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NOTE ON STOPPED AVERAGES OF MARTINGALES

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Let $\{f_n, F_n; n \geqq 1\}$ be a martingale. Kazamaki and Tsuchikura ([2] Th. 1) showed that the two conditions

$$\sup_n E|f_n| < \infty \tag{1}$$

and

$$\sup_n E|u_n| < \infty \tag{2}$$

are equivalent to each other, where

$$u_n = \frac{f_1 + f_2 + \cdots + f_n}{n} .$$

Let ST be the class of all stopping times with respect to $\{F_n\}$. It is well known ([1] p. 300), that the condition

$$\sup_{\tau \in ST} E|f_\tau| < \infty \tag{3}$$

is equivalent to (1). Here $E|f_\tau|$ is defined by $\int_{(\tau<\infty)} |f_\tau| dP$.

In this note, we show that there is not always an equivalent relation between the two inequalities (3) and $\sup_{\tau \in ST} E|u_\tau| < \infty$.

LEMMA. *Let $\{a_n\}$ be a positive, non-increasing sequence, which converges to 0, and $\{b_n\}$ be a non-negative non-increasing sequence. Then*

$$\sum_{n=2}^{\infty} \frac{a_{n-1} - a_n}{n} \sum_{j=1}^{n-1} \frac{b_j}{a_j} \geqq \frac{1}{4} \sum_{n=1}^{\infty} \frac{a_{2^n} - a_{2^{n+1}}}{a_{2^n}} b_{2^n} .$$

PROOF. We put $\alpha_n = a_n - a_{n+1}$ $(n = 1, 2, \cdots)$, then

$$\sum_{n=2}^{\infty} \frac{a_{n-1} - a_n}{n} \sum_{j=1}^{n-1} \frac{b_j}{a_j} = \sum_{n=2}^{\infty} \frac{\alpha_{n-1}}{n} \sum_{j=1}^{n-1} \frac{b_j}{\sum_{k=j}^{\infty} \alpha_k} = \sum_{j=1}^{\infty} \frac{\sum_{n=j}^{\infty} \frac{\alpha_n}{n+1}}{\sum_{k=j}^{\infty} \alpha_k} b_j .$$

It is easily verified, that $\sum_{n=j}^{\infty}(\alpha_n/n+1)/\sum_{n=j}^{\infty}\alpha_n$ is non-increasing as j increases. This, with non-increasingness of the sequence $\{b_n\}$, gives

$$\sum_{j=1}^{\infty} \frac{\sum_{n=j}^{\infty} \frac{\alpha_n}{n+1}}{\sum_{k=j}^{\infty} \alpha_k} b_j \geqq \sum_{i=0}^{\infty} \sum_{n=2^i}^{2^{i+1}-1} \left\{ \frac{\sum_{k=2^{i+1}}^{\infty} \frac{\alpha_k}{k+1}}{\sum_{k=2^{i+1}}^{\infty} \alpha_k} \right\} b_{2^{i+1}}$$

$$\geqq \sum_{i=0}^{\infty} 2^i \frac{b_{2^{i+1}}}{a_{2^{i+1}}} \sum_{k=2^{i+1}}^{2^{i+2}-1} \frac{\alpha_k}{k+1}$$

$$\geqq \sum_{i=0}^{\infty} 2^i \frac{b_{2^{i+1}}}{a_{2^{i+1}}} \sum_{k=2^{i+1}}^{2^{i+2}-1} \alpha_k \frac{1}{2^{i+2}}$$

$$= \frac{1}{4} \sum_{i=1}^{\infty} \frac{a_{2^i} - a_{2^{i+1}}}{a_{2^i}} b_{2^i} . \qquad \text{q.e.d.}$$

THEOREM. *For any positive integer valued random variable τ, which is not essentially bounded and such that $P(\tau < \infty) = 1$, there exists a martingale $\{f_n, F_n;\ n \geqq 1\}$ such that*

(a) *τ is a stopping time with respect to $\{F_n\}$.*

(b) *$\{f_n, F_n;\ n \geqq 1\}$ is uniformly integrable,*

and

(c) $E|u_\tau| = \infty$.

PROOF. Note that from the assumptions on τ we have

(4) $\qquad P(\tau > n) > 0 \ (n = 1, 2, \cdots)$ and $P(\tau > n) \downarrow 0 \ (n \to \infty)$.

We construct a counter example of a martingale $\{f_n, F_n;\ n \geqq 1\}$. Let F_n be the σ-field generated by the sets $(\tau = 1), (\tau = 2), \cdots,$ and $(\tau = n)$; clearly τ is a stopping time with respect to $\{F_n\}$. We denote by $\{b_n\}$ a positive non-increasing sequence whose additional restrictions will be given step by step.

First we assume that

$$\frac{b_{n-1} - b_n}{P(\tau = n)} = 0 \qquad \text{and} \qquad b_{n-1} - b_n = 0$$

if $P(\tau = n) = 0$. Put

$$f_n = \sum_{k=1}^{n} \frac{b_{k-1} - b_k}{P(\tau = k)} I_{(\tau = k)} + \frac{b_n}{P(\tau > n)} I_{(\tau > n)} ,$$

where $I_{(.)}$ is the indicator function of the set $(.)$. Then $\{f_n, F_n;\ n \geqq 1\}$ is a non-negative martingale, because

$$\int_{(\tau = k)} (f_n - f_{n-1}) dP = 0 \qquad (k = 1, 2, \cdots, n-1)$$

$$\int_{(\tau > n-1)} (f_n - f_{n-1}) dP = (b_{n-1} - b_n) + b_n - b_{n-1} = 0$$

and the sets $(\tau=1), (\tau=2), \cdots, (\tau=n-1)$ and $(\tau>n-1)$ are atoms of F_{n-1}. Furthermore,

$$E(f_n)=b_0 \ (n=1, 2, \cdots) \quad \text{and} \quad E(f_\infty)=\lim_{n\to\infty}(b_0-b_n)\,.$$

So $\{f_n\}$ is uniformly integrable, if and only if $\{b_n\}$ converges to 0 (c.f. [1] p. 319 Th. 4.1 (ii)). Now,

$$u_\tau \geqq \frac{1}{\tau}\sum_{k=1}^{\tau-1} f_k = \frac{1}{\tau}\sum_{k=1}^{\tau-1}\frac{b_k}{P(\tau>k)}I_{(\tau>k)} \geqq 0\,,$$

and

$$\begin{aligned} E(u_\tau) &\geqq \sum_{n=2}^{\infty}\int_{(\tau=n)}\frac{1}{n}\sum_{k=1}^{n-1}\frac{b_k}{P(\tau>k)}I_{(\tau>k)}dP \\ &= \sum_{n=2}^{\infty}\frac{P(\tau=n)}{n}\sum_{k=1}^{n-1}\frac{b_k}{P(\tau>k)} \\ &= \sum_{n=2}^{\infty}\frac{P(\tau>n-1)-P(\tau>n)}{n}\sum_{k=1}^{n-1}\frac{b_k}{P(\tau>k)}\,. \end{aligned}$$

Note that the sequences $\{a_n\}$ $(a_n=P(\tau>n))$ and $\{b_n\}$ satisfy the assumptions of Lemma from (4). So,

$$E(u_\tau) \geqq \frac{1}{4}\sum_{n=1}^{\infty}\frac{P(\tau>2^n)-P(\tau>2^{n+1})}{P(\tau>2^n)}\,b_{2^n}\,. \tag{5}$$

Let N be an integer such that

$$P(\tau>2^N)-P(\tau>2^{N+1})>0$$

(such an integer always exists), and define

$$b_{2^n}=\left\{\sum_{k=N}^{n-1}\frac{P(\tau>2^k)-P(\tau>2^{k+1})}{P(\tau>2^k)}\right\}^{-1}, \qquad (n=N+1, N+2, \cdots)\,.$$

From (4), $\prod_{k=N}^{n} P(\tau>2^{k+1})/P(\tau>2^k)=P(\tau>2^{n+1})/P(\tau>2^N)$ diverges to 0 as $n\to\infty$, so $b_{2^n}^{-1}\uparrow\infty$ and $b_{2^n}\downarrow 0\,(n\to\infty)$. And we define the remaining terms $b_n(n\neq 2^{N+1}, 2^{N+2}, \cdots)$, such that all the assumptions imposed on $\{b_n\}$ in this proof are satisfied. The martingale with this $\{b_n\}$ satisfies (b), and it remains only to show that $E|u_\tau|=\infty$. From (5), it follows that

$$\begin{aligned} E|u_\tau| &\geqq \frac{1}{4}\sum_{n=N+1}^{\infty}\frac{P(\tau>2^n)-P(\tau>2^{n+1})}{P(\tau>2^n)}\,b_{2^n} \\ &= \frac{1}{4}\sum_{n=N+1}^{\infty}\left(\frac{1}{b_{2^{n+1}}}-\frac{1}{b_{2^n}}\right)b_{2^n} = \frac{1}{4}\sum_{n=N+1}^{\infty}\left(\frac{b_{2^n}}{b_{2^{n+1}}}-1\right). \end{aligned}$$

But from the fact that

$$\prod_{k=N+1}^{n} \frac{b_{2^k}}{b_{2^{k+1}}} = \frac{b_{2^{N+1}}}{b_{2^{n+1}}}$$

diverges as $n \to \infty$,

$$\sum_{n=N+1}^{\infty} \left(\frac{b_{2^n}}{b_{2^{n+1}}} - 1 \right) = \infty . \qquad \text{q.e.d.}$$

It is easily checked, if we put

$$P(\tau = n) = \frac{1}{2^n} \quad \text{and} \quad b_n = \frac{1}{n+1}$$

then the martingale $\{f_n, F_n; n \geqq 1\}$ defined as in the proof gives an example such that

$$\sup_{\tau \in ST} E \,|\, u_\tau \,| < \infty \quad \text{and} \quad E\,[\sup_n |\, f_n \,|] = \infty .$$

REFERENCES

[1] J. L. DOOB, Stochastic Processes, Wiley, New York, 1953.
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