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HELSON-SZEGÖ-SARASON THEOREM FOR DIRICHLET ALGEBRAS

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Two linear subspaces in a Hilbert space are said to be at positive angle if

$$\sup |(f, g)| < 1$$

where f and g range over the elements of the respective linear subspaces with norm at most 1. Let μ be a finite positive Borel measure on the unit circle T in the complex plane. Let χ be the function on T defined by $\chi(e^{ix}) = e^{ix}$. For each integer n, we form in the Hilbert space $L^2(d\mu)$ the linear subspace $\mathscr{F}_n$ spanned by the functions $\chi^n, \chi^{n+1}, \cdots$. For any set S of complex-valued functions, write $\bar{S} = \{\bar{f} \mid f \in S\}$.

Helson and Szegö [6] and Helson and Sarason [5] have proved the following result.

THEOREM 1. *Let n be a natural number. In order for $\mathscr{F}_0$ and $\bar{\mathscr{F}}_n$ to be at positive angle in $L^2(d\mu)$ it is necessary and sufficient that $d\mu$ is absolutely continuous with respect to the Lebesgue measure $d\theta$ on T, strictly so that $d\mu = wd\theta$, $w \in L^1(d\theta)$, and w has the form*

$$w = |P|^2 e^{r+Cs} ,$$

where P is a polynomial of degree less than n, r is a real bounded function, and Cs is the conjugate of a real function s with bound smaller than $\pi/2$.

For $n = 1$, Devinatz [1, 2] and Ohno [9] extended this result to a Dirichlet algebra setting. See also Merrill [7] as regards another result of this prediction type. The main object of this note is to show that the result is valid for a general natural number n.

Let X be a compact Hausdorff space and let A be a Dirichlet algebra on X, i.e., A is a uniform algebra on X such that the real parts of the functions in A are uniformly dense in the real continuous functions on X. Let m be the unique representing measure on X for a complex homomorphism of A. We assume that the Gleason part of m contains more than one point. If $0<p<\infty$, H^p shall be the closure of A in $L^p(dm)$ and H^∞ shall be the weak*-closure of A in $L^\infty(dm)$. We put $A_0=\left\{f\in A\,\middle|\,\int f dm=0\right\}$ and $H_0^p=\left\{f\in H^p\,\middle|\,\int f dm=0\right\}$ $(1\leqq p\leqq\infty)$. We denote by $(H_0^\infty)^n$(resp. A_0^n) the ideal generated by products of n elements in H_0^∞(resp. A_0). We shall make use of the theory of the abstract Hardy spaces. We refer to Gamelin [4] in this connection.

Let ν be a positive finite measure on X and consider a condition for A and $\overline{A_0^n}$ to be at positive angle in $L^2(d\nu)$. For the sake of simplicity we shall, at the outset, assume that $d\nu$ is absolutely continuous with respect to dm, i.e., $d\nu=wdm$, where w is a non-negative function in $L^1(dm)$.

For $n=1,2,\cdots$, we define

$$\rho_n=\sup\left|\int fgwdm\right|,$$

where f and g range over the elements of A and A_0^n, respectively, subject to the restriction

$$\int|f|^2wdm\leqq 1\quad\text{and}\quad\int|g|^2wdm\leqq 1\,.$$

PROPOSITION 2. *If* $\rho_n<1$, *then* $\int\log wdm>-\infty$.

PROOF. We denote the norm in $L^p(wdm)$ by $\|\cdot\|_p$. Suppose that $\int\log wdm=-\infty$. By Szegö's theorem ([4], Theorem 8.2),

$$\inf\left\{\int|1-f|^pwdm\,|\,f\in A_0\right\}=0$$

for $0<p<\infty$. We consider the case $p=4n$ and choose $f_k\in A_0$ such that $\|f_k-1\|_{4n}\to 0$ $(k\to\infty)$. Then there exists a constant M such that $\|f_k\|_{4n}\leqq M$ $(k=1,2,\cdots)$. By Hölder's inequality and Minkowski's inequality,

$$\begin{aligned}&\int|f_k^n-1|^2wdm\\&\quad\leqq\left\{\int|f_k-1|^4wdm\right\}^{1/2}\left\{\int|f_k^{n-1}+f_k^{n-2}+\cdots+1|^4wdm\right\}^{1/2}\\&\quad\leqq\|f_k-1\|_4^2\{\|f_k^{n-1}\|_4+\|f_k^{n-2}\|_4+\cdots+\|1\|_4\}^2\\&\quad\leqq\|f_k-1\|_4^2\{KM^{n-1}+K^2M^{n-2}+\cdots+K^n\}^2\end{aligned}$$

where $K=\left\{\int wdm\right\}^{1/4n}$. It follows that $\|f_k^n-1\|_2\to 0(k\to\infty)$. This shows that 1, hence any constant function lies in the $L^2(wdm)$-closure of A_0^n. It follows that $\rho_n = 1$. This contradicts the hypothesis and we have $\int \log w dm > -\infty$.

Thus, in order to characterize w with $\rho_n < 1$, we may as well assume, from the outset, that $\int \log w dm > -\infty$. In this case, it is easy to see that

$$\rho_n = \sup \left| \int f g w dm \right|$$

where f and g range over the elements of H^∞ and $(H_0^\infty)^n$, respectively, subject to the restriction

$$\int |f|^2 w dm \leqq 1 \quad \text{and} \quad \int |g|^2 w dm \leqq 1 .$$

We begin by following an idea of Helson and Szegö [6]. Such a function w is of the form $|h|^2$, h being an outer function in H^2 ([3], Theorem 6). Define the function $e^{-i\phi}$ by equation $w = h^2 e^{-i\phi}$. Then ρ_n is given by

$$(1) \qquad \rho_n = \sup \left| \int (fh)(gh) e^{-i\phi} dm \right|$$

where the supremum is taken over all $f \in H^\infty$ and $g \in (H_0^\infty)^n$ such that

$$\int |fh|^2 dm \leqq 1 \quad \text{and} \quad \int |gh|^2 dm \leqq 1 .$$

By Wermer's embedding theorem ([4], Theorem 7.2), there exists an inner function Z in H^∞ such that $H_0^\infty = ZH^\infty$. Then $(H_0^\infty)^n = Z^n H^\infty$ and we have

$$(2) \qquad \rho_n = \sup \left| \int (fh)(gh) Z^n e^{-i\phi} dm \right|$$

where f and g range over the elements of H^∞ subject to the respective restriction

$$\int |fh|^2 dm \leqq 1 \quad \text{and} \quad \int |gh|^2 dm \leqq 1 .$$

Since h is outer in H^2, $\{fh \mid f \in H^\infty\}$ is dense in H^2 and more specifically $\left\{fh \mid f \in H^\infty, \int |fh|^2 dm \leqq 1\right\}$ is dense in the unit ball of H^2. Thus $\left\{fgh^2 \,\middle|\, f, g \in H^\infty, \int |fh|^2 dm \leqq 1, \int |gh|^2 dm \leqq 1\right\}$ is dense in the unit ball of H^1(cf. [2], Lemma 6). Therefore (2) can be written in the form

$$(3) \qquad \rho_n = \sup \left| \int f Z^n e^{-i\phi} dm \right|$$

where f ranges over the functions in H^1 such that $\int |f| dm \leqq 1$. Evidently (3) expresses ρ_n as the norm of the linear functional on H^1 defined by

$$(4) \qquad \int f Z^n e^{-i\phi} dm$$

for $f \in H^1$. By the Hahn-Banach theorem, this linear functional has a norm-preserving extension to the whole $L^1(dm)$, Since $L^1(dm)^* = L^\infty(dm)$, we can identify the above extension with $Z^n e^{-i\phi} - g_0 \in L^\infty(dm)$. In this case $g_0 \in H_0^\infty$, because

$$\int f Z^n e^{-i\phi} dm = \int f(Z^n e^{-i\phi} - g_0) dm \quad (f \in A)$$

and so $\int f g_0 dm = 0$ $(f \in A)$, implying $g_0 \in H_0^\infty$. Furthermore, for every $g \in H_0^\infty$, $Z^n e^{-i\phi} - g$ gives an extension of the linear functional given by (4) to the whole $L^1(dm)$ and so

$$\rho_n = \| Z^n e^{-i\phi} - g_0 \| \leqq \| Z^n e^{-i\phi} - g \| \quad (g \in H_0^\infty)$$

where $\| \cdot \|$ denotes the norm in $L^\infty(dm)$. It follows that

$$(5) \qquad \rho_n = \inf_{g \in H_0^\infty} \| Z^n e^{-i\phi} - g \| = \inf_{F \in H^\infty} \| 1 - Z^{1-n} e^{i\phi} F \| .$$

PROPOSITION 3. *$\rho_n < 1$ if and only if for some $\varepsilon > 0$ and $F \in H^\infty$, we have*

$$(6) \qquad |F| > \varepsilon$$

$$(7) \qquad |\operatorname{Arg}(F h^2 Z^{1-n})| < \pi/2 - \varepsilon$$

where $-\pi \leqq \operatorname{Arg} z < \pi$.

PROOF. If $\rho_n < 1$, take $\varepsilon > 0$ such that $\rho_n \leqq 1 - 2\varepsilon$. Then by (5), there exists an $F \in H^\infty$ such that $\| 1 - Z^{1-n} e^{i\phi} F \| < 1 - \varepsilon$. This implies $|F| > \varepsilon$, and then (7) is geometrically obvious, perhaps with a smaller value of ε.

Conversely if F satisfies (6) and (7), then it is easy to see that

$$\| Z^{n-1} e^{-i\phi} - \lambda F \| < 1$$

for some $\lambda > 0$, and so $\rho_n < 1$.

PROPOSITION 4. *If $F \in H^1$ and*

$$(8) \qquad |\operatorname{Arg}(F Z^{-k})| < \pi/2 - \varepsilon$$

then there exist an integer $m(0 \leqq m \leqq k)$ *and* $B \in H^1$ *such that* $\int B dm \neq 0$ *and* $F = Z^m B$.

PROOF. Since $H_0^1 = ZH^1$, it suffices to show that if $F = Z^k B$ and $B \in H^1$, then $\int B dm \neq 0$. By (8),

$$|\operatorname{Arg} B| = |\operatorname{Arg}(FZ^{-k})| < \pi/2 - \varepsilon$$

and we have $\operatorname{Re} B \geqq 0$. Since $B \in H^1$, it follows from Theorem 12 of Devinatz [2] that B is outer in H^1. Hence we have $\int B dm \neq 0$.

We denote by $\mathscr{H}^p$ the closure in $L^p(dm)$ of the set of polynomials in Z and denote by $\mathscr{L}^p$ the closure in $L^p(dm)$ of the set of polynomials in Z and $\bar{Z}$ (the norm closure for $1 \leqq p < \infty$; the weak*-closure for $p = \infty$). For $1 \leqq p \leqq \infty$, we put

$$I^p = \left\{ f \in H^p \,\middle|\, \int f \bar{Z}^k dm = 0 \ \ (k = 0, 1, 2, \cdots) \right\}.$$

LEMMA 5. (Merrill and Lal [8], Lemma 5.) *If* $1 \leqq p \leqq \infty$, *then*

$$H^p = \mathscr{H}^p \oplus I^p$$
$$L^p = \mathscr{L}^p \oplus N^p$$

where $\oplus$ *denotes the algebraic direct sum and* N^p *denotes the closure of* $\overline{I^p} + I^p$ *in* $L^p(dm)$ (*the norm closure for* $1 \leqq p < \infty$; *the weak*-closure for* $p = \infty$).

THEOREM 6. *In order for* A *and* $\overline{A_0^n}$ *to be at positive angle in* $L^2(wdm)$, *it is necessary and sufficient that* w *has the form*

$$w = |P|^2 e^{r + Cs} \tag{9}$$

where P *is a function in* H^∞ *such that* $P \perp A_0^n$ *in* $L^2(dm)$, $r, s \in L_R^\infty(dm)$, $\|s\| < \pi/2$ *and* Cs *is the conjugate of* s.

PROOF. We assume $\rho_n < 1$. By Proposition 3, there exist $\varepsilon > 0$ and $F \in H^\infty$ such that $|F| > \varepsilon$ and

$$|\operatorname{Arg}(Fh^2 Z^{1-n})| < \pi/2 - \varepsilon\,. \tag{10}$$

Let s be the function bounded by $\pi/2 - \varepsilon$ such that

$$s + \operatorname{Arg}(Fh^2 Z^{1-n}) = 0\,. \tag{11}$$

We put

$$S = Fh^2 Z^{1-n} e^{-Cs + is}\,, \tag{12}$$

then, by (11), $S \geqq 0$. From Theorem 10 of [2], we conclude that $e^{-Cs+is} \in H^1$ is outer. By (10) and Proposition 4, we may write $Fh^2 = Z^m B$, where $B \in H^1$, $\int B dm \neq 0$ and $0 \leqq m \leqq n-1$. Therefore

$$S = BZ^{-k}e^{-Cs+is} \geqq 0 \tag{13}$$

and so

$$Z^k S = Be^{-Cs+is} \in H^{1/2} \tag{14}$$

where $k = n - m - 1$. Furthermore, by Jensen's inequality,

$$\begin{aligned} \int \log |Z^k S| dm &= \int \log |B| dm + \int \log |e^{-Cs+is}| dm \\ &\geqq \log \left| \int B dm \right| + \log \left| \int e^{-Cs+is} dm \right| > -\infty \end{aligned}$$

and so we have

$$\exp \int \log |Z^k S| dm > 0 \, . \tag{15}$$

Using Theorem 2 of [3], it follows from (14) and (15) that there exist an outer function P in H^1 and an inner function q in H^∞ such that

$$Z^k S = qP^2 \, . \tag{16}$$

Since $S = |S|$ and $|S| = |P|^2$, we have from (16) that

$$qP^2 = Z^k |P|^2 \, . \tag{17}$$

Since P is outer, it follows that P is not zero. Thus we may divide (17) by P and we obtain

$$qP = Z^k \bar{P} \, . \tag{18}$$

By Lemma 5, we may write

$$P = \sum_{j=0}^{\infty} a_j Z^j + \alpha_I \in \mathscr{H}^1 \oplus I^1$$

where α_I belongs to I^1. Now

$$\begin{aligned} Z^k \bar{P} &= \bar{a}_0 Z^k + \bar{a}_1 Z^{k-1} + \cdots + \bar{a}_{k-1} Z + \bar{a}_k \\ &\quad + \bar{a}_{k+1} \bar{Z} + \bar{a}_{k+2} \bar{Z}^2 + \cdots + Z^k \bar{\alpha}_I \, . \end{aligned} \tag{19}$$

Because $a_{k+1} Z + a_{k+2} Z^2 + \cdots \in H_0^1$ and $\bar{Z}^k \alpha_I \in I^1 \subset H_0^1$, we have

$$g = a_{k+1} Z + a_{k+2} Z^2 + \cdots + \bar{Z}^k \alpha_I \in H_0^1 \, .$$

By (18), $Z^k \bar{P} \in H^1$ and we conclude $\bar{g} \in H^1$ by (19). Hence $g \in \overline{H^1} \cap H_0^1$. Since $\bar{A} + A_0$ is weak*-dense in $L^\infty(dm)$, we have $g = 0$ and

$$Z^k\bar{P} = \bar{a}_0 Z^k + \bar{a}_1 Z^{k-1} + \cdots + \bar{a}_{k-1}Z + \bar{a}_k .$$

Hence P has the form

$$P = a_0 + a_1 Z + \cdots + a_k Z^k$$

where $0 \leqq k \leqq n-1$. Therefore $P \in H^\infty$ and $P \perp A_0^n$ in $L^2(dm)$. Indeed, if $G \in A_0^n \subset (H_0^\infty)^n$, then $G = Z^n K$ for some $K \in H^\infty$ and we have

$$(P, G) = \int\Bigl(\sum_{j=0}^k a_j Z^j\Bigr)\bar{Z}^n \bar{K} dm = \sum_{j=0}^k a_j \int \bar{Z}^{n-j}\bar{K} dm$$
$$= \sum_{j=0}^k a_j \int \bar{Z} dm \int \bar{Z}^{n-j-1}\bar{K} dm = 0 ,$$

since m is multiplicative on H^∞ and $n-1 \geqq k$. Now by (16) and (12) we have

$$|P|^2 = S = |S| = |F|\,|h|^2 e^{-Cs}$$

and since $w = |h|^2$,

$$w = |P|^2 |F|^{-1} e^{Cs} = |P|^2 e^{r+Cs}$$

where $r = -\log|F|$. In this case $r, s \in L_R^\infty(dm)$ and $\|s\| < \pi/2$.

Conversely, suppose w has the form (9). We put $S = |P|^2$. Since $Z^{n-1}Pf \in (H_0^\infty)^n$ for $f \in I^\infty$, we have

$$\int Z^{n-1} S f dm = (Z^{n-1}Pf, P) = 0 \qquad (f \in I^\infty) . \tag{20}$$

If $f \in I^\infty$, then it is easy to see that $\bar{Z}^{2(n-1)}f$ is also in I^∞. Therefore, by (20),

$$\int \bar{Z}^{n-1} S f dm = \int Z^{n-1} S \bar{Z}^{2(n-1)} f dm = 0 \qquad (f \in I^\infty) .$$

Since $S = \bar{S}$,

$$\int Z^{n-1} S \bar{f} dm = 0 \qquad (f \in I^\infty) . \tag{21}$$

It follows from (20) and (21) that

$$\int Z^{n-1} S f dm = 0 \qquad (f \in \overline{I^\infty} \oplus I^\infty) .$$

By Lemma 5, $Z^{n-1}S \in \mathscr{L}^1$. Furthermore, we have

$$\int Z^{n-1} S \bar{Z}^k dm = \begin{cases} (Z^{n-1-k}P, P) = 0 \ (n-1-k \geqq n, \text{ i.e., } k = -1, -2, \cdots) \\ (P, Z^{k+1-n}P) = 0 \ (k+1-n \geqq n, \text{ i.e., } k = 2n-1, 2n, \cdots) . \end{cases}$$

We conclude that $Z^{n-1}S$ has the form

$$Z^{n-1}S = a_0 + a_1 Z + \cdots + a_{2n-2} Z^{2n-2} .$$

We put

$$k = \max\{m \mid 0 \leqq m \leqq n-1,\ a_{m+n-1} \neq 0\} .$$

Since $S \neq 0$ and $\bar{S} = S$, such k exists. Then $Z^k S \in H^\infty$ and $\int Z^k S dm \neq 0$, therefore by Theorem 2 of [3], $Z^k S$ has the factoring

$$Z^k S = qG^2$$

where q is inner and G is outer in H^∞. If we take an outer function F in H^∞ such that $|F| = e^{-r}$, then

$$Z^k \bar{q} S e^{Cs - is} = Fh^2 , \tag{22}$$

up to constant factors of modulus 1. Indeed, by Theorem 10 of [2], e^{Cs-is} is outer in H^1, and $Z^k \bar{q} S = G^2$ is also outer in H^∞, so that the left hand side of (22) is outer in H^1. Furthermore, since F is outer in H^∞ and h is outer in H^2, the right hand side of (22) is also outer in H^1. Now by the assumption on w

$$|Z^k \bar{q} S e^{Cs-is}| = Se^{Cs} = |P|^2 e^{Cs} = we^{-r} = |h|^2 |F| = |Fh^2| .$$

Since an outer function is determined up to a constant factor by its modulus, (22) follows. By (22), $S = Fh^2 Z^{-k} q e^{-Cs+is}$ and $S \geqq 0$, it follows that

$$\operatorname{Arg}(Fh^2 Z^{-k} q e^{-Cs+is}) = 0 .$$

Hence for sufficiently small $\varepsilon > 0$,

$$|\operatorname{Arg}(Fh^2 Z^{-k} q) = |s| \leqq ||s|| < \pi/2 - \varepsilon$$

and

$$|F| = e^{-r} > \varepsilon .$$

If we put $B = FqZ^{n-1-k}$, then $B \in H^\infty$ and

$$|\operatorname{Arg}(Bh^2 Z^{1-n})| = |\operatorname{Arg}(Fh^2 Z^{-k} q)| < \pi/2 - \varepsilon$$
$$|B| = |F| > \varepsilon .$$

The assertion follows from Proposition 3.

COROLLARY. *In order for A and $\overline{A_0^n}$ to be at positive angle in $L^2(wdm)$, it is necessary and sufficient that w has the form*

$$w = |P|^2 e^{r+Cs}$$

where P is a polynomial in Z of degree less than n, $r, s \in L_R^\infty(dm)$, $||s|| < \pi/2$ and Cs is the conjugate of s.

EXAMPLE. Put $S = \{(k, l) \in \boldsymbol{Z}^2 | k > 0\} \cup \{(0, l) \in \boldsymbol{Z}^2 | l \geqq 0\}$. Let $A = A(T^2)$ be the Dirichlet algebra of continuous functions on T^2 which are uniform limits of polynomials in $e^{ikx}e^{ily}$ where $(k, l) \in S$. Let m denote the normalized Haar measure on T^2. Then the Gleason part of m can be identified with $\{(0, \alpha) \in \boldsymbol{C}^2 \mid |\alpha| < 1\}$ and is non-trivial. Wermer's embedding function Z is given by $Z(e^{ix}, e^{iy}) = e^{iy}$. In this case, A_0^n is the uniformly closed linear span of $\{e^{ikx}e^{i(l+n)y} | (k, l) \in S\}$ and the function P in Theorem 6 is a polynomial in e^{iy} of degree less than n.

REMARK. We used the setting such that A is a Dirichlet algebra and the Gleason part of the unique representing measure m is non-trivial since it is easier to work with. However, similar proofs will show that all results are valid for a setting such that A is a weak*-Dirichlet algebra on a given probability measure space (X, m) and there exists a non-zero weak*-continuous multiplicative linear functional on A which is different from dm (for the relevant definition, see, Srinivasan and Wang [10]).

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