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TRANSLATION INVARIANT OPERATORS ON GROUPS

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1. Notations. Let G be an arbitrary, but fixed, locally compact abelian group, and Γ its dual. The Fourier transform of a function f in $L^1(G)$ is defined by

$$\hat{f}(\gamma) = \int_G f(x) < x, -\gamma > dx \quad (\gamma \in \Gamma), \tag{1.1}$$

and the inverse Fourier transform of a function u in $L^1(\Gamma)$ by

$$\check{u}(x) = \int_\Gamma u(\gamma) < x, \gamma > d\gamma \quad (x \in G). \tag{1.2}$$

We denote by $AL^1(G)$ the space of those continuous functions in $L^1(G)$ whose Fourier transforms are also in $L^1(\Gamma)$. It will be always assumed that the Haar measures of G and Γ are so adjusted that the inversion formula holds:

$$f(x) = \int_\Gamma \hat{f}(\gamma) < x, \gamma > d\gamma \quad (x \in G) \tag{1.3}$$

for every function f in $AL^1(G)$. For each p with $1 \leqq p \leqq 2$, $T_p(G)$ denotes the space of all bounded linear operators from $L^p(G)$ into $L^p(G)$ that commute with all translations. It is well-known ([3: p. 100]) that to every operator Φ in $T_p(G)$ there corresponds a function in $L^\infty(\Gamma)$, denoted by $\widehat{\Phi}$, such that

$$\Phi(f)^\wedge = \widehat{\Phi} \cdot \hat{f} \quad (f \in L^p(G)). \tag{1.4}$$

We can easily show that any function in $L^\infty(\Gamma)$ is $\widehat{\Phi}$ for some Φ in $T_2(G)$. It is also known ([5: p. 73]) that a function on Γ is $\widehat{\Phi}$ for some Φ in $T_1(G)$ if and only if it is the Fourier-Stieltjes transform of a bounded (regular) measure on G. We denote by $M_p(\Gamma)$ the space of all $\widehat{\Phi}$ with Φ in $T_p(G)$. Finally, for any Φ in $T_p(G)$ and any f in $L^p(G)$, we shall often write $f*\Phi$ to denote $\Phi(f)$.

This paper is motivated by that of de Leeuw [5], where some characterizations of L^p-multipliers on the real line and the circle group are obtained. We shall here give some generalizations of his results.

2. Some remarks and lemmas. Throughout the remainder part of this paper, let us fix p arbitrarily so that $1 \leqq p \leqq 2$, and put $q = p/(p-1)$. We shall assume that every function whose definition may be changed on a locally null set to make it continuous is always so redefined on such a set to be continuous. Recall that the spectrum of a function φ in $L^\infty(\Gamma)$ is the closed subset of G defined by

$$S(\varphi) = \bigcap_u \{x \in G;\ \check{u}(x) = 0\}\ ,$$

where the intersection is taken over all functions u in $L^1(\Gamma)$ such that $u*\varphi=0$.

Let now Φ be any operator in $T_p(G)$, and put $\varphi = \widehat{\Phi}$. Then the spectrum of φ is called the support of Φ and denoted by supp Φ. For any closed subset K of G such that $\operatorname{supp}\Phi \subset K$, we say that Φ is concentrated on K. We also define $\|\varphi\| = \|\varphi\|_{M_p(\Gamma)}$ to be the operator norm of Φ. Then for every function u in $L^1(\Gamma)$ it is well-known that $\varphi*u$ is in $M_p(\Gamma)$, that $\|\varphi*u\| \leqq \|\varphi\| \cdot \|u\|_1$, and that $S(\varphi*u) \subset S(\varphi) \cap \operatorname{supp}(\check{u})$. We shall write $\Phi\check{u}$ to denote the operator in $T_p(G)$ defined by $(\Phi\check{u})^\wedge = \varphi*u$. Finally, observe that if f is in $AL^1(G)$, then $f*\Phi$ is continuous since we have

$$(f*\Phi)(x) = \int_\Gamma \hat{f}(\gamma)\,\widehat{\Phi}(\gamma) <x, \gamma> d\gamma \qquad (x \in G).$$

LEMMA 2.1. *Let Φ be any operator in $T_p(G)$ concentrated on a closed subgroup H of G, and let f and g be any functions in $AL^1(G)$ such that $f=g$ on H. Then we have $f*\Phi = g*\Phi$ on H.*

PROOF. It suffices to prove that if h is in $AL^1(G)$ and if $h=0$ on H, then $h*\Phi=0$ on H. But this follows from the fact that every closed subgroup of G is an S-set for the algebra $L^1(\Gamma)$ (see [6; p. 170]).

We shall now denote by $AL^1_c(G)$ the space of all functions in $AL^1(G)$ having compact support, and observe then that there exists a net s_n in $AL^1_c(G)$ satisfying the following conditions:

(2. A) $$\|\hat{s}_n\|_1 \leqq 1 \quad \text{for all } n;$$

(2. B) If φ is a function in $L^\infty(\Gamma)$, then the net $\varphi * \hat{s}_n$ converges to φ in the weak-star topology of $L^\infty(\Gamma)$. If, in addition, φ is continuous, then the net $\varphi * \hat{s}_n$ converges to φ uniformly on each compact subset of Γ.

LEMMA 2.2. *Let Φ be any operator in $T_p(G)$ and let s_n be a net in $AL^1{}_c(G)$ satisfying (2. A) and (2. B). Then we have $\|\Phi s_n\| \leqq \|\Phi\|$ and* $\operatorname{supp} \Phi s_n \subset \operatorname{supp} \Phi \cap \operatorname{supp} s_n$ *for all n. Furthermore, the net Φs_n in $T_p(G)$ converges to Φ in the topology of weak convergence of operators.*

PROOF. Trivial from the above observations.

3. Operators concentrated on a closed subgroup. In this section, we shall characterize the operators in $T_p(G)$ concentrated on a closed subgroup.

Let H be any closed subgroup of G, and Λ its annihilator. For each element x of G, $\tilde{x}$ denotes the coset of H containing x. Similarly, for each element γ of Γ, $\tilde{\gamma}$ denotes the coset of Λ containing γ. We shall fix the Haar measures of G, H, and the quotient group G/H so that

$$\int_G f\,dx = \int_{G/H}\int_H f(x+t)\,dt\,d\tilde{x} \quad (f \in L^1(G)). \tag{3. 1}$$

Then we have

$$\int_\Gamma u\,d\gamma = \int_{\Gamma/\Lambda}\int_\Lambda u(\gamma+\lambda)\,d\lambda\,d\tilde{\gamma} \quad (u \in L^1(\Gamma)). \tag{3. 2}$$

(Note that the Haar measures of a locally compact abelian group and its dual are always assumed to be so adjusted that the inversion formula holds. Since we have fixed the Haar measures of G, H, and G/H, it follows that those of Γ, Γ/Λ, and Λ are automatically determined by this requirement.) For a function f on G and an element x of G, the function f_x is defined by $f_x(y) = f(y-x)$ $(y \in G)$. It is then easy to see that

$$\int_H f_{-x}(t)\langle t, -\tilde{\gamma}\rangle\,dt = \int_\Lambda (f_{-x})^\wedge(\gamma+\lambda)\,d\lambda \tag{3. 3}$$

for all f in $AL^1{}_c(G)$ and all x in G, where the integral in the right side exists for almost all $\tilde{\gamma}$ in Γ/Λ, and the equality holds for almost all $\tilde{\gamma}$ in Γ/Λ.

LEMMA 3.1. *Let Ψ be any operator in $T_p(H)$. Then there exists a unique operator Φ in $T_p(G)$ such that $\widehat{\Phi} = \widehat{\Psi} \circ \pi_\Lambda$ where π_Λ denotes the quotient mapping from Γ to Γ/Λ. In this case we have also $\|\Phi\| = \|\Psi\|$.*

PROOF. For any function f on G, let $f|_H$ be the restriction of f to H. It is readily seen that the restrictions to H of the functions in $AL^1_c(G)$ coincides with $AL^1_c(H)$.

Let now f be any fixed function in $AL^1_c(G)$. It then follows from (3.3) that

$$(f_{-x}|_H)^\wedge(\tilde{\gamma}) = \int_\Lambda (f_{-x})^\wedge(\gamma+\lambda)\,d\lambda \quad (x \in G) \tag{3.1.1}$$

for almost all $\tilde{\gamma}$ in Γ/Λ, and so

$$(\Psi(f_{-x}|_H))^\wedge(\tilde{\gamma}) = \widehat{\Psi}(\tilde{\gamma})\int_\Lambda (f_{-x})^\wedge(\lambda+\lambda)\,d\lambda \quad (x \in G) \tag{3.1.2}$$

for almost all $\tilde{\gamma}$ in Γ/Λ. Integrating both sides of (3.1.2), we have by (3.2)

$$\begin{aligned} \Psi(f_{-x}|_H)(0) &= \int_{\Gamma/\Lambda} \widehat{\Psi}(\tilde{\gamma}) \int_\Lambda (f_{-x})^\wedge(\gamma+\lambda)\,d\lambda\,d\tilde{\gamma} \\ &= \int_\Gamma (\widehat{\Psi}\circ\pi_\Lambda)(\gamma)(f_{-x})^\wedge(\gamma)\,d\gamma \\ &= \int_\Gamma (\widehat{\Psi}\circ\pi_\Lambda)(\gamma)\hat{f}(\gamma)<x,\gamma>\,d\gamma \quad (x \in G). \end{aligned} \tag{3.1.3}$$

This shows that the function defined by

$$\Phi(f)(x) = \Psi(f_{-x}|_H)(0) \quad (x \in G) \tag{3.1.4}$$

is continuous. We have also by (3.1) and (3.1.4)

$$\begin{aligned} \int_G |\Phi(f)(x)|^p\,dx &= \int_{G/H}\int_H |\Psi(f_{-x}|_H)(t)|^p\,dt\,dx \\ &\leqq \|\Psi\|^p \int_{G/H}\int_H |f(x+t)|^p\,dt\,dx \\ &= \|\Psi\|^p \int_G |f(x)|^p\,dx, \end{aligned}$$

which shows that $\Phi(f)$ is in $L^p(G)$ and

$$\|\Phi(f)\|_p \leqq \|\Psi\|\cdot\|f\|_p. \tag{3.1.5}$$

It also follows from (3.1.3) and (3.1.4) that

$$(3.1.6)\qquad (\Phi(f))^{\wedge}(\gamma) = (\widehat{\Psi}\circ\pi_{\Lambda})(\gamma)\hat{f}(\gamma)\quad (\gamma\in\Gamma).$$

Since (3.1.5) and (3.1.6) are valid for every function f in $AL^1{}_c(G)$, and since $AL^1{}_c(G)$ is dense in $L^p(G)$, Φ can be uniquely extended to an operator in $T_p(G)$, which we also denote by Φ, so that

$$(3.1.7)\qquad \|\Phi\|\leqq\|\Psi\|,\quad\text{and}\quad \widehat{\Phi}=\widehat{\Psi}\circ\pi_{\Lambda}.$$

It remains only to show that $\|\Phi\|=\|\Psi\|$. If we note that $\operatorname{supp}\Phi\subset\operatorname{supp}\Psi\subset H$, which follows at once from (3.1.4), this equality will follows from the following:

LEMMA 3.2. *Let Φ be any operator in $T_p(G)$ concentrated on H. Then there exists a unique operator Ψ in $T_p(H)$ such that $\widehat{\Psi}\circ\pi_{\Lambda}=\widehat{\Phi}$. In this case we have $\|\Psi\|=\|\Phi\|$.*

PROOF. Let F be any function in $AL^1{}_c(H)$. Then there exists a function f in $AL^1{}_c(G)$ such that $F=f|_H$. We put

$$(3.2.1)\qquad \Psi(F)(t)=\Phi(f)(t)\quad (t\in H),$$

which is well-defined by Lemma 2.1. We then claim that Ψ can be extended to an operator in $T_p(H)$ having the required properties.

First assume that Φ has compact support. It is then easy to see that Φ maps $AL^1{}_c(G)$ into $AL^1{}_c(G)$. Let us now fix any function f in $AL^1{}_c(G)$. The inequality $\|\Phi(f)\|_p\leqq\|\Phi\|\cdot\|f\|_p$ is equivalent to

$$\int_G|\Phi(f)(x)|^p\,dx\leqq\|\Phi\|^p\int_G|f(x)|^p\,dx,$$

and this can be rewritten in the form

$$(3.2.2)\quad \int_{G/H}\left\{\int_H|\Phi(f)(x+t)|^p\,dt\right\}d\tilde{x}\leqq\|\Phi\|^p\int_{G/H}\left\{\int_H|f(x+t)|^p\,dt\right\}d\tilde{x}.$$

Take now any compact neighborhood K of 0 in G/H, and any open neighborhood U of K. Denote by π_H the quotient mapping from G to G/H. We can find a function g on G, which is the Fourier-Stieltjes transform of a bounded measure concentrated on Λ, so that

(3.2.3) $\qquad \operatorname{supp} g \subset \pi_H^{-1}(U);\ g=1 \text{ on } \pi_H^{-1}(K);\ |g| \leqq 1 \text{ on } G.$

Replacing f by gf in (3.2.2), we have

$$\int_{G/H}\left\{\int_H |\Phi(gf)(x+t)|^p\,dt\right\}d\tilde{x} \leqq \|\Phi\|^p \int_{G/H}\left\{\int_H |(gf)(x+t)|^p\,dt\right\}d\tilde{x}. \tag{3.2.4}$$

But, the integral in the bracket in the right side, as a function on G/H, vanishes outside U by (3.2.3). Thus (3.2.4) together with (3.2.3) yields

$$\int_K\left\{\int_H |\Phi(gf)(x+t)|^p\,dt\right\}d\tilde{x} \leqq \|\Phi\|^p \int_U\left\{\int_H |f(x+t)|^p\,dt\right\}d\tilde{x}. \tag{3.2.5}$$

One more application of (3.2.3) also shows that $(gf)(x+t)=f(x+t)$ for all x in $\pi_H^{-1}(K)$ and all t in H. It follows from Lemma 2.1 that $\Phi(gf)(x+t) = \Phi(f)(x+t)$ for all such x and t, since gf is in $AL^1(G)$. Thus (3.2.5) can be written as

$$\int_K\left\{\int_H |\Phi(f)(x+t)|^p\,dt\right\}d\tilde{x} \leqq \|\Phi\|^p \int_U\left\{\int_H |f(x+t)|^p\,dt\right\}d\tilde{x}.$$

Since U was an arbitrary neighborhood of K, this yields

$$\int_K\left\{\int_H |\Phi(f)(x+t)|^p\,dt\right\}d\tilde{x} \leqq \|\Phi\|^p \int_K\left\{\int_H |f(x+t)|^p\,dt\right\}d\tilde{x}. \tag{3.2.6}$$

Observe now that the integrals in the brackets in both sides of (3.2.6), as functions of $\tilde{x}$, are continuous. Since (3.2.6) holds for all compact neighborhoods K of 0 in G/H, this assures that

$$\int_H |\Phi(f)(t)|^p\,dt \leqq \|\Phi\|^p \int_H |f(t)|^p\,dt,$$

which combined with (3.2.1) yields

$$\|\Psi(F)\|_p \leqq \|\Phi\| \cdot \|F\|_p \qquad (F \in AL^1_c(H)).$$

Thus Ψ can be uniquely extended to an operator in $T_p(H)$, which we also denote by Ψ, with $\|\Psi\| \leqq \|\Phi\|$.

In order to show that $\widehat{\Psi} \circ \pi_\Lambda = \widehat{\Phi}$, it suffices to note that (3.2.1) implies (3.1.4). In fact, then (3.1.6) shows $\widehat{\Psi} \circ \pi_\Lambda = \widehat{\Phi}$.

Let now Φ be any operator in $T_p(G)$ concentrated on H whose support is not necessarily compact. Take a net s_n in $AL^1_c(G)$ satisfying (2.A) and (2.B). It then follows from Lemma 2.2 that every operator Φs_n has a compact support contained in H and norm $\leqq \|\Phi\|$. By what has already been proved, we can find a net Ψ_n in $T_p(H)$ so that

$$\|\Psi_n\| \leqq \|\Phi\|, \quad \text{and} \quad \widehat{\Psi}_n \circ \pi_\Lambda = \widehat{\Phi} * \hat{s}_n \tag{3.2.7}$$

for all n. Then we have for any function u in $L^1(\Gamma)$

$$\begin{aligned} \int_\Gamma \widehat{\Phi}\cdot u\,d\gamma &= \lim_n \int_\Gamma (\widehat{\Phi}*\hat{s}_n)\cdot u\,d\gamma \\ &= \lim_n \int_\Gamma (\widehat{\Psi}_n \circ \pi_\Lambda)\cdot u\,d\gamma \\ &= \lim_n \int_{\Gamma/\Lambda} \widehat{\Psi}_n(\tilde{\gamma}) \int_\Lambda u(\gamma+\lambda)\,d\lambda\,d\tilde{\gamma}. \end{aligned} \tag{3.2.8}$$

Since

$$\|\widehat{\Psi}_n\|_\infty = \|\widehat{\Phi} * \hat{s}_n\|_\infty \leqq \|\widehat{\Phi}\|_\infty$$

for all n, and since the mapping

$$u \longrightarrow u'(\tilde{\gamma}) = \int_\Lambda u(\gamma+\lambda)\,d\lambda$$

carries $L^1(\Gamma)$ onto $L^1(\Gamma/\Lambda)$, (3.2.8) shows that the net $\widehat{\Psi}_n$ converges to some ψ of $L^\infty(\Gamma/\Lambda)$ in the weak-star topology of $L^\infty(\Gamma/\Lambda)$ such that

$$\int_\Gamma \widehat{\Phi}\cdot u\,d\gamma = \int_{\Gamma/\Lambda} \psi\cdot u'\,d\tilde{\gamma} \quad (u \in L^1(\Gamma)),$$

which clearly implies that $\widehat{\Phi} = \psi \circ \pi_\Lambda$ locally almost everywhere.

To show that ψ is in $M_p(\Gamma/\Lambda)$ and $\|\psi\| \leqq \|\Phi\|$, let f and g be any functions in $AL^1(H)$. Since the net $\widehat{\Psi}_n$ converges to ψ in the weak-star topology of $L^\infty(\Gamma/\Lambda)$, and since $\|\Psi_n\| \leqq \|\Phi\|$ for all n, it follows that

$$\begin{aligned} \left|\int_{\Gamma/\Lambda} \hat{g}\cdot\hat{f}\cdot\psi\,d\tilde{\gamma}\right| &= \lim_n \left|\int_{\Gamma/\Lambda} \hat{g}\cdot\hat{f}\cdot\widehat{\Psi}_n\,d\tilde{\gamma}\right| \\ &\leqq \|g\|_q\cdot\|f\|_p\cdot\|\Phi\|. \end{aligned}$$

Thus we conclude that ψ is in $M_p(\Gamma/\Lambda)$ and $\|\psi\| \leqq \|\Phi\|$. Summarizing up, we have found an operator Ψ in $T_p(H)$ such that

$$\|\Psi\| \leqq \|\Phi\|, \quad \text{and} \quad \widehat{\Psi} \circ \pi_\Lambda = \widehat{\Phi}. \tag{3.2.9}$$

Combining (3.2.9) and (3.1.7), we have also $\|\Psi\| = \|\Phi\|$. This completes the proof.

THEOREM 3.3. *There exists an isometrical isomorphism α from the Banach algebra $T_p(H)$ to the Banach algebra $T_p(G)$ such that*

$$(\alpha(\Psi))^\wedge = \widehat{\Psi} \circ \pi_\Lambda \qquad (\Psi \in T_p(H)).$$

The range of α is precisely the space of all operators in $T_p(G)$ concentrated on H.

PROOF. This is trivial from Lemma 3.1 and Lemma 3.2.

LEMMA 3.4. *Let φ be a function in $L^\infty(\Gamma)$. Then the spectrum of φ is contained in H if and only if φ may be redefined on a locally null set to be constant on each coset of Λ.*

PROOF. Let u be any function in $L^1(\Gamma)$. Then for all λ in Γ and all x in G, we have $\check{u}_\lambda(x) = \langle x, \lambda \rangle \check{u}(x)$. Therefore, if λ is in Λ, we have $\check{u}_\lambda = \check{u}$ on H.

Suppose now that the spectrum of φ is contained in H. Since H is an S-set for the algebra $L^1(\Gamma)$, it follows from the above observation that for all λ in Λ

$$\varphi_\lambda * u = \varphi * u_\lambda = \varphi * u \qquad (u \in L^1(\Gamma)).$$

This clearly implies that for every λ in Λ, $\varphi_\lambda = \varphi$ locally almost everywhere, from which we can easily deduce that φ may be redefined on a locally null set to be constant on each coset of Λ.

Conversely assume that φ is constant on each coset of Λ. Let ψ be the function on Γ/Λ defined by $\psi \circ \pi_\Lambda = \varphi$. We have then by (3.2)

$$\int_\Gamma \varphi \cdot \hat{f}\, d\gamma = \int_{\Gamma/\Lambda} \psi(\tilde{\gamma}) \int_\Lambda \hat{f}(\gamma+\lambda)\, d\lambda\, d\tilde{\gamma} \tag{3.4.1}$$

for all f in $AL^1_c(G)$. Substituting (3.1.1) to (3.4.1), we see that

$$\int_\Gamma \varphi \cdot \hat{f}\, d\gamma = \int_{\Gamma/\Lambda} \psi(\tilde{\gamma}) (f|_H)^\wedge(\tilde{\gamma})\, d\tilde{\gamma} \tag{3.4.2}$$

for all f in $AL^1_c(G)$. Thus we have

$$\int_\Gamma \varphi \cdot \hat{f} d\gamma = 0$$

for all f in $AL^1_c(G)$ such that $f = 0$ on H. In other words, the spectrum of φ is contained in H. This completes the proof.

COROLLARY 3.5. *Let ψ be any function on Γ/Λ. Set $\varphi = \psi \circ \pi_\Lambda$. Then φ is in $M_p(\Gamma)$ if and only if ψ is in $M_p(\Gamma/\Lambda)$. In this case, we have $\|\varphi\| = \|\psi\|$.*

PROOF. If ψ is in $M_p(\Gamma/\Lambda)$, then φ is in $M_p(\Gamma)$ and $\|\varphi\| = \|\psi\|$ by Theorem 3.3. Conversely, if φ is in $M_p(\Gamma)$, then the operator in $T_p(G)$ corresponding to φ is concentrated on H by Lemma 3.4 since φ is constant on each coset of Λ. Applying Theorem 3.3, we see that ψ is in $M_p(\Gamma/\Lambda)$ and $\|\psi\| = \|\varphi\|$. This completes the proof.

4. Restrictions of L^p-multipliers to a subgroup. We shall denote by $\overline{G}$ the Bohr compactification of G. Thus the dual group of $\overline{G}$ is Γ_d, the group Γ endowed with the discrete topology. Let R be the group of real numbers, Z the group of integers, and T the circle group.

LEMMA 4.1. *Let φ be in $L^\infty(\Gamma)$, and u in $L^1(\Gamma)$. If φ is in $M_p(\Gamma_d)$, then so is $\varphi * u$, and $\|\varphi * u\|_{M_p(\Gamma_d)} \leqq \|\varphi\|_{M_p(\Gamma_d)} \cdot \|u\|_1$.*

LEMMA 4.2. *Let ψ be any continuous function on the group $\Gamma' = R^N \times D$, where N is a non-negative integer and D a discrete abelian group. Then ψ is in $M_p(\Gamma')$ if and only if ψ is in $M_p(\Gamma'_d)$. In this case we have $\|\psi\|_{M_p(\Gamma')} = \|\psi\|_{M_p(\Gamma'_d)}$.*

The first lemma is due to de Leeuw [5], and the second one is also due to him in case $\Gamma' = R^N$. The modifications needed in the proofs are obvious.

THEOREM 4.3. *Let φ be any continuous function on Γ. Then φ is in $M_p(\Gamma)$ if and only if φ is in $M_p(\Gamma_d)$. In this case we have $\|\varphi\|_{M_p(\Gamma)} = \|\varphi\|_{M_p(\Gamma_d)}$.*

PROOF. We shall prove this in three steps.

Step 1. Suppose here that G has the form $G = R^m \times Z^n \times K$ for some non-negative integers m and n and some compact abelian group K. Then Γ is of the form $\Gamma=R^m\times T^n\times D$, where D is the discrete dual of K. Let now $\Gamma'=R^m\times R^n\times D$. Then we can construct a continuous homomorphism π from Γ' onto Γ so that $\Gamma=\Gamma'/\Lambda$, where Λ is the kernel of π. Let φ' be the function on Γ' defined by $\varphi'=\varphi\circ\pi$. Applying Corollary 3.5, we see that φ' is in $M_p(\Gamma')$ if and only if φ is in $M_p(\Gamma)$, and that then $\|\varphi'\|_{M_p(\Gamma')}=\|\varphi\|_{M_p(\Gamma)}$. The same is true even if Γ and Γ' are replaced by Γ_d and Γ'_d respectively. It also follows from Lemma 4.2 that φ is in $M_p(\Gamma')$ if and only if φ' is in $M_p(\Gamma'_d)$ and that then $\|\varphi'\|_{M_p(\Gamma')}=\|\varphi'\|_{M_p(\Gamma'_d)}$. Combining these facts, we have the desired conclusion.

Step 2. Suppose here that the spectrum of φ is compact. Let H be any compactly generated open subgroup of G containing the spectrum of φ. Then H has the form $H=R^m\times Z^n\times K$ for some non-negative integers m and n and some compact abelian group K (see [2: Theorem 9.8, p. 90]). Let Λ be the annihilator of H, then we see from Lemma 3.4 that φ is constant on each coset of Λ. Let ψ be the function on Γ/Λ defined by $\varphi=\psi\circ\pi_\Lambda$. It follows from Corollary 3.5 that φ is in $M_p(\Gamma)$ if and only if ψ is in $M_p(\Gamma/\Lambda)$ and that in this case $\|\varphi\|_{M_p(\Gamma)}=\|\psi\|_{M_p(\Gamma/\Lambda)}$. The same is true even if Γ and Γ/Λ are replaced by Γ_d and $(\Gamma/\Lambda)_d$. By Step 1, we have also that ψ is in $M_p(\Gamma/\Lambda)$ if and only if ψ is in $M_p(\Gamma_d/\Lambda_d)$. From these facts we have the desired conclusion.

Step 3. Let s_n be a net in $AL^1{}_c(G)$ satisfying (2. A) and (2. B). Suppose that φ is in $M_p(\Gamma)$. Then we see from Lemma 2.2 that every $\varphi*\hat{s}_n$ has norm $\leqq\|\varphi\|$ and its spectrum is compact. It follows from Step 2 that every $\varphi*\hat{s}_n$ is in $M_p(\Gamma_d)$ and has norm $\leqq\|\varphi\|_{M_p(\Gamma)}$. Since the net $\varphi*\hat{s}_n$ converges to φ pointwise, it is easy to see that φ is in $M_p(\Gamma_d)$ and $\|\varphi\|_{M_p(\Gamma_d)}\leqq\|\varphi\|_{M_p(\Gamma)}$. Conversely, assume that φ is in $M_p(\Gamma_d)$. We then see from Lemma 4.1 that $\varphi*\hat{s}_n$ is in $M_p(\Gamma_d)$ and has norm $\leqq\|\varphi\|_{M_p(\Gamma_d)}$. Thus the proof proceeds as before, and we conclude that φ is in $M_p(\Gamma)$ and $\|\varphi\|_{M_p(\Gamma)}\leqq\|\varphi\|_{M_p(\Gamma_d)}$. This establishes our theorem.

COROLLARY 4.4. *Let φ be any measurable function on Γ. If φ is in $M_p(\Gamma_d)$, then φ is in $M_p(\Gamma)$ and* $\|\varphi\|_{M_p(\Gamma)}\leqq\|\varphi\|_{M_p(\Gamma_d)}$.

PROOF. Take a net s_n in $AL^1{}_c(G)$ satisfying (2. A) and (2. B). If φ is in $M_p(\Gamma_d)$, then by Lemma 4.1 so is every $\varphi*\hat{s}_n$ and $\|\varphi*\hat{s}_n\|_{M_p(\Gamma_d)}\leqq\|\varphi\|_{M_p(\Gamma_d)}$. Since every $\varphi*\hat{s}_n$ is continuous, Theorem 4.3 applies, and we see that $\varphi*\hat{s}_n$ is in $M_p(\Gamma)$ and

$$\|\varphi*\hat{s}_n\|_{M_p(\Gamma)}\leqq\|\varphi\|_{M_p(\Gamma_d)}\,.$$

Therefore, we have at once the desired conclusion.

We say that a bounded measurable function φ is regulated, if there exists a net u_n in $L^1(\Gamma)$ satisfying the following conditions:

(a) $\|u_n\|_1 \leqq 1$ for all n;

(b) $\varphi * u_n$ converges to φ in the weak-star topology of $L^\infty(\Gamma)$ and also pointwisely.

We shall now state two corollaries without proof. These two results are immediate from Theorem 4.3, etc.

COROLLARY 4.5. *Let φ be a bounded measurable function on Γ which is regulated. Then φ is in $M_p(\Gamma)$ if and only if φ is in $M_p(\Gamma_d)$. In this case we have* $\|\varphi\|_{M_p(\Gamma)} = \|\varphi\|_{M_p(\Gamma_d)}$.

COROLLARY 4.6. *Let φ be as in Corollary 4.5, and Λ an algebraic subgroup of Γ. If φ is in $M_p(\Gamma)$, and if ψ is the restriction of φ to Λ, we have:*

(a) *ψ is in $M_p(\Lambda_d)$, and* $\|\psi\|_{M_p(\Lambda_d)} \leqq \|\varphi\|_{M_p(\Gamma)}$;

(b) *If Λ is closed, and if ψ is measurable with respect to the Haar measure of Λ, then ψ is in $M_p(\Lambda)$ and* $\|\psi\|_{M_p(\Lambda)} \leqq \|\varphi\|_{M_p(\Gamma)}$.

REMARKS. (a) In some special cases, we can weaken the assumption in the "if" part of Theorem 4.3. See [4].

(b) Let Λ be a closed subgroup of Γ, and for any topological space X, let $C(X)$ be thes pace of all continuous functions. Part (b) of Corollary 4.6 shows that the restrictions to Λ of the functions in $M_p(\Gamma) \cap C(\Gamma)$ belong to $M_p(\Lambda) \cap C(\Lambda)$. Conversely, is it true that every function in $M_p(\Lambda) \cap C(\Lambda)$ is the restriction of a function in $M_p(\Gamma) \cap C(\Gamma)$? This is the case if $p = 1$ or 2. The author conjectures that this is also true for all p with $1 < p < 2$.

ADDED IN PROOF. (c) The Answer to the question in (b) is Yes if Λ is a closed discrete subgroup of Γ (see [1]).

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