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SATURATION OF POSITIVE LINEAR OPERATORS

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1. **Introduction.** Roughly speaking, the phenomenon of saturation of approximation is that there exists an "optimal" order of approximation, called "saturation order," such that better approximation occurs only in trivial cases. This is exactly defined as follows (cf. P. L. Butzer and R. J. Nessel [2]):

Let B be a Banach space with norm $\|\cdot\|$, and let (L_i) be a net of bounded linear operators of B into itself, converging strongly to the identity operator, which will be called a strong approximation process on B. Denote by $T[B; (L_i)]$ the closed linear subspace of B, consisting of all f in B for which $L_i(f) = f$ for all i. Suppose that there exists a net (ϕ_i) of positive real numbers, converging to zero such that every f in B for which $\|L_i(f) - f\| = o(\phi_i)$ belongs to $T[B; (L_i)]$, and there exists a g in B but not in $T[B; (L_i)]$ such that $\|L_i(g) - g\| = O(\phi_i)$. Then the strong approximation process (L_i) is said to be saturated in B with order (ϕ_i). The set $T[B; (L_i)]$ and the net (ϕ_i) are called the trivial class of (L_i) and the saturation order of (L_i), respectively. Furthermore, the set $S[B; (L_i)]$ consisting of all f in B for which $\|L_i(f) - f\| = O(\phi_i)$ is called the saturation class of (L_i).

The saturation problem may actually consist of two different questions: firstly, the question of whether saturation holds, that is, the establishment of the existence of the saturation order of a given strong approximation process (L_i) on B; secondly, the characterization of the saturation class $S[B; (L_i)]$.

The purpose of this paper is to establish a result concerning the first problem of saturation of positive linear operators on $C(X)$, the Banach space of all real-valued continuous functions on a compact Hausdorff space X with sup-norm $\|\cdot\|$. The applications will be made to the Bernstein-Schnabl functions constructed by M. W. Grossman [3]. The arguments of this paper can be based on the author [4].

Throughout this paper, $\mathscr{F}$ will be a subset of $C(X)$, separating the points of X. 1 will denote the unit function on X.

2. **A saturation theorem in $C(X)$.** The main result of this paper

is the following:

THEOREM 1. *Let* (L_i) *be a net of positive linear operators of* $C(X)$ *into itself, and let* M *be a proper linear subspace of* $C(X)$, *containing* 1 *and* $\mathscr{F}$. *Suppose that there exists a net* (ϕ_i) *of positive real numbers, converging to zero and a positive projection* L *of* $C(X)$ *onto* M *such that* $L_i \circ L = L$ *and* $L_i(f^2) = f^2 + \phi_i(L(f^2) - f^2)$ *for all* i *and for every* f *in* $\mathscr{F}$. *Then* (L_i) *is saturated in* $C(X)$ *with order* (ϕ_i) *and* $T[C(X); (L_i)] = M$.

In order to prove the above theorem, we shall now begin with the following.

PROPOSITION 1. *Let* (L_i) *be a net of positive linear operators of* $C(X)$ *into itself, and let* L *be a positive linear operator of* $C(X)$ *into itself, satisfying* $L(1) = 1$ *and* $L(f) = f$ *for all* f *in* $\mathscr{F}$. *Suppose that* $L_i \circ L = L$ *for all* i *and* $(L_i(f^2))$ *converges to* $L(f^2)$ *for all* f *in* $\mathscr{F}$. *Then* (L_i) *converges strongly to* L.

PROOF. Since X is compact and $\mathscr{F}$ separates the points of X, the original topology on X is identical with the weak topology on X induced by $\mathscr{F}$. Without loss of generality we may assume $\mathscr{F}$ contains $\lambda\mathscr{F}$ for all $\lambda > 0$. Let g in $C(X)$ and $\varepsilon > 0$ be given. Then there exists a finite subset S of $\mathscr{F}$ such that

$$|g(x) - g(y)| \leqq \varepsilon + \sum_{f \in S} (f(x) - f(y))^2 \tag{1}$$

for all x, y in X. Since L is a positive linear operator with $L(1) = 1$, we can operate on the variable x in (1) and obtain

$$|L(g)(y) - g(y)| \leqq \varepsilon + \sum_{f \in S} L((f(x) - f(y))^2, y) \tag{2}$$

for all y in X, which yields

$$|L(g) - g| \leqq \varepsilon + \sum_{f \in S} (L(f^2) - f^2) \tag{3}$$

since $L(1) = 1$ and $L(h) = h$ whenever h in $\mathscr{F}$. By the positivity and linearity of L_i and $L_i \circ L = L$ (therefore, $L_i(1) = 1$ since $L(1) = 1$), we obtain from (3) that

$$|L(g) - L_i(g)| \leqq \varepsilon + \sum_{f \in S} (L(f^2) - L_i(f^2)),$$

and so

$$\|L(g) - L_i(g)\| \leqq \varepsilon + \sum_{f \in S} \|L(f^2) - L_i(f^2)\| \tag{4}$$

for all i. Therefore, the hypothesis and (4) complete the proof.

Note that Theorem 1 in Grossman [3] remains valid for a net of positive linear operators of $C(X)$ into itself (we omit the proof). That is,

PROPOSITION 2 (cf. H. Bauer [1; Proposition 2.9]). *Let* (L_i) *be a net of positive linear operators of* $C(X)$ *into itself. Suppose that* $(L_i(1))$ *converges to* 1 *and* $(L_i(f^k))$ *converges to* f^k *for every* f *in* $\mathscr{F}$ *and* $k = 1, 2$. *Then* (L_i) *converges strongly to the identity operator.*

As an immediate consequence of Proposition 2, we have the following.

COROLLARY. *Let* L *be a positive linear operator of* $C(X)$ *into itself. If* $L(1) = 1$ *and* $L(f^k) = f^k$ *for* $k = 1, 2$ *and for all* f *in* $\mathscr{F}$, *then* L *coincides with the identity operator.*

REMARK 1. Propositions 1 and 2 can be reformulated with respect to pointwise convergence.

With the help of the previous results we can now prove Theorem 1.

PROOF OF THEOREM 1. By the hypotheses, we have $\lim_i L_i(f^k) = f^k$ for $k = 0, 1, 2$ and for all f in $\mathscr{F}$, where $f^0 = 1$. Therefore, by Proposition 2, (L_i) is a strong approximation process on $C(X)$.

Suppose that $\| L_i(g) - g \| = o(\phi_i)$. Denote by D the directed set of all elements i, and for each i in D, set $\phi_i^{-1} \| L_i(g) - g \| = a_i$. Since $\lim_i a_i = 0$ and $\lim_i \phi_i = 0$, we can choose a countable subset $\{s_n; n = 1, 2, \cdots\}$ of D so that $\lim_n a_{s_n} = 0$ and $\lim_n \phi_{s_n} = 0$. Note now that for all i in D and for every positive integer k

$$\| L_i^k(g) - g \| \leqq k a_i \phi_i \, , \tag{5}$$

where L_i^k denotes the k-th iteration of L_i. We now choose a sequence (k_n) of positive integers so that

$$\lim_n k_n \phi_{s_n} = +\infty \quad \text{and} \quad \lim_n a_{s_n} k_n \phi_{s_n} = 0 \, .$$

Putting $k = k_n$ and $i = s_n$ in (5), and letting n tend to ∞, we obtain

$$\lim_n L_{s_n}^{k_n}(g) = g \, . \tag{6}$$

By induction on k and the hypotheses, we see that

$$L_i^k \circ L = L \quad \text{and} \quad L_i^k(f^2) = L(f^2) + (1 - \phi_i)^k (f^2 - L(f^2)) \tag{7}$$

for all i in D, f in $\mathscr{F}$ and k. Putting $k = k_n$ and $i = s_n$ in (7), and letting n tend to ∞, since $\lim_n (1 - \phi_{s_n})^{k_n} = 0$, we have $\lim_n L_{s_n}^{k_n}(f^2) = L(f^2)$ for every f in $\mathscr{F}$. Therefore, by Proposition 1, we have

$$\lim_n L_{s_n}^{k_n}(g) = L(g) . \tag{8}$$

Therefore, (6) and (8) yield $g = L(g)$, and so we have $L_i(g) = L_i(L(g)) = L(g) = g$ for all i.

We can choose an element f_0 in $\mathscr{F}$ so that $L(f_0^2) \neq f_0^2$. Indeed, if $L(f^2) = f^2$ for all f in $\mathscr{F}$, by Corollary, L must agree with the identity operator, and so $M = C(X)$, which contradicts that M is proper. Now, set $g_0 = f_0^2$. Then g_0 is not in $T[C(X); (L_i)]$ and the equality $\| L_i(g_0) - g_0 \| = \| L(g_0) - g_0 \| \phi_i$ holds for all i.

Finally, since L is a projection of $C(X)$ onto M with $L_i \circ L = L$ for all i, M is contained in $T[C(X); (L_i)]$. On the one hand, from the above argument, it is seen that $T[C(X); (L_i)]$ is contained in M. Thus the theorem is completely proved.

REMARK 2. Our results can be applied in the following situation: $X =$ compact subset of a locally convex Hausdorff space F over the field of real numbers.

$\mathscr{F} = \{f \mid X; f \in F^*\}$, where F^* denotes the dual space of F and $f \mid X$ the restriction of f to X. If X is a compact convex subset of F, then $\mathscr{F}$ can be taken as the space of all real-valued continuous affine functions on X.

3. Saturation and limit of the iterations of the Bernstein-Schnabl functions. In the first place, let us introduce the Bernstein-Schnabl functions, which have been constructed by Grossman [3]. Let F be a locally convex Hausdorff space over the field of real numbers, and let K be a compact convex subset of F. Denote by $A(K)$ the space of all real-valued continuous affine functions on K. For a point x in K, an $A(K)$-representing measure for x is a probability measure μ_x on K such that $f(x) = \int_K f d\mu_x$ for all f in $A(K)$. Let E be a closed subset of K, containing the extreme points of K, and let $\mathscr{U}(E) = \{\mu_x\}_{x \in K}$ be a selection of $A(K)$-representing measures supported by E. Let $P = (p_{nj})_{n,j \geq 1}$ be a lower triangular stochastic matrix, that is, an infinite real matrix satisfying: $p_{nj} \geqq 0$ for all $n \geqq 1$ and $j \geqq 1$, $p_{nj} = 0$ whenever $j > n$, and $\sum_{j \geqq 1} p_{nj} = 1$ for each $n \geqq 1$. Let a g in $C(K)$ be given. Then the n-th Bernstein-Schnabl function of g with respect to the matrix P and the selection $\mathscr{U}(E)$ is defined by:

$$B_{n,P}^{\mathscr{U}(E)}(g)(x) = \int_K g d\pi_{n,P}(\mu_x^{\otimes n}) \text{ for each } x \text{ in } K ,$$

where $\pi_{n,P}: E^n \to K$ is defined by $\pi_{n,P}(x_1, \cdots, x_n) = \sum_{j \geqq 1} p_{nj} x_j$, $\otimes$ denotes tensor product and $\pi_{n,P}(\mu_x^{\otimes n})$ is the induced measure on K.

We suppose now that $\int_K g d\mu_x$ belongs to $A(K)$ for every g in $C(K)$, and for brevity, we write $B_{n,P}^{\mathscr{U}(E)} = B_n$. Then we have the following.

THEOREM 2. *Let B_n^k be the k-th iteration of B_n. Then we have:*

(a) *If n is fixed, then $(B_n^k)_{k \geqq 1}$ converges strongly to B_1.*

(b) *If k is fixed and $\lim_n \sum_{j \geqq 1} p_{nj}^2 = 0$, then $(B_n^k)_{n \geqq 1}$ converges strongly to the identity operator.*

(c) *If $\lim_n \sum_{j \geqq 1} p_{nj}^2 = 0$, then $(B_n)_{n \geqq 1}$ is saturated in $C(K)$ with order $(\sum_{j \geqq 1} p_{nj}^2)_{n \geqq 1}$ and $T[C(K); (B_n)_{n \geqq 1}] = A(K)$.*

PROOF. $A(K)$ is the linear subspace of $C(K)$, separating the points of K and containing 1. It can now be verified that $B_n \circ B_1 = B_1$, $B_n(f) = f$ and $B_n(f^2) = f^2 + \sum_{j \geqq 1} p_{nj}^2 (B_1(f^2) - f^2)$ for all $n \geqq 1$ and f in $A(K)$. Furthermore, an induction argument reveals that $B_n^k \circ B_1 = B_1$, $B_n^k(f) = f$ and $B_n^k(f^2) = B_1(f^2) + (1 - \sum_{j \geqq 1} p_{nj}^2)^k (f^2 - B_1(f^2))$ for all $k, n \geqq 1$ and f in $A(K)$ (cf. (7) in the proof of Theorem 1). Therefore, (a) and (b) follow from Propositions 1 and 2, respectively. Since B_1 is a positive projection of $C(K)$ onto $A(K)$, (c) follows from Theorem 1.

REMARK 3. Let (k_n) be a sequence of positive integers. Then, from the proofs of Theorems 1 and 2, we see:

(d) $\lim_n k_n \sum_{j \geqq 1} p_{nj}^2 = 0$ if and only if $\lim_n \| B_n^{k_n}(g) - g \| = 0$ for all g in $C(K)$.

(e) $\lim_n k_n \sum_{j \geqq 1} p_{nj}^2 = +\infty$ if and only if $\lim_n \| B_n^{k_n}(g) - B_1(g) \| = 0$ for all g in $C(K)$.

Theorem 2 should also be compared with the results of the author [4].

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