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QUASI-TAUBERIAN THEOREMS, APPLIED TO THE SUMMABILITY OF FOURIER SERIES BY RIESZ'S LOGARITHMIC MEANS

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1. In [6] N. Wiener has introduced the quasi-Tauberian method to prove some problems concerning the summability of Fourier series and integrals by Cesàro sums, which had been proposed and solved partially by Hardy and Littlewood and completely by L. S. Bosanquet and R. Paley [1], [2]. He gave there some problems to which it is desirable to apply the quasi-Tauberian method. G. Sunouchi has given some applications to the summability of the conjugate or derived Fourier series etc. We show in this note that Wiener's method is also applicable to the summability of Fourier series by Riesz's logarithmic means, though F. T. Wang has solved this problem by another method.

Now let $f(t)$ be a summable and periodic function with period 2π, and let

$$f(t) \sim a_0/2 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt). \tag{1, 1}$$

The Fourier series (1, 1) is said to be summable (R, α), for $t = x$, to sum s, provided

$$R_\omega^\alpha = \frac{a_0}{2} + \frac{1}{(\log\omega)^\alpha} \sum_{n<\omega} \left(\log \frac{\omega}{n}\right)^\alpha (a_n \cos nx + b_n \sin nx)$$

tends to a limit s, when $\omega \to \infty$.

Let

$$\phi(u) = \frac{1}{2}(f(x+u) + f(x-u) - 2s).$$

We write

$$\phi(t) \to 0 \qquad (R, \alpha)$$

as $t \to 0$, provided

$$\psi_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_t^1 \left(\log \frac{u}{t}\right)^{\alpha-1} \frac{\phi(u)}{u} du = o\left(\left(\log \frac{1}{t}\right)^\alpha\right)$$

as $t \to 0$.

F. T. Wang has proved the following theorems.

THEOREM 1. *If* $\alpha > 0$ *and*

$$\psi_\alpha(t) = o\left(\left(\log \frac{1}{t}\right)^\alpha\right), \tag{1,2}$$

when $t \to 0$, *then the Fourier series* (1, 1) *is summable* $(R, \alpha+\delta)$ $(\delta > 0)$, *for* $t = x$, *to sum* s.

THEOREM 2. *If* $\alpha > 0$ *and the Fourier series* (1, 1) *is summable* (R, α), *for* $t = x$, *to sum* s, *then*

$$\psi_{\alpha+1+\delta}(t) = o\left(\left(\log \frac{1}{t}\right)^{\alpha+1+\delta}\right),$$

when $t \to 0$, *for every* $\delta > 0$.

2. We shall prove Theorem 1 and Theorem 2 by the quasi-Tauberian method. To prove the theorems we may assume clearly that $f(t)$ is even and $s = 0$, $x = 0$, $a_0 = 0$. Thus if $\alpha > 0$, we have

$$R_\omega^\alpha = \frac{2\omega}{\pi(\log \omega)^\alpha} \int_0^\infty L_\alpha(\omega t)\phi(t)dt$$

where

$$L_\alpha(t) = \int_0^1 \left(\log \frac{1}{u}\right)^\alpha \cos ut \, du.$$

By simple calculation one can see that if $\alpha \geqq 1$

$$R_\omega^\alpha = o(1) + \frac{2\omega}{\pi(\log \omega)^\alpha} \int_0^1 L_\alpha(\omega t)\phi(t)dt,$$

and that if $1 > \alpha > 0$

$$R_\omega^\alpha = O(1) + \frac{2\omega}{\pi(\log \omega)^\alpha}\int_0^1 L_\alpha(\omega t)\phi(t)dt$$

as $\omega \to \infty$.

One can see easily that

$$\lim_{\omega\to\infty} \frac{2\omega}{\pi(\log \omega)^\alpha}\int_0^1 L_\alpha(\omega t)\phi(t)dt = 0$$

implies

$$\lim_{\omega\to\infty} \frac{2\omega}{\pi(\log \omega)^\beta}\int_0^1 L_\beta(\omega t)\phi(t)dt = 0$$

for every $\beta > \alpha > 0$. (For instance, we can show it in the same way as in the proof of the following Theorem 3 quite easily.)

Now there is a lemma which was obtained by M. Riesz [3],

LEMMA 1. *If* $0 < \alpha < \beta$ *and if*

$$R_\omega^\alpha = O(1) \quad \textit{and} \quad R_\omega^\beta = o(1)$$

for $\omega \to \infty$, *then* $R_\omega^{\alpha+\delta} = o(1)$ *for* $\omega \to \infty$ *and for every* $\delta > 0$.

Combining this lemma with the above considerations, we have the following proposition,

PROPOSITION 1. *If* $\alpha > 0$, $\phi(t)$ *as in the Theorem* 1, *and if*

$$\lim_{\omega\to\infty} \frac{2\omega}{\pi(\log \omega)^\alpha}\int_0^1 L_\alpha(\omega t)\phi(t)dt = 0\,,$$

then we have

$$\lim_{\omega\to\infty} R_\omega^\beta = 0$$

for every $\beta > \alpha$.

This proposition reduces Theorem 1 and Theorem 2 to the following Theorem 3 and Theorem 4, respectively.

THEOREM 3. *Let* $\alpha > 0$ *and* $\phi(t) \in L^1(0, 1)$. *Then*

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon(\log 1/\varepsilon)^{\alpha}}\int_{\varepsilon}^{1}\left(\log\frac{x}{\varepsilon}\right)^{\alpha-1}\frac{\varepsilon}{x}\phi(x)dx=0$$

implies

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon(\log 1/\varepsilon)^{\beta}}\int_{0}^{1}\phi(x)dx\int_{0}^{1}\left(\log\frac{1}{z}\right)^{\beta}\cos\frac{xz}{\varepsilon}\,dz=0$$

for every $\beta>\alpha$.

THEOREM 4. *Let* $\beta>0$ *and* $\phi(x)\in L^{1}(0,1)$. *Then for every* $\alpha>\beta+1$

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon(\log 1/\varepsilon)^{\beta}}\int_{0}^{1}\phi(x)dx\int_{0}^{1}\left(\log\frac{1}{z}\right)^{\beta}\cos\frac{xz}{\varepsilon}\,dz=0$$

implies

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon(\log 1/\varepsilon)^{\alpha}}\int_{\varepsilon}^{1}\left(\log\frac{x}{\varepsilon}\right)^{\alpha-1}\frac{\varepsilon}{x}\phi(x)dx=0\,.$$

3. Proof of Theorem 3. Let

$$\begin{aligned}K_{1}^{\alpha}(x)&=(\log x)^{\alpha-1}x^{-1},\quad x>1\\&=0,\qquad\qquad 0<x\leqq 1,\end{aligned}$$

and

$$K_{2}^{\beta}(x)=\int_{0}^{1}\left(\log\frac{1}{z}\right)^{\beta}\cos xz\,dz\,.$$

Then their asymptotic properties are as follows;

$$K_{2}^{\beta}(x)=O(1)\qquad(\beta>-1)\,,\tag{3,1}$$

$$K_{2}^{\beta}(x)=O((\log x)^{\beta-1}x^{-1})\qquad(\beta>0,\ x\geqq 2)\,,\tag{3,2}$$

$$K_{1}^{\alpha}(x)=O(1)\qquad(\alpha\geqq 1),\tag{3,3}$$

$$\int_{0}^{\infty}|K_{1}^{\alpha}(x)|x^{-1}dx<\infty\qquad(\alpha>0)\,.\tag{3,4}$$

We shall first find the solution $R(x)$ of the following integral equation by

Mellin transform,

$$K_2^{\beta}(x) = \int_0^{\infty} R(y) K_1^{\alpha}\left(\frac{x}{y}\right)\frac{dy}{y}\,. \tag{3, 5}$$

The Mellin transforms of $K_1^{\alpha}(x)$ and $K_2^{\beta}(x)$ are given as follows;

$$k_1^{\alpha}(w) = \int_0^{\infty} x^{-w} K_1^{\alpha}(x)dx = \Gamma(\alpha) w^{-\alpha}\,,$$

$$k_2^{\beta}(w) = \int_0^{\infty} x^{-w} K_2^{\beta}(x)dx = -\Gamma(\beta+1) w^{-\beta}\Gamma(-w)\,\sin\frac{\pi}{2}w\,.$$

Let

$$r(w) = \frac{k_2^{\beta}(w)}{k_1^{\alpha}(w)} = -\frac{\Gamma(\beta+1)}{\Gamma(\alpha)} w^{\alpha-\beta}\Gamma(-w)\,\sin\frac{\pi}{2}w\,.$$

Then we have

$$r(w) = \int_0^{\infty} x^{-w} R(x) dx$$

where

$$R(x) = \frac{\Gamma(\beta+1)}{\Gamma(\alpha)\Gamma(\beta-\alpha+1)} \int_0^{\infty}\left(\log\frac{1}{z}\right)^{\beta-\alpha} \cos xz\, dz\,.$$

If $\alpha<\beta$, the unicity of Mellin trnsform shows that this function $R(x)$ satisfies (3, 5), because there exists $1>b>0$, such that $K_1^{\alpha}(x)x^{-b}$, $K_2^{\beta}(x)x^{-b}$, $R(x)x^{-b} \in L^1(0, \infty)$.

Considering the following Lemma 2, we have, for every $\phi(x) \in L^1(0,1)$,

$$\begin{aligned}\frac{1}{\varepsilon}\int_0^1 \phi(x) K_2^{\beta}\left(\frac{x}{\varepsilon}\right)dx &= \frac{1}{\varepsilon}\int_0^1 \phi(x)dx \int_0^{\infty} R\left(\frac{y}{\varepsilon}\right)K_1^{\alpha}\left(\frac{x}{y}\right)\frac{dy}{y} \\ &= \frac{1}{\varepsilon}\int_0^{\infty} R\left(\frac{y}{\varepsilon}\right)\frac{dy}{y}\int_0^1 K_1^{\alpha}\left(\frac{x}{y}\right)\phi(x)dx\,.\end{aligned} \tag{3, 7}$$

LEMMA 2. *Suppose that $N(x)$ and $R(x)$ satisfy the following four conditions,*

1) *there exist N, $\theta>0$ such that $|N(x)|<N$ for $0<x<\theta$,*
2) *for every $\delta>0$, there exists $M(\delta)>0$ such that $|R(x)|<M(\delta)$ for*

$0 < x < \delta$,

3) *there exists* $\eta > 0$ *such that* $\int_\eta^\infty |R(x)|x^{-1}dx < \infty$,

4) $\int_0^\infty |N(x)|x^{-1}dx < \infty$.

Then for every ε, $0 < \varepsilon < \frac{1}{\theta\eta}$, *we have*

$$\left|\int_0^\infty R\left(\frac{y}{\varepsilon}\right)N\left(\frac{x}{y}\right)\frac{dy}{y}\right| \leqq C_\varepsilon < \infty \quad (0 < x < 1).$$

In fact we have

$$\begin{aligned}\left|\int_0^\infty R\left(\frac{y}{\varepsilon}\right)N\left(\frac{x}{y}\right)\frac{dy}{y}\right| &\leqq M\left(\frac{1}{\varepsilon\theta}\right)\int_0^{1/\theta}\left|N\left(\frac{x}{y}\right)\right|\frac{dy}{y} + N\int_{1/\theta}^\infty\left|R\left(\frac{y}{\varepsilon}\right)\right|\frac{dy}{y} \\ &\leqq M\left(\frac{1}{\varepsilon\theta}\right)\int_{x\theta}^\infty |N(y)|\frac{dy}{y} + N\int_{1/\varepsilon\theta}^\infty |R(y)|\frac{dy}{y} \\ &< \infty .\end{aligned}$$

Now we shall finish the proof. By the assumption, for any $\eta > 0$, there exists $\delta > 0$ such that if $0 < y < \delta$

$$\left|\frac{1}{y(\log 1/y)^\alpha}\int_0^1 K_1^\alpha\left(\frac{x}{y}\right)\phi(x)dx\right| < \eta . \tag{3, 8}$$

We may assume $2\varepsilon < \delta$. Let

$$\int_0^\infty R\left(\frac{y}{\varepsilon}\right)\frac{dy}{y}\int_0^1 K_1^\alpha\left(\frac{x}{y}\right)\phi(x)dx = \int_0^{2\varepsilon} + \int_{2\varepsilon}^\delta + \int_\delta^\infty = I_1 + I_2 + I_3, \text{ say.}$$

Then, since $R(x)$ is bounded as seen in (3, 1) and

$$\int_0^\varepsilon \left(\log\frac{1}{y}\right)^\alpha dy = O\left(\varepsilon\left(\log\frac{1}{\varepsilon}\right)^\alpha\right)$$

as $\varepsilon \to 0$, we have

$$|I_1| \leqq C\eta \int_0^{2\varepsilon} \left(\log \frac{1}{y}\right)^{\alpha} dy = \eta \cdot O\left(\varepsilon\left(\log \frac{1}{\varepsilon}\right)^{\alpha}\right).$$

Since $R(x) = O((\log x)^{\beta-\alpha-1} x^{-1})$, $x \geqq 2$, we have

$$\begin{aligned} |I_2| &\leqq C\eta \int_{2\varepsilon}^{\delta} \frac{\varepsilon}{y}\left(\log \frac{y}{\varepsilon}\right)^{\beta-\alpha-1}\left(\log \frac{1}{y}\right)^{\alpha} dy \\ &\leqq C\eta\varepsilon\left(\log \frac{1}{2\varepsilon}\right)^{\alpha} \int_{2\varepsilon}^{\delta}\left(\log \frac{y}{\varepsilon}\right)^{\beta-\alpha-1} y^{-1} dy \\ &\leqq C\eta\varepsilon\left(\log \frac{1}{2\varepsilon}\right)^{\alpha} (\beta-\alpha)^{-1}\left[\left(\log \frac{\delta}{\varepsilon}\right)^{\beta-\alpha} - (\log 2)^{\beta-\alpha}\right] \\ &\leqq \eta \cdot O\left(\varepsilon\left(\log \frac{1}{\varepsilon}\right)^{\beta}\right). \end{aligned}$$

And since $K_1^{\alpha}\left(\frac{x}{y}\right) = 0$ for $y > 1$, we have

$$\begin{aligned} |I_3| &\leqq \int_{\delta}^{1} \left|R\left(\frac{y}{\varepsilon}\right)\right| \frac{dy}{y} \left|\int_0^1 K_1^{\alpha}\left(\frac{x}{y}\right)\phi(x)dx\right| \\ &\leqq C_1\varepsilon \int_{\delta}^{1}\left(\log \frac{y}{\varepsilon}\right)^{\beta-\alpha-1} y^{-2} dy \int_0^1 \left|K_1^{\alpha}\left(\frac{x}{y}\right)\right| |\phi(x)|\, dx \\ &\leqq C_1\varepsilon\left(\log \frac{1}{\varepsilon}\right)^{\beta-\alpha}\left(\log \frac{\delta}{\varepsilon}\right)^{-1} \int_{\delta}^{1} y^{-2} dy \int_0^1 \left|K_1^{\alpha}\left(\frac{x}{y}\right)\right| |\phi(x)|\, dx \\ &\leqq C\varepsilon\delta^{-1}\left(\log \frac{1}{\varepsilon}\right)^{\beta-\alpha-1} \int_0^1 |\phi(x)|\, dx \int_{\delta}^{1} \left|K_1^{\alpha}\left(\frac{x}{y}\right)\right| y^{-1} dy \\ &\leqq C\varepsilon\delta^{-1}\left(\log \frac{1}{\varepsilon}\right)^{\beta-\alpha-1} \int_0^1 |\phi(x)|\, dx \int_x^{x/\delta} |K_1^{\alpha}(y)|\, y^{-1} dy \\ &\leqq O\left(\varepsilon\left(\log \frac{1}{\varepsilon}\right)^{\beta-\alpha-1}\right). \end{aligned}$$

These estimations, combined with (3, 7), complete the proof.

4. Proof of Theorem 4. We may assume $2>\alpha-\beta>1$, because evidently

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon(\log 1/\varepsilon)^{\alpha_1}}\int_0^1 K_1^{\alpha_1}\left(\frac{x}{\varepsilon}\right)\phi(x)dx=0$$

implies

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon(\log 1/\varepsilon)^{\alpha_2}}\int_0^1 K_1^{\alpha_2}\left(\frac{x}{\varepsilon}\right)\phi(x)dx=0$$

if $\alpha_2\geqq\alpha_1$.

Now let

$$r(w)=\frac{k_1^{\alpha}(w)}{k_2^{\beta}(w)}=-\frac{2\Gamma(\alpha)}{\pi\Gamma(\beta+1)}w^{\beta-\alpha}\Gamma(1+w)\cos\frac{\pi}{2}w,$$

then we have

$$r(w)=\int_0^{\infty}x^{-w}R(x)dx \qquad (0<\operatorname{Re} w<1),$$

where

$$\begin{aligned}R(x)=-\frac{2\Gamma(\alpha)}{\pi\Gamma(\beta+1)}\Bigg\{&\frac{1}{\Gamma(\alpha-\beta-1)}\int_1^{\infty}\frac{(\log z)^{\alpha-\beta-2}}{z^2}\sin\frac{z}{x}\,dz\\&-\frac{1}{\Gamma(\alpha-\beta-2)}\int_1^{\infty}(\log z)^{\alpha-\beta-3}\frac{z-1}{z^3}\sin\frac{z}{x}\,dz\\&-\frac{2}{\Gamma(\alpha-\beta-1)}\int_1^{\infty}\frac{(\log z)^{\alpha-\beta-2}}{z^3}\sin\frac{z}{x}\,dz\\&+\frac{1}{\Gamma(\alpha-\beta-1)}\frac{1}{x}\int_1^{\infty}\frac{(\log z)^{\alpha-\beta-2}}{z^2}\cos\frac{z}{x}\,dz\Bigg\}\\=-\frac{2\Gamma(\alpha)}{\pi\Gamma(\beta+1)}\Big\{&R_1(x)-R_2(x)-R_3(x)+R_4(x)\Big\},\ \text{say.}\end{aligned}$$

We first write down the properties of these functions, which are obtained by simple calculation, considering the periodicity of trigonometric functions.

(4, 1) Let $R_\delta(x) = \int_1^\infty (\log z)^\delta z^{-2} \sin \frac{z}{x} dz$, then we have

1) $R_\delta(x) = O(1)$, $\delta > -1$,

2) $R_\delta(x) = O((\log x)^{1+\delta} x^{-1})$, $x \geqq 2, \delta > -1$,

3) $R_\delta(x) = O(x)$, $x \to 0$, $\delta \geqq 0$,

$= O(x^{1+\delta})$, $x \to 0$, $0 > \delta > -1$.

(4, 2) Let $Q_\delta(x) = \int_1^\infty (\log z)^\delta z^{-3} \sin \frac{z}{x} dz$, then we have the same estimationsas for $R_\delta(x)$.

(4, 3) Let $P_\delta(x) = \int_1^\infty (\log z)^\delta z^{-3}(z-1) \sin \frac{z}{x} dz$, then we have

1) $P_\delta(x) = O((\log x)^{1+\delta} x^{-1})$, $x \geqq 2$, $\delta > -2$,

2) $P_\delta(x) = O(x^{2+\delta})$, $x \to 0$, $-1 \geqq \delta > -2$,

$= O(x^{1+\delta})$, $x \to 0$, $0 \geqq \delta > -1$,

$= O(x)$, $x \to 0$, $\delta > 0$.

(4, 4) Let $S_\delta(x) = x^{-1} \int_1^\infty (\log z)^\delta z^{-2} \cos \frac{z}{x} dz$, then we have

1) $S_\delta(x) = O(x^{-1})$, $x \geqq 2$, $\delta > -1$,

2) $S_\delta(x) = O(1)$, $x \to 0$, $\delta \geqq 0$,

3) $S_\delta(x) = O(x^\delta)$, $x \to 0$, $0 > \delta > -1$.

We give here only the proof of (4, 4), 3). The proofs of another propositions are obtained in a similar way. Since $(\log z)^\delta z^{-2}$ decreases monotonously as $z \to \infty$ and $\cos \frac{z}{x}$ is a periodic function with period $2\pi x$, we have

$$|S_\delta(x)| \leqq x^{-1} \int_1^{1+2\pi x} (\log z)^\delta z^{-2} dz,$$

which shows (4, 4), 3).

Now, if we set $b=\dfrac{\alpha-\beta-1}{2}$, then considering (4, 1), (4, 2), (4, 3) and (4, 4) we have

$$K_1^{\alpha}(x)x^{-b},\ K_2^{\beta}(x)x^{-b},\ R(x)x^{-b}\in L^1(0,\infty)\,.$$

Hence K_1^{α}, K_2^{β} and $R(x)$ satisfy the following integral equation,

$$K_1^{\alpha}(x)=\int_0^{\infty}R(y)K_2^{\beta}\left(\frac{x}{y}\right)\frac{dy}{y}\,. \tag{4, 5}$$

By (4, 1), (4, 2) and (4, 3) we have

$$R_1(x)x^{-1},\ R_2(x)x^{-1},\ R_3(x)x^{-1}\in L^1(0,\infty)\,.$$

Hence we have

$$\begin{aligned}&\int_0^{\infty}\left(R\left(\frac{y}{\varepsilon}\right)-R_4\left(\frac{y}{\varepsilon}\right)\right)\frac{dy}{y}\int_0^1 K_2^{\beta}\left(\frac{x}{y}\right)\phi(x)dx\\&=\int_0^1\phi(x)dx\int_0^{\infty}\left(R\left(\frac{y}{\varepsilon}\right)-R_4\left(\frac{y}{\varepsilon}\right)\right)K_2^{\beta}\left(\frac{x}{y}\right)\frac{dy}{y}\,.\end{aligned} \tag{4, 6}$$

Next we shall show that (4, 6) remains valid if $R(x)-R_4(x)$ is replaced by $R_4(x)$. Let $\delta>0$ and $k=\alpha-\beta-2$. Let

$$\begin{aligned}I_\delta(x)&=\int_\delta^{\infty}R_4\left(\frac{y}{\varepsilon}\right)K_2^{\beta}\left(\frac{x}{y}\right)\frac{dy}{y}\\&=\varepsilon\int_\delta^{\infty}\frac{dy}{y^2}\int_1^{\infty}\frac{(\log t)^k}{t^2}\cos\frac{\varepsilon t}{y}\,dt\int_0^1\left(\log\frac{1}{z}\right)^{\beta}\cos\frac{xz}{y}\,dz\\&=\varepsilon\int_1^{\infty}\int_0^1\frac{(\log t)^k}{t^2}\left(\log\frac{1}{z}\right)^{\beta}dt\,dz\int_\delta^{\infty}\frac{\cos\varepsilon t/y\cos xz/y}{y^2}\,dy\,.\end{aligned}$$

Then $I_\delta(x)$ is bounded for $0<\delta<\infty$, $0\leqq x\leqq 1$. In fact we have

$$J_\delta(x,t,z)=\int_\delta^{\infty}\frac{\cos\varepsilon t/y\cos xz/y}{y^2}\,dy=\int_0^{1/\delta}\cos\varepsilon ty\cos xzy\,dy$$

$$= \frac{1}{2}\int_0^{1/\delta} \{\cos(\varepsilon t + xz)y + \cos(\varepsilon t - xz)y\}\,dy$$

and thus we have $|J_\delta(x, t, z)| \leqq 1$ if $\varepsilon t \notin \{\pm xz\}$. This shows that $I_\delta(x)$ is bounded for $0<\delta<\infty$ and $0 \leqq x \leqq 1$, because $(\log t)^k\ t^{-2}\left(\log \frac{1}{z}\right)^\beta$ is summable in $(1, \infty)\times(0, 1)$. Now applying Lebesgue's dominated convergence theorem and Fubini's theorem, we have

$$(4, 7)\quad \int_0^\infty R_4\left(\frac{y}{\varepsilon}\right)\frac{dy}{y}\int_0^1 K_2^\beta\left(\frac{x}{y}\right)\phi(x)dx$$

$$= \lim_{\delta\to 0}\int_\delta^\infty R_4\left(\frac{y}{\varepsilon}\right)\frac{dy}{y}\int_0^1 K_2^\beta\left(\frac{x}{y}\right)\phi(x)dx$$

$$= \lim_{\delta\to 0}\int_0^1 \phi(x)dx\int_\delta^\infty R_4\left(\frac{y}{\varepsilon}\right)K_2^\beta\left(\frac{x}{y}\right)\frac{dy}{y} = \int_0^1 \phi(x)dx \int_0^\infty R_4\left(\frac{y}{\varepsilon}\right)K_2^\beta\left(\frac{x}{y}\right)\frac{dy}{y},$$

which, combined with (4, 5) and (4, 6), shows that

$$(4, 8)\qquad \int_0^\infty R\left(\frac{y}{\varepsilon}\right)\frac{dy}{y}\int_0^1 K_2^\beta\left(\frac{x}{y}\right)\phi(x)dx$$

$$= \int_0^1 K_1^\alpha\left(\frac{x}{\varepsilon}\right)\phi(x)dx.$$

We finish now the proof. First note that

$$R(x) = O((\log x)^{\alpha-\beta-1}x^{-1}),\quad x \geqq 2,$$

$$= O(x^{\alpha-\beta-2}),\qquad x\to 0.$$

By the assumption, for every $\eta>0$, there exists $\delta>0$ such that

$$\left|\frac{1}{y(\log 1/y)^\beta}\int_0^1 K_2^\beta\left(\frac{x}{y}\right)\phi(x)dx\right| < \eta,\ (0<y<\delta).$$

Let

$$\int_0^\infty R\left(\frac{y}{\varepsilon}\right)\frac{dy}{y}\int_0^1 K_2^\beta\left(\frac{x}{y}\right)\phi(x)dx = \int_0^{2\varepsilon} + \int_{2\varepsilon}^\delta + \int_\delta^\infty = I_1+I_2+I_3,\quad \text{say.}$$

Then we have

$$|I_1| \leqq \left|\int_0^{2\varepsilon} R\left(\frac{y}{\varepsilon}\right)\left(\log\frac{1}{y}\right)^{\beta} dy\right|\eta \leqq C\eta\int_0^{2\varepsilon}\left(\frac{y}{\varepsilon}\right)^{\alpha-\beta-2}\left(\log\frac{1}{y}\right)^{\beta} dy$$

$$\leqq C\eta\varepsilon^{2-\alpha+\beta}O\left(\varepsilon^{\alpha-\beta-1}\left(\log\frac{1}{\varepsilon}\right)^{\beta}\right)$$

$$\leqq \eta\cdot O\left(\varepsilon\left(\log\frac{1}{\varepsilon}\right)^{\beta}\right),$$

because

$$\int_0^{\varepsilon} y^{\alpha}\left(\log\frac{1}{y}\right)^{\beta} dy = O\left(\varepsilon^{\alpha+1}\left(\log\frac{1}{\varepsilon}\right)^{\beta}\right), \quad \varepsilon\to 0, \ (\alpha>-1, \beta>-1).$$

For I_2 we have

$$|I_2| \leqq \eta\left|\int_{2\varepsilon}^{\delta} R\left(\frac{y}{\varepsilon}\right)\left(\log\frac{1}{y}\right)^{\beta} dy\right| \leqq C\eta\varepsilon\int_{2\varepsilon}^{\delta}\left(\log\frac{y}{\varepsilon}\right)^{\alpha-\beta-1} y^{-1}\left(\log\frac{1}{y}\right)^{\beta} dy$$

$$\leqq C\eta\varepsilon\left(\log\frac{1}{2\varepsilon}\right)^{\beta}\int_{2\varepsilon}^{\delta}\left(\log\frac{y}{\varepsilon}\right)^{\alpha-\beta-1} y^{-1}dy$$

$$\leqq C\eta\varepsilon\left(\log\frac{1}{2\varepsilon}\right)^{\beta}\left\{\left(\log\frac{\delta}{\varepsilon}\right)^{\alpha-\beta} - (\log 2)^{\alpha-\beta}\right\}(\alpha-\beta)^{-1}$$

$$\leqq \eta\cdot O\left(\varepsilon\left(\log\frac{1}{\varepsilon}\right)^{\alpha}\right).$$

For I_3, since $K_2^{\beta}(x)$ is bounded and $\phi(x)$ is summable, we have

$$|I_3| \leqq C\varepsilon\int_{\delta}^{\infty}\left(\log\frac{y}{\varepsilon}\right)^{\alpha-\beta-1} y^{-2}dy$$

$$= C\int_0^{\varepsilon/\delta}\left(\log\frac{1}{t}\right)^{\alpha-\beta-1} dt = O\left(\frac{\varepsilon}{\delta}\left(\log\frac{\delta}{\varepsilon}\right)^{\alpha-\beta-1}\right).$$

These estimations, combined with (4, 8), show that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon(\log 1/\varepsilon)^{\alpha}}\int_0^1 K_1^{\alpha}\left(\frac{x}{\varepsilon}\right)\phi(x)dx = 0,$$

which is the desired conclusion.

REMARK. In the case $\alpha=\beta$ in Theorem 3, $R(x)$ is given by $\frac{\sin x}{x}$. Thus in a similar way to the proof of Theorem 3, we can show that

$$\int_\varepsilon^1 \left(\log\frac{x}{\varepsilon}\right)^{\alpha-1} x^{-1}\phi(x)dx = O\left(\left(\log\frac{1}{\varepsilon}\right)^{\alpha}\right)$$

implies

$$\frac{1}{\varepsilon}\int_0^1 \phi(x)dx \int_0^1 \left(\log\frac{1}{z}\right)^{\alpha}\cos\frac{xz}{\varepsilon}\,dz = O\left(\left(\log\frac{1}{\varepsilon}\right)^{\alpha+1}\right).$$

In the case $\alpha=\beta+1$ in Theorem 4, $R(x)$ is given by $\frac{1}{x}\cos\frac{1}{x}$. By calculation we can check that (4, 5) and (4, 8) remain valid. However, we can not obtain by our method the same result as in the above remark. We have not succeeded in estimating the following integral,

$$\int_0^{2\varepsilon}\frac{\cos\varepsilon/y}{y^2}\,dy\int_0^1 K_2^\beta\left(\frac{x}{y}\right)\phi(x)dx\,.$$

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