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MULTIPLICITY OF HELICES OF A SPECIAL FLOW

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0. The purpose of this note is to show that the multiplicity of helices of a special flow is equal to that of helices of the basic automorphism.

1. Throughout this note $(\Omega, \mathscr{F}, P)$ denotes a complete and separable probability space. An automorphism T of Ω is a one-to-one transformation of Ω onto itself which is bimeasurable and measure-preserving. A flow $\{T_t, -\infty < t < +\infty\}$ on Ω is a one-parameter group of automorphisms of Ω; $T_t T_s = T_{t+s}$, $-\infty < t, s < +\infty$.

As a special type of flows, which we deal with later, we define the following: Let θ be an integrable function on Ω, bounded below by some positive constant. Define a new probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P})$ by

$$\tilde{\Omega} = \{\tilde{\omega} = (\omega, u); \omega \in \Omega, 0 \leqq u < \theta(\omega)\} ,$$

$$d\tilde{P}(\tilde{\omega}) = \frac{1}{E(\theta)} du dP(\omega) ,$$

$$\tilde{\mathscr{F}} = \text{the completion of} \quad \mathscr{F} \times \mathscr{B}^1|_{\tilde{\Omega}}$$

where $\mathscr{B}^1$ is the σ-field of Lebesgue measurable sets of $\mathscr{R}^1$ and du is the Lebesgue measure. It is also a complete and separable probability space. For an automorphism T of Ω, a flow $\{S_t, -\infty < t < +\infty\}$ on $\tilde{\Omega}$ is defined by

$$S_t(\omega, u) = \begin{cases} (\omega, u + t) & \text{for} \quad 0 \leqq t < \theta(\omega) - u \\ (T\omega, 0) & \text{for} \quad t = \theta(\omega) - u \end{cases}$$

and for other value of t, the automorphism S_t is defined by the group property. The flow $\{S_t\}$ is called *a special flow* with the ceiling function θ, the basic space Ω and the basic automorphism T.

In this note, we deal with a pair $(\{T_t\}, \mathscr{F}_0)$ of a flow $\{T_t\}$ on Ω and a complete sub-σ-field $\mathscr{F}_0$ of $\mathscr{F}$ which satisfies

(a) $\mathscr{F}_0 \subset T_t \mathscr{F}_0$ for all $t > 0$,

(b) $\bigvee_{-\infty < t < +\infty} T_t \mathscr{F}_0 = \mathscr{F}$.

The pair is called *a system* on Ω. If $\mathscr{F}_0$ is a proper sub-σ-field, the system is said to be non-trivial. It is well-known that there is always a proper sub-σ-field $\mathscr{F}_0$ with (a) and (b) for a flow with a positive entropy (cf. [1], [2]).

Also for an automorphism T of Ω, a system $(T, \mathscr{F}_0)$ is similarly defined.

Let $(T, \mathscr{F}_0)$ be a non-trivial system and $\{S_t\}$ a special flow with the basic automorphism T and the ceiling function θ which is measurable with respect to $\mathscr{F}_0$. Let $\tilde{\mathscr{F}}_0$ denote the completion of $\mathscr{F}_0 \times \mathscr{B}^1|_{\tilde{\Omega}}$. Then $(\{S_t\}, \tilde{\mathscr{F}}_0)$ is obviously a non-trivial system. We denote it by $(\{S_t\}, \tilde{\mathscr{F}}_0, T, \mathscr{F}_0, \theta)$ and call it *a special system* and $(T, \mathscr{F}_0)$ the basic system.

2. Let $(\{T_t\}, \mathscr{F}_0)$ be a non-trivial system on Ω. Let us denote by $\mathscr{H} = L_0^2(\Omega)$ a Hilbert space of all squarely integrable real random variables with zero-expectations. For each t, $-\infty < t < +\infty$, let $\mathscr{H}_t$ be the subspace of $\mathscr{H}$ consisting of all elements measurable with respect to $T_t\mathscr{F}_0$. We assume that the unitary operators of $\mathscr{H}$ defined by $x \mapsto x \circ T_t^{-1}$ for $x \in \mathscr{H}$ are strongly continuous.

DEFINITION 2.1 ([3]). A process $X = (x_t)$, $-\infty < t < +\infty$, is called a helix with orthogonal increments, or simply an HOI, if the following conditions are satisfied:

(a) $x_0 = 0$ and trajectories are right-continuous,

(b) $x_t - x_s \in \mathscr{H}_t$ for any s, t, $-\infty < s < t < +\infty$,

(c) $x_t - x_s \in \mathscr{H}_s^{\perp}$ for any s, t, $-\infty < s < t < +\infty$ where $\perp$ indicates the orthogonal complement in $\mathscr{H}$,

(d) $(x_t - x_s) \circ T_u^{-1} = x_{t+u} - x_{s+u}$ for any s, t, u, $-\infty < s, t, u < +\infty$.

Note that any HOI $X = (x_t)$ has the property of a martingale, namely, $(x_{t+s} - x_s, T_{t+s}\mathscr{F}_0)$, $t \geqq 0$, is a squarely integrable martingale for fixed s, $-\infty < s < +\infty$. Thus by Doob-Meyer decomposition theorem for martingales, there is a unique adapted process $\langle X \rangle = (\langle X \rangle_t)$, $-\infty < t < +\infty$, so that $(\langle X \rangle_t)$, $t \geqq 0$, is previsible with respect to $(T_t\mathscr{F}_0)$, $t \geqq 0$, and $(x_t^2 - \langle X \rangle_t, T_t\mathscr{F}_0)$, $t \geqq 0$, is a martingale. We call $\langle X \rangle$ *an increasing helix* of X. It has the following properties:

(a) $\langle X \rangle_0 = 0$ and trajectories are right-continuous and increasing,

(b) $\langle X \rangle_t - \langle X \rangle_s$ is measurable with respect to $T_t\mathscr{F}_0$ for any s, t, $-\infty < s < t < +\infty$, and integrable,

(c) $(\langle X \rangle_t - \langle X \rangle_s) \circ T_u^{-1} = \langle X \rangle_{t+u} - \langle X \rangle_{s+u}$ for any s, t, u, $-\infty < s, t, u < +\infty$.

For HOI's X and X', we put

$$\langle X, X' \rangle_t = \frac{1}{2}(\langle X + X' \rangle_t - \langle X \rangle_t - \langle X' \rangle_t) .$$

If $X = X'$, we have clearly $\langle X, X\rangle = \langle X\rangle$.

DEFINITION 2.2. Two HOI's X and X' are said to be strictly orthogonal if $\langle X, X'\rangle = (\langle X, X'\rangle_t)$ vanishes.

Also for a non-trivial system $(T, \mathscr{F}_0)$ of discrete time, the HOI and others are similarly defined. They are considerably simplified as follows. Any HOI $X = (x_i)$ can be written as

$$x_i = \sum_{k=0}^{i-1} x \circ T^{-k} \quad (i > 0)$$

for some $x \in \mathscr{H}_1 \cap \mathscr{H}_0^{\perp}$ and the increasing helix of X is

$$\langle X\rangle_i = \sum_{k=0}^{i-1} E[x^2 | \mathscr{F}_0] \circ T^{-k} \quad (i > 0)\,.$$

Thus two HOI's X and X' for $(T, \mathscr{F}_0)$ are strictly orthogonal if $\langle X, X'\rangle_1 = (\langle X + X'\rangle_1 - \langle X\rangle_1 - \langle X'\rangle_1)/2$ vanishes, where $\langle X\rangle_1 = E[x^2 | \mathscr{F}_0]$.

For a special flow, the following result was obtained by J. de Sam Lazaro. Any HOI $\tilde{X} = (\tilde{x}_t)$ for a special system $(\{S_t\}, \tilde{\mathscr{F}}_0, T, \mathscr{F}_0, \theta)$ can be written in the form:

$$\tilde{x}_t(\omega, u) = \sum_{k=0}^{\infty} x(T^{-k}\omega) 1_{\{R_k \leq t\}}(\omega, u) \quad (t > 0)$$

for some $x \in \mathscr{H}_1 \cap \mathscr{H}_0^{\perp}$ in the basis, where

$$R_k(\omega, u) = \begin{cases} u & (k = 0) \\ \sum_{j=1}^{k} \theta(T^{-j}\omega) + u & (k > 0) \end{cases}.$$

We note that any HOI $\tilde{X}$ corresponds uniquely to an HOI X for the basic system, associated to x. When another HOI $\tilde{X}'$ is given similarly with x' in the place of x, then $\tilde{X}$ and $\tilde{X}'$ are strictly orthogonal if and only if $E[xx' | \mathscr{F}_0] = 0$. Further, the increasing helix $\langle \tilde{X}\rangle$ of $\tilde{X}$ is given by

$$\langle \tilde{X}\rangle_t(\omega, u) = \sum_{k=0}^{\infty} E[x^2 | \mathscr{F}_0](T^{-k}\omega) 1_{\{R_k \leq t\}}(\omega, u) \quad (t > 0)\,.$$

3. We now define the multiplicity of helices for a system and show that the multiplicity of a special system coincides with that of the basic system.

Let $(\{T_t\}, \mathscr{F}_0)$ be a non-trivial system and $\mathscr{G}_0$ a sub-σ-field of $\mathscr{F}_0$ consisting of all $A \in \mathscr{F}_0$ such that the process $(1_A \circ T_t^{-1})$, $t \geq 0$, is previsible with respect to $(T_t \mathscr{F}_0)$, $t \geq 0$.

DEFINITION 3.1. For HOI's X and X' for $(\{T_t\}, \mathscr{F}_0)$, let $\mu_{\langle X, X'\rangle}$ be a measure on $(\Omega, \mathscr{G}_0)$ such that

$$\mu_{\langle X,X'\rangle}(A) = E\left[\int_0^1 1_A \circ T_t^{-1} d\langle X, X'\rangle_t\right] \quad \text{for} \quad A \in \mathscr{G}_0 .$$

Clearly, $\mu_{\langle X,X'\rangle}$ is a finite measure. If X and X' are strictly orthogonal, $\mu_{\langle X,X'\rangle}$ is a null measure, that is, $\mu_{\langle X,X'\rangle}(A) = 0$ for any $A \in \mathscr{G}_0$.

LEMMA 3.1. *For any positive number α and $A \in \mathscr{G}_0$, we have*

$$\mu_{\langle X\rangle}(A) = \frac{1}{\alpha} E\left[\int_0^\alpha 1_A \circ T_t^{-1} d\langle X\rangle_t\right] .$$

PROOF. If we put

$$f(\alpha) = E\left[\int_0^\alpha 1_A \circ T_t^{-1} d\langle X\rangle_t\right] ,$$

then $f(\alpha)$ is an increasing function and for $\alpha, \beta > 0$

$$\begin{aligned} f(\alpha+\beta) &= E\left[\int_0^{\alpha+\beta} 1_A \circ T_t^{-1} d\langle X\rangle_t\right] \\ &= E\left[\int_0^\alpha 1_A \circ T_t^{-1} d\langle X\rangle_t\right] + E\left[\int_\alpha^{\alpha+\beta} 1_A \circ T_t^{-1} d\langle X\rangle_t\right] \\ &= f(\alpha) + f(\beta) \end{aligned}$$

by the stationarity of the increments of $\langle X\rangle$. Thus we obtain

$$f(\alpha) = \alpha f(1) .$$

For an HOI, we can define a concept similar to the stochastic integral by the martingale.

DEFINITION 3.1. For any HOI $X = (x_t)$ for $(\{T_t\}, \mathscr{F}_0)$ and a squarely integrable random variable ν on $(\Omega, \mathscr{G}_0, \mu_{\langle X\rangle})$, we set a new HOI $Y = (y_t)$ by

$$y_t = \int_0^t \nu \circ T_s^{-1} dx_s \quad (t > 0) ,$$

where this integral means the stochastic integral by the martingale. Denote Y by $\nu * X$ and call it a stochastic integral by an HOI X.

By the definition, for any HOI X',

$$\langle \nu * X, X'\rangle_t = \int_0^t \nu \circ T_s^{-1} d\langle X, X'\rangle_s .$$

Thus we see easily that

$$d\mu_{\langle \nu * X, X'\rangle} = \nu d\mu_{\langle X,X'\rangle}$$

and

$$d\mu_{\langle \nu * X \rangle} = \nu^2 d\mu_{\langle X \rangle}$$

on the sub-σ-field $\mathscr{G}_0$.

Conversely, applying a theorem of projection for martingales, we have the following.

LEMMA 3.2. *Let X be an* HOI. *For any* HOI Y, *there exists a squarely integrable random variable ν on $(\Omega, \mathscr{G}_0, \mu_{\langle X \rangle})$ such that*

$$\langle Y, X \rangle = \langle \nu * X, X \rangle ,$$

and so we have

$$d\mu_{\langle Y, X \rangle} = \nu d\mu_{\langle X \rangle} .$$

Thus for HOI's X *and* Y, *the measure* $\mu_{\langle Y, X \rangle}$ *is absolutely continuous with respect to* $\mu_{\langle X \rangle}$ *and* ν *is the Radon-Nikodym derivative.*

Now we can state a representation theorem for HOI's of a system.

THEOREM 3.1. *For any non-trivial system* $(\{T_t\}, \mathscr{F}_0)$, *there exists a finite or countable sequence of strictly orthogonal* HOI's $\mathscr{X} = (X^{(n)})$ *such that for any* HOI X, *there exist stochastic integrals* $\nu^{(n)} * X^{(n)}$ *with*

$$X = \sum_n \nu^{(n)} * X^{(n)}$$

where

$$\mu_{\langle X \rangle}(\Omega) = \sum_n \int_\Omega \nu^{(n)2} d\mu_{\langle X^{(n)} \rangle} < +\infty$$

and $\mu_{\langle X^{(n)} \rangle} \succ \mu_{\langle X^{(n+1)} \rangle}$ *for all* n, *where* $\succ$ *denotes the relation of absolute continuity of measures. If another sequence* $\mathscr{Y} = (Y^{(n)})$ *is also one stated above, then* $\mu_{\langle X^{(n)} \rangle} \sim \mu_{\langle Y^{(n)} \rangle}$ *for all* n, *where* $\sim$ *denotes the relation of equivalence of measures.*

DEFINITION 3.3. The length of such a sequence as in Theorem 3.1 is called the multiplicity of the system $(\{T_t\}, \mathscr{F}_0)$ and is denoted by $M(\{T_t\}, \mathscr{F}_0)$.

For an HOI X for a system $(T, \mathscr{F}_0)$, we can also define a helix-transform $\nu * X$ of X, which corresponds to a martingale-transform, by a random variable $\nu \in L^2(\Omega, \mathscr{F}_0, \mu_{\langle X \rangle})$ and so a projection of HOI. A theorem of the same type as Theorem 3.1 for $(T, \mathscr{F}_0)$ was given in [4]. Theorem 3.1 can be proved by the same method as in [4].

Now we are in the position to state the main theorem in this note.

THEOREM 3.2. *The multiplicity of a special system is equal to that of the basic system.*

PROOF. Let $(\{S_t\}, \tilde{\mathscr{F}}_0, T, \mathscr{F}_0, \theta)$ be a special system. We consider the sets in $\tilde{\mathscr{F}}_0$ of the following type:

$$\tilde{A} = A \times \mathscr{R}^1|_{\tilde{\Omega}} \quad \text{for some} \quad A \in \mathscr{F}_0 .$$

Since the process $(1_{\tilde{A}} \circ S_t^{-1})$, $t \geqq 0$, has left-continuous paths, we have $\tilde{A} \in \tilde{\mathscr{G}}_0$. Let X be an HOI for the special system

$$\tilde{X}_t(\omega, u) = \sum_{k=0}^{\infty} x(T^{-k}\omega) 1_{\{R_k \leqq t\}}(\omega, u) \quad (t > 0) .$$

Then, for any $\tilde{A}$ of the above type,

$$\mu_{\langle\tilde{X}\rangle}(\tilde{A}) = \frac{1}{\alpha}\int_\Omega dP(\omega)\int_0^{\theta(\omega)}\left[\int_0^\alpha 1_{\tilde{A}} \circ S_t^{-1}(\omega, u) d\langle\tilde{X}\rangle_t\right]du$$

by Lemma 3.1. Let α be sufficiently small. If $0 \leqq t \leqq \alpha$, then

$$\langle\tilde{X}\rangle_t = E[x^2 | \mathscr{F}_0] 1_{\{R_0 \leqq t\}}$$

and so

$$\begin{aligned}\int_0^\alpha 1_{\tilde{A}} \circ S_t^{-1}(\omega, u) d\langle\tilde{X}\rangle_t(\omega, u) &= 1_{\tilde{A}} \circ S_{R_0}^{-1}(\omega, u)(E[x^2 | \mathscr{F}_0] 1_{\{R_0 \leqq \alpha\}})(\omega, u) \\ &= 1_{\tilde{A}} \circ S_u^{-1}(\omega, u)(E[x^2 | \mathscr{F}_0] 1_{\Omega \times [0,\alpha]})(\omega, u) \\ &= 1_{\tilde{A}}(\omega, 0)(E[x^2 | \mathscr{F}_0](\omega) 1_{\Omega \times [0,\alpha]})(\omega, u) \\ &= E[x^2 | \mathscr{F}_0](\omega) 1_{A \times [0,\alpha]}(\omega, u) .\end{aligned}$$

Hence

$$\mu_{\langle\tilde{X}\rangle}(\tilde{A}) = \frac{1}{\alpha}\int_A dP \int_0^\alpha E[x^2 | \mathscr{F}_0] du = \int_A x^2 dP .$$

Thus, if we denote by X the corresponding HOI for the basic system $(T, \mathscr{F}_0)$ associated to x, we have

$$\mu_{\langle\tilde{X}\rangle}(\tilde{A}) = \mu_{\langle X\rangle}(A) .$$

Consequently, if $\tilde{X}$ and $\tilde{X}'$ are HOI's for the special system such that $\mu_{\langle\tilde{X}\rangle} \succ \mu_{\langle\tilde{X}'\rangle}$, then we have $\mu_{\langle X\rangle} \succ \mu_{\langle X'\rangle}$, where X and X' are the corresponding HOI's for the basic system.

Let $\tilde{\mathscr{X}} = (\tilde{X}^{(n)})$ be a sequence of HOI's for a special system $(\{S_t\}, \tilde{\mathscr{F}}_0, T, \mathscr{F}_0, \theta)$ in the Theorem 3.1 and $\mathscr{X} = (X^{(n)})$ the corresponding HOI's for the basic system $(T, \mathscr{F}_0)$. We have seen that $\mu_{\langle X^{(n)}\rangle} \succ \mu_{\langle X^{(n+1)}\rangle}$ for all n. By the result of Sam Lazaro stated in Section 2, $\mathscr{X}$ is maximal and so any HOI for the basic system is represented by $\mathscr{X}$. Thus we have

$$M(\{S_t\}, \tilde{\mathscr{F}}_0, T, \mathscr{F}_0, \theta) = M(T, \mathscr{F}_0) .$$

4. We now apply the preceding result to a class of special flows. Let $(T, \mathscr{A})$ be a B-system, *i.e.*, (a) $\bigvee_i T^i \mathscr{A} = \mathscr{F}$ and (b) $\{T^i \mathscr{A}; -\infty < i < +\infty\}$ is an independent sequence. Putting $\mathscr{A}_0 = \bigvee_{i<0} T^i \mathscr{A}$, we obtain a K-system $(T, \mathscr{A}_0)$, *i.e.*, $\bigcap_i T^i \mathscr{A}_0 =$ trivial.

Let $(\{S_t\}, \tilde{\mathscr{A}_0}, T, \mathscr{A}_0, \theta)$ be a special system constructed by the basic system $(T, \mathscr{A}_0)$ whose ceiling function θ is measurable with respect to $\mathscr{A}$. If θ is not lattice-distributed, then the special system is a K-system, *i.e.*, $\bigcap_t S_t \tilde{\mathscr{A}_0} =$ trivial ([5]).

In [4] we proved that the multiplicity of the system $(T, \mathscr{A}_0)$ is equal to the dimension of the subspace of $\mathscr{H}$ consisting of all elements measurable with respect to $\mathscr{A}$. Thus the multiplicity of the special K-system $(\{S_t\}, \tilde{\mathscr{A}_0}, T, \mathscr{A}_0, \theta)$ is equal to the dimension of the subspace of $\mathscr{H}$ mentioned above.

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REFERENCES

[1] V. A. ROHLIN AND JA. G. SINAI, Construction and properties of invariant measurable partitions, Dokl. Akad. Nauk. SSSR, 141 (1962), 1038-1041.

[2] M. S. PINSKER, Dynamical systems with completely positive or zero entropy, Dokl. Akad. Nauk. SSSR, 133 (1960), 1025-1026.

[3] J. DE SAM LAZARO AND P. A. MEYER, Méthodes de martingales et théorie de flots, Z. Wahrsch. verw. Geb., 18 (1971), 116-140.

[4] T. SHIMANO, An invariant of systems in the ergodic theory, Tohoku Math. Journ., 30 (1978), 337-350.

[5] H. TOTOKI, On a class of special flows, Z. Wahrsch. verw. Geb., 15 (1970), 157-167.

[6] J. DE SAM LAZARO, Sur les helices du flot special sous une fonction, Z. Wahrsch. verw. Geb., 30 (1974), 279-302.

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