H_{ji}) - \eta_i(kg_{jh} + \alpha H_{jh}), \tag{3. 8}$$

because of (2. 10) and (3. 4).

On the other hand, in §2 we have seen that η^i is a Killing vector and consequently an infinitesimal affine transformation. Therefore it follows that

$$\underset{\eta}{\mathcal{L}} \begin{Bmatrix} h \\ ji \end{Bmatrix} = \nabla_j \nabla_i \eta^h + R_{kji}{}^h \eta^k = 0, \tag{3. 9}$$

where $\underset{\eta}{\mathcal{L}}$ means the operator of Lie derivation with respect to the vector η^i. Comparing (3. 8) and (3. 9), we have

$$-\nabla_j \nabla_i \eta_h = \eta_h(kg_{ji} + \alpha H_{ji}) - \eta_i(kg_{jh} + \alpha H_{jh}).$$

Transvecting the above equation with η^h and making use of (2. 13), we get

$$\nabla_j \eta^h \nabla_i \eta_h = kg_{ji} + \alpha H_{ji} - (k + \alpha^2)\eta_i \eta_j,$$

which implies that

$$H_{ir} H_j{}^r = \alpha H_{ji} + k(g_{ji} - \eta_j \eta_i) \tag{3. 10}$$

6) For example, Schouten, J. A. [10].

by virtue of (2. 6) and (2. 10).

LEMMA 3. 1. *Let M^{2n-1} be an analytic normal almost contact hypersurface in a Kaehlerian manifold of constant holomorphic sectional curvature M^{2n}, then one of the following two relations must be satisfied.*
1) *α in (2. 10) is a constant;*
2) *The Kaehlerian manifold in consideration is a locally Euclidean manifold.*

PROOF. Suppose that the scalar α is not constant. Applying the operator ∇_j to (2. 10) and making use of (2. 6) we have

$$\nabla_j H_{kr}\eta^r + \phi_j{}^s H_k{}^r H_{sr} = \nabla_j \alpha \eta_k - \alpha \phi_k{}^r H_{rj},$$

from which

$$\nabla_j H_{kr}\eta^r + k\phi_{jk} = \nabla_j \alpha \eta_k, \tag{3. 11}$$

because of (3. 10) and (2. 11). Making similar equation to (3. 11) by interchanging of the indices j and k, we get

$$(\nabla_j H_{kr} - \nabla_k H_{jr})\eta^r + 2k\phi_{jk} = \nabla_j \alpha \eta_k - \nabla_k \alpha \eta_j,$$

which implies, together with (3. 7), that

$$\nabla_j \alpha \eta_k = \nabla_k \alpha \eta_j,$$

from which

$$\nabla_j \alpha = \beta \eta_j, \qquad (\beta = \eta^r \nabla_r \alpha), \tag{3. 12}$$

and therefore

$$\nabla_k \nabla_j \alpha = \beta \nabla_k \eta_j + \nabla_k \beta \eta_j.$$

Since $\nabla_j \alpha$ is a gradient vector and η_j is a unit Killing vector, we get by contraction with $\nabla^k \eta^j$,

$$\beta \nabla_k \eta_j \nabla^k \eta^j = 0. \tag{3. 13}$$

The Riemannian metric being positive definite, from our assumption we have $\nabla_j \eta_i = 0$ or from (2. 6)

$$H_{jr}\phi_i{}^r = 0, \tag{3. 14}$$

which implies that $H_{ji} = \alpha \eta_j \eta_i$ because of (1. 4).

Differentiating the last equation covariantly and taking account of (3. 12), we get $\nabla_k H_{ji}=\beta\eta_i\eta_j\eta_k$. From which we have $k=0$ because of (3. 6). This means that M^{2n} is a locally Euclidean manifold.

4. **Principal curvatures of the hypersurface**. In this section we consider the principal curvatures of the hypersurface M^{2n-1} and give some fundamental formulas. In the following discussions we only consider the normal almost contact hypersurfaces in non-Euclidean Kaehlerian manifold of constant holomorphic sectional curvature, because we have already discussed the normal almost contact hypersurfaces in Euclidean space [5].

By means of Lemma 3. 1, the scalar function in (2. 10) being constant, we have the following identity for the second fundamental tensor.

$$H_{ir}H_j{}^r = cH_{ij} + k(g_{ji} - \eta_j\eta_i)\,, \tag{4. 1}$$

where $c=H_{ji}\eta^j\eta^i=$const.. From (2. 10) c is a characteristic root of the second fundamental tensor $H_j{}^i$ and η^i is a corresponding eigenvector to the root c.

Let λ be a characteristic root of the matrix $(H_j{}^i)$ which is distinct to c and v^i corresponding eigenvector to the root. Then transvecting (4. 1) with v^j and making use of the orthogonality of v^j and η^j, we have

$$(\lambda^2 - c\lambda - k)v_i = 0\,,$$

by virtue of $H_j{}^i v^j=\lambda v^i$. Thus, the principal curvatures of the hypersurface must satisfy the following algebraic equation of the third order,

$$(\lambda - c)(\lambda^2 - c\lambda - k) = 0\,. \tag{4. 2}$$

Furthermore, since k and c are both constants, the characteristic roots are all constants. Thus we have the

THEOREM 4. 1. *Let M^{2n-1} be a normal almost contact hypersurface in a Kaehlerian manifold of constant holomorphic sectional curvature. Then M^{2n-1} has at most three distinct principal curvatures and they are all constants.*

If the hypersurface M^{2n-1} admits only one principal curvature $\lambda=c$, then with respect to a suitable frame, the second fundamental tensor has the form $(H_j{}^i) = \begin{pmatrix} c & & 0 \\ & \ddots & \\ 0 & & c \end{pmatrix} = c\,(\delta_j{}^i)$. So, M^{2n-1} is totally umbilical. However, we have known that there is no umbilical hypersurface in a non-Euclidean Kaehlerian manifold of constant holomorphic sectional curvature[7]. Hence we deduce that the hypersurface admits two or three distinct principal curvatures.

LEMMA 4.2. *There exists no other vector than η^i which corresponds to the characteristic root c of the matrix $(H_j{}^i)$.*

PROOF. Let v^j be an eigenvector corresponding to the characteristic root c. Transvecting (4.1) with v^j and making use of $H_j{}^i v^j = cv^i$, we have

$$k(v_i - (\eta_j v^j)\eta_i) = 0,$$

which implies the lemma.

From Theorem 4.1 and Lemma 4.2, it follows that the second fundamental tensor $H_j{}^i$ and the Riemannian metric tensor g_{ji} have the components of the form

$$(H_j{}^i) = \begin{array}{c} \begin{array}{cc} \overbrace{\qquad\qquad\qquad}^{r+1} & \overbrace{\qquad\qquad\qquad}^{s} \end{array} \\ \left(\begin{array}{cccc|cccc} c & & & & & & & \\ & \lambda_1 & & 0 & & & & \\ & & \ddots & & & 0 & & \\ & 0 & & \lambda_1 & & & & \\ \hline & & & & \lambda_2 & & & \\ & & & & & \ddots & 0 & \\ & 0 & & & & & & \\ & & & & & 0 & & \lambda_2 \end{array}\right) \end{array}, \qquad (g_{ji}) = \begin{pmatrix} 1 & & & & \\ & 1 & & 0 & \\ & & \ddots & & \\ & 0 & & \ddots & \\ & & & & 1 \end{pmatrix},$$

with respect to a suitable orthonormal frame which will be called in the following an adapted frame, where λ_1, λ_2 are given by

$$\lambda_1 = \frac{1}{2}(c + \sqrt{c^2+4k}\,), \qquad \lambda_2 = \frac{1}{2}(c - \sqrt{c^2+4k}\,), \tag{4.3}$$

because of (4.2). Since the characteristic roots are constants and $r+s = 2n-2$, the multiplicities of the roots are also constants. From these facts, $H_r{}^r = \text{const}$. The trace of a matrix being invariant under the change of the frame, we have

7) Tashiro, Y. and S. Tachibana [13].

THEOREM 4.3. *The mean curvature of the normal almost contact hypersurface M^{2n-1} in a Kaehlerian manifold of constant holomorphic sectional curvature is a constant.*

Using (3.4) and (4.1), as a corollary of the theorem we get

COROLLARY 4.4. *Let M^{2n-1} be a normal almost contact hypersurface in a Kaehlerian manifold of constant holomorphic sectional curvature. Then the scalar curvature of M^{2n-1} with respect to the associated Riemannian metric is a constant.*

Suppose that the hypersurface M^{2n-1} admits two distinct principal curvatures c and λ. Then with respect to the adapted frame, the second fundamental tensor $H_j{}^i$ has the components

$$(H_j{}^i)=\begin{pmatrix} c & & & & & \\ & \lambda & & & 0 & \\ & & \cdot & & & \\ & & & \cdot & & \\ & 0 & & & \cdot & \\ & & & & & \lambda \end{pmatrix} \tag{4.4}$$

from which we get

$$(H_j{}^i)=\lambda\begin{pmatrix} 1 & & & & \\ & 1 & & 0 & \\ & & \cdot & & \\ & 0 & & \cdot & \\ & & & & 1 \end{pmatrix}+(c-\lambda)\begin{pmatrix} 1 & & & & \\ & 0 & & 0 & \\ & & \cdot & & \\ & 0 & & \cdot & \\ & & & & 0 \end{pmatrix},$$

that is

$$H_j{}^i=\lambda\delta_j^i+(c-\lambda)\eta^i\eta_j. \tag{4.5}$$

However, since (4.5) is a tensor equation, it does hold for any frame, especially for natural frame. If we substitute (4.5) into (2.6) we have $\phi_{ji}=\lambda\nabla_j\eta_i$. As η^i is a Killing vector, this means that the almost contact structure is a normal contact metric structure. Substituting (4.5) into Gauss equation (3.4), we have the curvature tensor of the hypersurface as follows:

$$\begin{aligned}R_{kjih}=&(k+\lambda^2)(g_{kh}g_{ji}-g_{jh}g_{ki})+k(\phi_{kh}\phi_{ji}-\phi_{jh}\phi_{ki}-2\phi_{kj}\phi_{ih})\\&+\lambda(c-\lambda)(g_{kh}\eta_j\eta_i-g_{jh}\eta_k\eta_i+g_{ji}\eta_k\eta_h-g_{ki}\eta_j\eta_h).\end{aligned} \tag{4.6}$$

An almost contact manifold which has the curvature tensor of the above form is called a locally C-Fubinian manifold[8]. Thus we have proved the

THEOREM 4.4. *If a normal almost contact hypersurface in a Kaehlerian manifold of constant holomorphic sectional curvature has only two distinct principal curvatures, then the almost contact structure is a normal contact metric structure and consequently the hypersurface M^{2n-1} is a locally C-Fubinian manifold.*

5. Hypersurfaces which admits three distinct principal curvatures. By means of Theorem 4.1, a normal almost contact hypersurface M^{2n-1} can admit at most three distinct principal curvatures. In this section we discuss the case that the hypersurface M^{2n-1} admits three distinct principal curvatures.

Let us denote by D_0, D_1 and D_2 the distributions spanned by the vectors corresponding to c, λ_1 and λ_2 respectively. Then the tangent bundle $T(M^{2n-1})$ satisfies

$$T(M^{2n-1}) = D_0 \oplus D_1 \oplus D_2 \qquad \text{(Whitney sum)},$$

over M^{2n-1}.

Let v^i be a vector belonging to D_1, that is v^i satisfies $H_j{}^i v^j = \lambda_1 v^i$. Then, owing to (2.11) we have

$$H_j{}^i \phi_k{}^j v^k = -\phi^{ij} H_{jk} v^k = -\lambda_1 \phi^{ik} v_k = \lambda_1 \phi_k{}^i v^k .$$

In exactly the same way, we get $H_j{}^i \phi_k{}^j w^k = \lambda_2 \phi_k{}^i w^k$ for any w^i belonging to D_2. This means that

$$\phi D_1 \subset D_1 , \qquad \phi D_2 \subset D_2 . \tag{5. 1}$$

Thus the following theorem is proved.

THEOREM 5.1. *The distributions D_1 and D_2 are both invariant under the mapping ϕ.*

Now, making use of the adapted frame we can easily see the following

LEMMA 5.2. *Let*

$$O_j{}^i = \eta_j \eta^i , \tag{5. 2}$$

8) Tashiro, Y. and S. Tachibana [13]. The definition of C-Fubinian manifold does not differ from that given in [13] except for a constant factor.

$$(5.3) \qquad P_j^{\ i} = \frac{1}{c-2\lambda_1}(-H_j^{\ i} + (c-\lambda_1)\delta_j^{\ i} + \lambda_1 \eta^i \eta_j),$$

$$(5.4) \qquad Q_j^{\ i} = \frac{1}{c-2\lambda_2}(-H_j^{\ i} + (c-\lambda_2)\delta_j^{\ i} + \lambda_2 \eta^i \eta_j),$$

then, at each point in M^{2n-1}, *the tensors* $O_j^{\ i}$, $P_j^{\ i}$ and $Q_j^{\ i}$ *are projections from* $T_p(M^{2n-1})$ *onto* $D_0(p)$, $D_1(p)$ *and* $D_2(p)$ *respectively.*

The tensors $O_j^{\ i}$, $P_j^{\ i}$ and $Q_j^{\ i}$ satisfy the following relations.

$$(5.5) \qquad O_j^{\ i} + P_j^{\ i} + Q_j^{\ i} = \delta_j^{\ i},$$

and

$$(5.6) \qquad O_j^{\ i} O_k^{\ j} = O_k^{\ i}, \qquad P_j^{\ i} P_k^{\ j} = P_k^{\ i}, \qquad Q_j^{\ i} Q_k^{\ j} = Q_k^{\ i},$$
$$O_j^{\ i} P_k^{\ j} = P_j^{\ i} Q_k^{\ j} = Q_j^{\ i} O_k^{\ j} = 0.$$

Next we shall prove the

THEOREM 5.3. *Let* D_0, D_1 and D_2 *be the distributions spanned by the vectors corresponding to the characteristic roots* c, λ_1 *and* λ_2 *of the second fundamental tensor* $(H_j^{\ i})$ *of the hypersurface respectively. Then distributions* $D_0 \oplus D_1$ *and* $D_0 \oplus D_2$ *are both integrable.*

PROOF. Since another case can be proved quite analogously, we shall only prove that $D_0 \oplus D_1$ is integrable. Denoting by u^i and v^i two arbitrary vectors belonging to D_1, we shall calculate $Q_j^{\ i}[u,v]^j$.
By definition of $Q_j^{\ i}$, it follows that

$$\begin{aligned} Q_j^{\ i}[u,v]^j &= \frac{1}{c-2\lambda_2}(-H_j^{\ i} + (c-\lambda_2)\delta_j^{\ i} + \lambda_2 \eta^i \eta_j)(u^r \nabla_r v^j - v^r \nabla_r u^j) \\ &= \frac{1}{c-2\lambda_2}\{v^r \nabla_r (H_j^{\ i} u^j) - u^r \nabla_r (H_j^{\ i} v^j) + (\nabla_r H_j^{\ i} - \nabla_j H_r^{\ i}) u^r v^j \\ &\qquad + (c-\lambda_2)(u^r \nabla_r v^i - v^r \nabla_r u^i) + 2\lambda_2 \eta^i u^j v^r \nabla_r \eta_j\} \end{aligned}$$

by virtue of (2.12) and $v^j \eta_j = u^j \eta_j = 0$. Making use of (3.5), this can be rewritten as

$$\begin{aligned} Q_j^{\ i}[u,v]^j = \frac{1}{c-2\lambda_2}\{&(c-\lambda_1-\lambda_2)(u^r \nabla_r v^i - v^r \nabla_r u^i) \\ &+ k(\eta_r \phi_j^{\ i} - \eta_j \phi_r^{\ i} - 2\phi_{rj}\eta^i) u^r v^j + 2\lambda_2 \eta^i \phi_j^{\ s} H_{rs} u^r v^j\}, \end{aligned}$$

where we have used the relations $H_j{}^i u^j = \lambda_1 u^i$ and $H_j{}^i v^j = \lambda_1 v^i$. Hence we get

$$Q_j{}^i [u, v]^j = \frac{1}{c-2\lambda_2}(-2k\phi_{rj}\eta^i u^r v^j + 2\lambda_1\lambda_2\phi_{js}\eta^i u^s v^j) = 0,$$

by virtue of (4. 3).

In exactly the same way, we can also prove that $Q_j{}^i[u, \eta]^j = 0$ and consequently, for two arbitrary vectors belonging to $D_0 \oplus D_1$, their bracket also belongs to $D_0 \oplus D_1$. Hence $D_0 \oplus D_1$ is integrable. This completes the proof.

6. Integral manifolds of $D_0 \oplus D_1$ and $D_0 \oplus D_2$. As we have seen in the previous section the distributions $D_0 \oplus D_1$ and $D_0 \oplus D_2$ are both integrable. Therefore through each point of the hypersurface there pass integral manifolds of $D_0 \oplus D_1$ and $D_0 \oplus D_2$. In the following we study almost contact strucatures of the integral manifolds of $D_0 \oplus D_1$ and $D_0 \oplus D_2$.

First of all notice that the mapping ϕ restricted to the vector space which spanned by vectors belonging to $D_1 \oplus D_2$ behaves just like an almost complex structure and that the distribution D_1 and D_2 are both invariant under ϕ. Hence the dimensions of D_1 and D_2 must be even[9]. From this fact we have

LEMMA 6. 1. *The integral submanifolds of both of the distributions $D_0 \oplus D_1$ and $D_0 \oplus D_2$ are odd dimensional.*

Denoting by r and s the dimensions of the distributions D_1 and D_2 respectively, we take r mutually orthonormal contravariant vectors $X_1{}^i, \cdots, X_r{}^i$ in D_1 and s mutually orthonormal contravariant vectors $Y_x{}^i$ $(x=1, 2, \cdots, s)$ in D_2. Moreover we put $X_{r+1}{}^i = \eta^i$. Then $2n-1$ vectors $X_a{}^i$ $(a=1, 2, \cdots, r, r+1)$ and $Y_x{}^i$ being linearly independent, we can construct the inverse of the matrix $(X_a{}^i, Y_x{}^i)$ which we denote by $(X^a{}_j, Y^x{}_j)$. Then we have the identities

$$(6.\ 1) \qquad X_b{}^i X^a{}_i = \delta_b{}^a, \qquad X_b{}^i Y^x{}_i = 0, \qquad Y_x{}^i X^a{}_i = 0, \qquad Y_x{}^i Y^y{}_i = \delta_x{}^y,$$

$$(6.\ 2) \qquad X_a{}^i X^a{}_j + Y_x{}^i Y^x{}_j = \delta_j{}^i,$$

from which we get $X^{r+1}{}_j = \eta_j$.

If we put $'\eta_a = X_a{}^i \eta_i$, $'\eta_a$ is a vector defined in the integral submanifold of $D_0 \oplus D_1$. The induced Riemannian metric of the submanifold is given by

9) Schouten, J. A. and K. Yano [11].

(6. 3) $$'g_{ba} = g_{ji} X_b{}^j X_a{}^i .$$

Making use of the metric $'g_{ba}$ and taking account of (6. 1), (6. 2), we have $'\eta^c = X^c{}_i \eta^i$ from which we can easily see that $'\eta_a$ is a unit vector.

Now, put

(6. 4) $$\Gamma_b{}^a{}_c = \left(X_b{}^j X_c{}^k \left\{ {i \atop jk} \right\} + X_b{}^j \partial_j X_c{}^i\right) X^a{}_i ,$$

then the covariant derivative of $'\eta_c$ along the integral submanifold of $D_0 \oplus D_1$ is given by

(6. 5) $$'\nabla_b {'\eta_c} = X_b{}^j \partial_j \eta_c - \Gamma_b{}^a{}_c \eta_a .$$ [10]

From this definition, we can easily see that

(6. 6) $$'\nabla_b {'\eta_a} = X_a{}^i X_b{}^j \nabla_j \eta_i ,$$

which implies that $'\eta_a$ is a Killing vector.

Now, we prove the

THEOREM 6. 2. *Let M^{2n-1} be a normal almost contact hypersurface in a Kaehlerian manifold of constant holomorphic sectional curvature and D_0, D_1 and D_2 be the distributions defined in the previous section. Then the integral submanifolds of $D_0 \oplus D_1$ and $D_0 \oplus D_2$ are both normal contact metric manifolds.*

PROOF. By virtue of (2. 6) and (6. 6) we have

(6. 7) $$'\nabla_b {'\eta_a} = -\phi_{ir} H_j{}^r X_a{}^i X_b{}^j .$$

For any vector belonging to $D_0 \oplus D_1$, the tensor $Q_j{}^i$ defined by (5. 4) behaves like a zero tensor and consequently it follows that

(6. 8) $$(c - 2\lambda_2) Q_j{}^i X_a{}^j = -H_j{}^i X_a{}^j + \lambda_1 X_a{}^i + \lambda_2 \eta^i \eta_j X_a{}^j = 0$$

by virtue of (4. 3). From (6. 7) and (6. 8) we get

(6. 9) $$'\nabla_b {'\eta_a} = -\lambda_1 \phi_{ij} X_a{}^i X_b{}^j .$$

Differentiating covariantly (6. 9) and taking account of Theorem 5. 1, we have

10) Yano, K. and E. T. Davies [15].

$$'\nabla_c{}'\nabla_b{}'\eta_a = -\lambda_1(\nabla_k\phi_{ij}X_a{}^iX_b{}^jX_c{}^k)$$

from which

$$'\nabla_c{}'\nabla_b{}'\eta_a = -\lambda_1(\eta_i H_{jk} - \eta_j H_{ik})X_a{}^iX_b{}^jX_c{}^k,$$

because of (2. 7). Therefore we have from (6. 8)

$$\begin{aligned}'\nabla_c{}'\nabla_b{}'\eta_a &= \lambda_1{}^2(\eta_j X_b{}^j g_{ki} X_c{}^k X_a{}^i - \eta_i X_a{}^i g_{kj} X_c{}^k X_b{}^j) \\ &= \lambda_1{}^2({}'\eta_b{}'g_{ca} - {}'\eta_a{}'g_{cb}).\end{aligned}$$

This implies that the vector $'\eta_a$ is a unit Killing vector satisfying (1. 9). Hence, the integral submanifold of the distribution $D_0 \oplus D_1$ has a normal contact metric structure by virtue of Theorem 1. 1. Entirely the same way we can also prove that the integral submanifold of the distribution $D_0 \oplus D_2$ admits a normal contact metric structure. This completes the proof.

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