Tôhoku Math. Journ.
22(1970), 109-116.

SOME REMARKS ON WIENER'S QUASI-TAUBERIAN THEOREMS AND THE SUMMABILITY OF FOURIER SERIES

KÔZÔ YABUTA

(Received September 11, 1969)

In [5] N. Wiener gave the so-called quasi-tauberian theorems and their applications to the problem on the summability of Fourier series and integrals by Cesàro sum of some order, which was discussed completely by L. Bosanquet [1] and R. Paley [3]. But we have found that Wiener's discussion on kernels vanishing for positive arguments is not correct, and that his Theorem 22′ is not stated correctly. In his Theorem 22′ the phrase $-\varepsilon<\mathrm{Re}\,u<\lambda+\varepsilon$ must be replaced by $-\lambda-\varepsilon<\mathrm{Re}\,u<\varepsilon$. The result of Bosanquet-Paley reads as follows;

THEOREM. *Let $f(x)$ be an integrable function of period 2π and let*

$$\phi(x)=\frac{1}{2}(f(x+y)+f(y-x)-2s).$$

Then if we write B_m for the proposition

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^\varepsilon\left(1-\frac{x}{\varepsilon}\right)^{m-1}\phi(x)dx=0$$

and C_m for the proposition

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^1\phi(x)dx\int_0^1(1-z)^m\cos\frac{xz}{\varepsilon}dz=0,$$

B_m implies $C_{m+\varepsilon}$ for $m\geqq 0$, while C_m implies $B_{m+1+\varepsilon}$ for $m\geqq 0$ and any $\varepsilon>0$.

If we apply Theorem 22′ of Wiener [5], we can obtain only that B_m implies $C_{m+\varepsilon}$ when $m\geqq 1$ and that C_m implies $B_{m+2+\varepsilon}$ when $m\geqq 0$. But applying Levinson's variant of Theorem 22′ of Wiener [2] and another quasi-tauberian theorem for some unbounded kernel, we can reprove the Bosanquet-Paley's result completely by Wiener's method. By his method we have also reproved F.T.Wang's results on the summability of Fourier series by Riesz's logarithmic means [4].

To prove Theorem we need two quasi-tauberian theorems.

LEMMA 1. (Wiener-Levinson) *Let*

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^1 N_1\left(\frac{x}{\varepsilon}\right)\phi(x)dx=0 \tag{0}$$

where

$$\int_0^1 |\phi(x)|\,dx<\infty \tag{1}$$

and

$$|N_1(x)|<A<\infty\,. \tag{2}$$

If $R(x)$ is a function such that

$$\int_0^\infty |R(x)|\,dx<\infty \tag{3}$$

and

$$\int_0^\infty \frac{|R(x)|}{x}\,dx<\infty\,, \tag{4}$$

and if

$$N_2(x)=\int_0^\infty R(y)\,N_1\left(\frac{x}{y}\right)\frac{dy}{y}\,, \tag{5}$$

then we have

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^1 N_2\left(\frac{x}{\varepsilon}\right)\phi(x)dx=0\,. \tag{6}$$

LEMMA 2. *Lemma* 1 *remains valid if* (2) *and* (3) *are replaced by*

$$\int_0^\infty |N_1(x)|\,dx<\infty\,, \tag{7}$$

(8) *there exist* $K>0$ *and* $A>0$ *such that*

$$|N_1(x)|<A \quad for \quad 0<x<K\,,$$

$$|R(x)|<B<\infty \tag{9}$$

and

(10) $$R(x)=O(1/x^{1+\alpha}) \text{ as } x\to\infty \text{ for some } \alpha>0$$

i.e., there exists $C>0$ *such that* $|R(x)|<C/x^{1+\alpha}$.

PROOF OF LEMMA 2. By simple calculation, the following repeated integral is absolutely convergent, so we can interchange the order of integration, giving

(11) $$\frac{1}{\varepsilon}\int_0^\infty R\left(\frac{y}{\varepsilon}\right)\frac{dy}{y}\int_0^1 N_1\left(\frac{x}{y}\right)\phi(x)dx$$

$$=\frac{1}{\varepsilon}\int_0^1 \phi(x)dx\int_0^\infty R\left(\frac{y}{\varepsilon}\right)N_1\left(\frac{x}{y}\right)\frac{dy}{y}=\frac{1}{\varepsilon}\int_0^1 \phi(x)N_2\left(\frac{x}{\varepsilon}\right)dx.$$

Since

$$\left|\int_\delta^\infty R\left(\frac{y}{\varepsilon}\right)N_1\left(\frac{x}{y}\right)\frac{dy}{y}\right|<C\varepsilon^{1+\alpha}/\delta^\alpha x\int_0^\infty |N_1(t)|dt$$

for every $\delta>0$, by (7), (8), (9), (10), it follows that

$$\left|\frac{1}{\varepsilon}\int_0^\infty R\left(\frac{y}{\varepsilon}\right)\frac{dy}{y}\int_0^1 N_1\left(\frac{x}{y}\right)\phi(x)dx\right|$$

$$\leqq\left|\frac{1}{\varepsilon}\int_0^\delta R\left(\frac{y}{\varepsilon}\right)dy\left\{\frac{1}{y}\int_0^1 N_1\left(\frac{x}{y}\right)\phi(x)dx\right\}\right|$$

$$+\left|\frac{1}{\varepsilon}\int_{K\delta}^1 \phi(x)dx\int_\delta^\infty R\left(\frac{y}{\varepsilon}\right)N_1\left(\frac{x}{y}\right)\frac{dy}{y}\right|$$

$$+\left|\frac{1}{\varepsilon}\int_0^{K\delta} \phi(x)dx\int_\delta^\infty R\left(\frac{y}{\varepsilon}\right)N_1\left(\frac{x}{y}\right)\frac{dy}{y}\right|$$

$$\leqq\left|\frac{1}{\varepsilon}\int_0^\delta R\left(\frac{y}{\varepsilon}\right)dy\left\{\frac{1}{y}\int_0^1 N_1\left(\frac{x}{y}\right)\phi(x)dx\right\}\right|$$

$$+\frac{C\varepsilon^\alpha A}{\delta^{1+\alpha}}\int_0^{K\delta}|\phi(x)|dx+\frac{C\varepsilon^\alpha}{\delta^\alpha}\int_{K\delta}^1\frac{|\phi(x)|}{x}dx\int_0^\infty |N_1(x)|dx.$$

Since for sufficiently small δ the first term on the right is arbitrarily small independently of ε, and any fixed δ, the second term and the third are arbitrarily small, we have the conclusion considering (11).

Now we shall first prove that B_m implies $C_{m+\varepsilon}$ for $m \geqq 0$. It is clearly sufficient to prove for $m>0$, because B_n implies $B_{n+\beta}$ for every $\beta>0$ and $n \geqq 0$. Let $n>m>0$ and

(12)
$$N_1(x) = \frac{(1-x)^{m-1}}{\Gamma(m)} \qquad 0<x<1 \qquad (m>0)$$
$$= 0 \qquad\qquad 1 \leqq x$$

and

(13)
$$N_2(x) = \frac{2}{\pi\Gamma(n+1)} \int_0^1 (1-z)^n \cos zx\, dz \qquad (n \geqq 0).$$

Their Mellin transforms are given by

(14)
$$n_1(w) = \int_0^\infty x^{-w} N_1(x) dx = \frac{\Gamma(1-w)}{\Gamma(m+1-w)}$$

(15)
$$n_2(w) = \int_0^\infty x^{-w} N_2(x) dx = \frac{1}{\Gamma(n+1+w)\cos \pi/2w}.$$

Set $r(w) = \dfrac{n_2(w)}{n_1(w)}$. Then $r(w)$ is holomorphic in the half plane Re $w<m+1$. And by Stirling's formula

(16)
$$|r(w)| \sim \frac{2}{\sqrt{2\pi}} |\mathrm{Im}\, w|^{-\mathrm{Re}w+m-n-1/2}$$

as Im $w \to \pm\infty$, uniformly in $-n-1<\mathrm{Re}\ w<m+1$. Hence $r(w)$ is uniformly L^2 in $-\dfrac{n-m}{2} \leqq \mathrm{Re}\ w \leqq 1+\dfrac{m}{2}$ and via Paley-Wiener theorem we have

$$\int_0^\infty |R(x)|\, dx < \infty \quad \text{and} \quad \int_0^\infty \frac{|R(x)|}{x} dx < \infty$$

where $R(x) = \underset{A\to\infty}{\mathrm{l.\,i.\,m}} \dfrac{1}{2\pi i} \displaystyle\int_{-iA}^{iA} r(w) x^{w-1} dw$. This function $R(x)$ satisfies the integral equation (5). Thus, since $N_1(x)$ is bounded for $m \geqq 1$, Lemma 1 shows that B_m implies $C_{m+\varepsilon}$ in this case. For the case $1>m>0$ we apply Lemma 2. We see easily that $N_1(x)$ satisfies (7), (8) and that $R(x)$ satisfies (4). Hence we have only to show that $R(x)$ is bounded and $O\left(\dfrac{1}{x^{1+\alpha}}\right)$ for some $\alpha>0$. The boundedness follows immediately from the fact $r(1+iv)\in L^1(-\infty,+\infty)$, easily seen by (16). To see the order of growth, we need the following lemma which follows from Stirling's formula.

LEMMA 3. *Let a, b, c be three numbers such that $a\leqq c$ and $-a\leqq b$. Then*

$$\frac{\Gamma(b-w)}{\Gamma(w+a)}-(-1)^{a-c}\frac{\Gamma(b-a+c-w)}{\Gamma(w+c)}=\frac{\Gamma(b-w)}{\Gamma(w+a)}\,O(1/w)$$

and

$$\frac{\Gamma(w+a)}{\Gamma(b-w)}-(-1)^{a-c}\frac{\Gamma(w+c)}{\Gamma(b-a+c-w)}=\frac{\Gamma(w+a)}{\Gamma(b-w)}\,O(1/w)$$

as $\operatorname{Im} w\to\pm\infty$, *uniformly in* $-a\leqq \operatorname{Re} w\leqq b$.

Set

$$r_1(w)=(-1)^m\frac{1}{\Gamma(n-m+1+w)\cos \pi/2w}=\int_0^\infty x^{-w}R_1(x)dx \tag{17}$$

where

$$R_1(x)=\frac{(-1)^m 2}{\pi\Gamma(n-m+1)}\int_0^1 (1-z)^{n-m}\cos zx\ dz.$$

Applying Lemma 3 to $r(w)$ and $r_1(w)$, we have

$$r(w)-r_1(w)=O(|\operatorname{Im} w|^{-\operatorname{Re} w+m-n-3/2})\,. \tag{18}$$

Hence $r(w)-r_1(w)$ is uniformly L^2 and L^1 in $-\frac{1}{2}\leqq \operatorname{Re} w\leqq \frac{1}{2}$. Since $r(w)-r_1(w)$ is holomorphic in $-1<\operatorname{Re} w<1$, there exists $C>0$ such that

$$|R(x)-R_1(x)|\leqslant Cx^{-3/2}\,. \tag{19}$$

By simple calculation we have

$$R_1(x)=O(x^{-1-\beta_1})\,, \tag{20}$$

where $\beta_1=\inf\,(1, n-m)$,

which, combined with (19), shows that

$$R(x) = O(1/x^{1+\beta_2}),$$

where $\beta_2 = \inf\left(\beta_1, \frac{1}{2}\right)$.

We have thus proved the first half of Theorem.

Next we shall show that C_m implies $B_{m+1+\varepsilon}$ for $m \geqq 0$. In this case, if we have (11) interchanging the rolls of $N_1(x)$ and $N_2(x)$, the proof goes in the same way as in Lemma 1 (Levinson [2], p. 140), because $N_2(x)$ is bounded for $n \geqq 0$. Let $m-1 > n \geqq 0$ and

$$r(w) = \frac{n_1(w)}{n_2(w)} = \frac{\Gamma(1-w)\Gamma(n+1+w)\cos \pi/2w}{\Gamma(m+1-w)}$$

and

$$r_1(w) = \frac{(-1)^{n+1}\Gamma(1-w)\Gamma(1+w)\cos \pi/2w}{\Gamma(m-n+1-w)}$$

$$= \int_0^\infty x^{-w} R_1(x)dx$$

where

$$R_1(x) = \frac{(-1)^{n+1}}{\Gamma(m-n)}\left\{1/x\int_0^1 (1-z)^{m-n-1}\cos\frac{z}{x}dz\right.$$

$$\left.-\frac{m-n-1}{x}\int_0^1 z(1-z)^{m-n-2}\cos\frac{z}{x}dz\right\}$$

$$= \frac{(-1)^{n+1}}{\Gamma(m-n)}\{R_2(x) - R_3(x)\}.$$

We shall show that we have (11) if we replace $R(x)$ by $R(x)-R_1(x)$ or $R_1(x)$ and thus (11) follows for $R(x)$. By Lemma 3, we have

$$r(w) - r_1(w) = O(|\mathrm{Im}\ w|^{\mathrm{Re}\ w+n-m-1/2}).$$

Since $r(w)-r_1(w)$ is holomorphic in $-1<\mathrm{Re}\ w<2$, via Paley-Wiener theorem we have

$$R(x)-R_1(x),\ (R(x)-R_1(x))/x \in L^1(0, \infty).$$

In the same way we have

$$R(x), R_1(x) \in L^1(0, \infty).$$

Since $N_2(x)$ is bounded, we have (11) for $R(x)-R_1(x)$.

For $R_2(x)$ we have by simple calculation

$$\begin{aligned} R_2(x) &= O(x^{-1}) \qquad (x \to \infty) \\ &= O(x^{\beta}) \qquad (x \to 0), \end{aligned}$$

where $\beta = \inf(1, m-n-1)$. Thus $R_2(x)/x \in L^1(0, \infty)$, which shows that (11) remains valid for $R_2(x)$.

Now consider $R_3(x)$. Let δ be an arbitrarily fixed positive number and $k = m-n-1$. Let

$$\begin{aligned} I_\delta(x) &= \int_\delta^\infty R_3\left(\frac{y}{\varepsilon}\right) N_2\left(\frac{x}{y}\right)\frac{dy}{y} \\ &= \int_\delta^\infty \frac{dy}{y}\left(\int_0^1 \frac{\varepsilon k z}{y}(1-z)^{k-1}\cos\frac{\varepsilon z}{y}\,dz\right)\left(\int_0^1 (1-t)^n \cos\frac{xt}{y}\,dt\right) \\ &= \varepsilon k\int_0^1\int_0^1 z(1-z)^{k-1}(1-t)^n dz dt \int_\delta^\infty \frac{\cos \varepsilon z/y \cos xt/y}{y^2}\,dy. \end{aligned}$$

The last term is equal to $\int_0^{1/\delta} \cos \varepsilon z y \cos x t y\, dy = \frac{1}{2}\int_0^{1/\delta} \{\cos(\varepsilon z + xt)y + \cos(\varepsilon z - xt)y\}\, dy$. Clearly it is less than or equal to 1 unless $\varepsilon z = \pm xt$. Hence $I_\delta(x)$ is finite with respect to $0 < \delta < \infty$ and $0 \leqq x \leqq 1$. Hence applying Lebesgue's dominated convergence theorem and Fubini's theorem, we have

$$\begin{aligned} &\int_0^\infty R_3\left(\frac{y}{\varepsilon}\right)\frac{dy}{y}\int_0^1 N_2\left(\frac{x}{y}\right)\phi(x)dx \\ &= \lim_{\delta\to 0}\int_\delta^\infty R_3\left(\frac{y}{\varepsilon}\right)\frac{dy}{y}\int_0^1 N_2\left(\frac{x}{y}\right)\phi(x)dx \\ &= \lim_{\delta\to 0}\int_0^1 \phi(x)dx\int_\delta^\infty R_3\left(\frac{y}{\varepsilon}\right)N_2\left(\frac{x}{y}\right)\frac{dy}{y} \\ &= \int_0^1 \phi(x)dx\int_0^\infty R_3\left(\frac{y}{\varepsilon}\right)N_2\left(\frac{x}{y}\right)\frac{dy}{y}\,, \end{aligned}$$

which shows that (11) holds for $R_3(x)$. Thus we have shown that (11) holds for $R(x)$, which completes the proof.

REFERENCES

[1] L. S. BOSANQUET, On the summability of Fourier series, Proc. London Math. Soc., (2)31 (1930), 144–164.
[2] N. LEVINSON, On the Poisson summability of Fourier series, Duke Math. J., 2(1936), 138–146.
[3] R. E. A. C. PALEY, On the Cesàro summability of Fourier series and allied series, Proc. Cambridge Philos. Soc., 26(1930), 173–203.
[4] F. T. WANG, On the summability of Fourier series by Riesz's logarithmic means, II, Tôhoku Math. J., 40(1935), 274–292.
[5] N. WIENER, Tauberian theorems, Ann. of Math., 32(1931), 1–100.

MATHEMATICAL INSTITUTE
TÔHOKU UNIVERSITY
SENDAI, JAPAN