Tôhoku Math. Journ.
32 (1980), 375-392.

FOLIATED COBORDISMS OF SUSPENDED FOLIATIONS

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(Received May 10, 1979, revised September 21, 1979)

Introduction. Let $(M^n, \mathscr{F})$ be a codimension q foliation on a closed oriented n-manifold M^n. Then the suspended foliation $\Sigma_f(M^n, \mathscr{F})$ is obtained from $(M^n, \mathscr{F}) \times [0, 1]$ by the identification of $(M^n, \mathscr{F}) \times \{0\}$ with $(M^n, \mathscr{F}) \times \{1\}$ by means of a foliation preserving diffeomorphism f. There are interesting examples of foliations of this type. For example, let $\mathscr{F}$ be a codimension 1 foliation on T^{n+1} whose leaves are transverse to the fibers of the canonical fibration $S^1 \to T^{n+1} = S^1 \times T^n \to T^n$. Then we can construct $\mathscr{F}$ from mutually commuting automorphisms $f_1, \cdots, f_n$ of S^1 (cf. Herman [5]). We denote $(T^{n+1}, \mathscr{F})$ by $\mathscr{F}(f_1, \cdots, f_n)$. Then it is easy to see that $\mathscr{F}(f_1, \cdots, f_n)$ is a suspended foliation of $\mathscr{F}(f_1, \cdots, \hat{f}_i, \cdots, f_n)$ by f_i, where the "hat" means that the term is left out.

Recently, Herman [5] and Morita-Tsuboi [22] proved that the Godbillon-Vey class of $\mathscr{F}(f_1, \cdots, f_n)$ is zero. Considering the conjecture that the map $\mathrm{GV}: \mathscr{F}\Omega_1(3) \to \boldsymbol{R}$ given by the Godbillon-Vey number is injective, we may ask if $\mathscr{F}(f_1, \cdots, f_n)$ is foliated null-cobordant. However, this seems to be very difficult even for the case $\mathscr{F}(f, g)$ on T^3 (see Tsuboi [23]). Moreover, $\Sigma_f(M, \mathscr{F})$ may not be null-cobordant in general even if $(M, \mathscr{F})$ is null-cobordant and $f \in \mathrm{LD}(M, \mathscr{F})$, where $\mathrm{LD}(M, \mathscr{F})$ is the group of all leaf preserving diffeomorphisms of $(M, \mathscr{F})$. We will give such an example in §6. But it seems to be natural to conjecture that $\Sigma_f(M, \mathscr{F})$ is null-cobordant for $f \in \mathrm{FD}(M, \mathscr{F})_0$, where $\mathrm{FD}(M, \mathscr{F})_0$ is the identity component of the group of all foliation preserving diffeomorphisms of $(M, \mathscr{F})$, because the elements in $\mathrm{FD}(M, \mathscr{F})_0$ are considerably restricted (see Lemma 10 in §4).

In this paper we consider this problem and verify the above conjecture for some codimension 1 foliations, i.e., in §3 for the Reeb foliation $(S^3, \mathscr{F}_R)$ and a modified Reeb foliation $(S^3, \bar{\mathscr{F}}_R)$, in §4 for the foliation $(S^3, \mathscr{F}_a)$ with the Godbillon-Vey number of $(S^3, \mathscr{F}_a) = a \neq 0$ constructed by Thurston [20], and in §2 for the foliation defined by a non-vanishing smooth closed 1-form. Concerning the last foliation, we will show that $\Sigma_f(M, \mathscr{F})$ is null-cobordant for $f \in \mathrm{FD}(M, \mathscr{F}) \cap \mathrm{Diff}^\infty_+(M)_0$. These results give some information on the relation between $\Sigma_f(M^3, \mathscr{F})$

and $\Sigma_g(s(M^3), s_\alpha(\mathscr{F}))$, where $(s(M^3), s_\alpha(\mathscr{F}))$ is obtained from $(M^3, \mathscr{F})$ by a foliated surgery (cf. Oshikiri [14]). These results also enable us to consider the problem for the foliation $(M^3, \mathscr{F}_S)$ obtained from a spinnable structure (cf. Fukui [2]). These are considered in §5. In §6, we will give some remarks on $\mathscr{F}(f_1, \cdots, f_n)$.

The author wishes to thank Professors H. Sato, T. Mizutani and T. Nishimori for helpful comments during the preparation of this paper.

1. Notations and results. In this paper we consider only C^∞ foliations and maps.

Let $(M_1^n, \mathscr{F}_1)$ and $(M_2^n, \mathscr{F}_2)$ be codimension q foliations on oriented closed n-manifolds M_1^n and M_2^n. Then $(M_1^n, \mathscr{F}_1)$ and $(M_2^n, \mathscr{F}_2)$ are said to be *foliated cobordant* or simply *cobordant* and denoted by $(M_1^n, \mathscr{F}_1) \sim (M_2^n, \mathscr{F}_2)$, if there exists a codimension q foliation $(N^{n+1}, \mathscr{F})$ of a compact oriented $(n+1)$-manifold N^{n+1} transverse to the boundary such that $\partial N = M_1 \cup (-M_2)$, $(M_1, \mathscr{F} | M_1) = (M_1, \mathscr{F}_1)$ and $(-M_2, \mathscr{F} | -M_2) = -(M_2, \mathscr{F}_2)$, where "$-$" means that the orientation of the manifold is reversed.

Let $\mathscr{F}\Omega_q(n)$ be the set of cobordism classes of codimension q foliations on closed oriented n-manifolds. Then $\mathscr{F}\Omega_q(n)$ becomes an abelian group under disjoint union (cf. Lawson [9]).

Let $(M^n, \mathscr{F})$ be a codimension q foliation on a closed oriented n-manifold M^n. We consider the following groups: $\mathrm{FD}(M, \mathscr{F})$ is the group of all diffeomorphisms of M which preserve the orientation of M and the foliation $\mathscr{F}$. $\mathrm{LD}(M, \mathscr{F})$ is the subgroup of $\mathrm{FD}(M, \mathscr{F})$ consisting of those which leave each leaf of $\mathscr{F}$ invariant. We also denote $\mathrm{FD}(M, \mathscr{F})$ (resp. $\mathrm{LD}(M, \mathscr{F})$) by FD (resp. LD) if there is no danger of confusion. We denote the identity component of $\mathrm{FD}(M, \mathscr{F})$ (resp. $\mathrm{LD}(M, \mathscr{F})$) by $\mathrm{FD}(M, \mathscr{F})_0$ (resp. $\mathrm{LD}(M, \mathscr{F})_0$).

REMARK. (i) When we refer to the topologies of $\mathrm{FD}(M, \mathscr{F})$ and $\mathrm{LD}(M, \mathscr{F})$, we always consider the C^∞-topologies induced by that of $\mathrm{Diff}^\infty(M)$.

(ii) We may assume a path in $\mathrm{FD}(M, \mathscr{F})$ or $\mathrm{LD}(M, \mathscr{F})$ to be smooth (cf. Leslie [10]).

DEFINITION. For each $(M^n, \mathscr{F}^q)$ and $f \in \mathrm{FD}(M, \mathscr{F})$ we define a foliated $(n+1)$-manifold $\Sigma_f(M, \mathscr{F})$ as follows: The $(n+1)$-manifold is defined by $M \times [0, 1]/(x, 1) \sim (f(x), 0) \equiv \Sigma_f M$ and the foliation is defined by $\mathscr{F} \times [0, 1]/(x, 1) \sim (f(x), 0)$, i.e.,

$$\bigcup_{m=-\infty}^{\infty} f^m(L) \times [0, 1]/(f^m(x), 1) \sim (f^{m+1}(x), 0)\,, \qquad x \in L$$

is a leaf for each $L \in \mathscr{F}$. We call this codimension q foliation a suspended foliation of $(M, \mathscr{F})$ by f. We also denote $\Sigma_f(M, \mathscr{F})$ by $\Sigma(f)$ if there is no danger of confusion.

Define a map $S\colon \mathrm{FD}(M^n, \mathscr{F}^q) \to \mathscr{F}\Omega_q(n+1)$ by

$$S(f) = [\Sigma_f(M, \mathscr{F})] = \text{the cobordism class of } \Sigma_f(M, \mathscr{F})\,.$$

PROPOSITION. *S is a homomorphism, i.e., $S(f \circ g) = S(f) + S(g)$. In particular, the kernel* $\mathrm{Ker}(S)$ *of S contains the commutator subgroup. Moreover, we have* $\mathrm{Ker}(S) \supset \mathrm{LD}_0$.

PROOF. Consider a foliated manifold $(M, \mathscr{F}) \times [0, 1] \times [0, 1]$ with a corner. Identify $(x, s, 1)$ with $(f \circ g(x), s, 0)$ for each $s \in [0, 1/4]$, and identify $(x, s, 1)$ with $(g(x), s, 0)$ for each $s \in [3/4, 1]$. By "straightening the angle", we have the desired foliated cobordism. The second part of Proposition follows from the result stated in Remark (ii) above, because we can construct smooth concordance between $\Sigma_f(M, \mathscr{F})$ and $\Sigma_{\mathrm{id}}(M, \mathscr{F})$ for $f \in \mathrm{LD}_0$. q.e.d.

Here are some examples of $\Sigma_f(M, \mathscr{F})$.

EXAMPLE 1. We consider the trivial foliation of M^n, i.e., $\mathscr{F} = \{x\}_{x \in M}$. Then we have

(a) If $f \in \mathrm{LD}_0$, then $\Sigma(f) \sim 0$. This follows from Proposition and the fact that $\mathrm{FD}_0 = \mathrm{Diff}^\infty_+(M)_0$ and $\mathrm{Diff}^\infty_+(M)_0$ is perfect (cf. Mather [12]).

(b) There is a non-trivial example (cf. §6). Let $M = \boldsymbol{C}P^2$ and $f\colon \boldsymbol{C}P^2 \to \boldsymbol{C}P^2$ be defined by $f([z_0, z_1, z_2]) = [\bar{z}_0, \bar{z}_1, \bar{z}_2]$ in homogeneous coordinates. Then $\Sigma_f \boldsymbol{C}P^2$ is a generator of $\Omega_5 \cong \boldsymbol{Z}_2$. Hence we have $\Sigma_f(\boldsymbol{C}P^2, \mathscr{F}) \not\sim 0$ in $\mathscr{F}\Omega_4(5)$.

EXAMPLE 2. Let $(M, \mathscr{F})$ be obtained from orbits of an Anosov flow with an integral invariant (see Leslie [11]). If $f \in \mathrm{FD}_0$, then $\Sigma_f(M, \mathscr{F}) \sim 0$. This follows from Proposition and the fact that $\mathrm{FD}_0 = \mathrm{LD}_0$ (Leslie [11]).

Now we state our results. In the following we consider only codimension 1 foliations.

THEOREM 1. *Let $(M, \mathscr{F})$ be a foliation defined by a nonvanishing smooth closed 1-form. If $f \in \mathrm{FD} \cap \mathrm{Diff}^\infty_+(M)_0$, then $S(f) = 0$. In particular, $S(\mathrm{FD}_0) = 0$.*

Let $(S^3, \mathscr{F}_R)$ be a Reeb foliation (cf. Lawson [9]). Replace the unique toral leaf $T^2 \in \mathscr{F}_R$ by toral leaves $T^2 \times \{t\}$, $t \in [0, 1]$. Then we have a modified Reeb foliation $(S^3, \bar{\mathscr{F}}_R)$.

THEOREM 2. *If f is in* $\mathrm{FD}(S^3, \mathscr{F}_R)_0$ *or in* $\mathrm{FD}(S^3, \bar{\mathscr{F}}_R)_0$, *then* $S(f) = 0$.

THEOREM 3. *Let* $(S^3, \mathscr{F}_a)$ *be the foliation with* $gv(\mathscr{F}_a) = a \neq 0$ *constructed by Thurston* [20]. *Then* $S(f) = 0$ *for* $f \in \mathrm{FD}_0$.

THEOREM 4. *Let* $(s(M^3), s_\alpha(\mathscr{F}))$ *be the foliation obtained from* $(M^3, \mathscr{F})$ *by a surgery (cf. Oshikiri* [14]). *If we use* $(S^3, \mathscr{F}_R)$, $(S^3, \bar{\mathscr{F}}_R)$ *or* $(S^3, \mathscr{F}_a)$ *in this surgery, then for each* $f \in \mathrm{FD}(s(M), s_\alpha(\mathscr{F}))_0$ *there exists a* $g \in \mathrm{FD}(M, \mathscr{F})_0$ *such that* $\Sigma_f(s(M), s_\alpha(\mathscr{F})) \sim \Sigma_g(M, \mathscr{F})$.

COROLLARY. *If we have* $\mathrm{FD}(M^3, \mathscr{F})_0 = \mathrm{LD}(M^3, \mathscr{F})_0$, *then for any* $(s(M^3), s_\alpha(\mathscr{F}))$ *in Theorem* 4 *and any* $f \in \mathrm{FD}(s(M^3), s_\alpha(\mathscr{F}))_0$ *we have* $\Sigma_f(s(M), s_\alpha(\mathscr{F})) \sim 0$. *In particular, let* $(M^3, \mathscr{F})$ *be an Anosov foliation (cf. Rosenberg-Thurston* [16], *Part* II). *Then* $\Sigma_f(s(M^3), s_\alpha(\mathscr{F})) \sim 0$ *for* $f \in \mathrm{FD}(s(M^3), s_\alpha(\mathscr{F}))_0$.

Now we generalize Theorem 2 as follows:

THEOREM 5. *Let* $(M^3, \mathscr{F}_S)$ *be a foliation obtained from a spinnable structure (cf. Fukui* [2]). *Then* $S(f) = 0$ *for* $f \in \mathrm{FD}_0$.

2. Proof of Theorem 1. Let $B_n \subset \mathscr{F}\Omega_1(n)$ be the subgroup generated by foliations defined by non-vanishing closed 1-forms. Then we have the following theorem by Thurston [21] and Koschorke [2] (see also Reinhart [15]).

THEOREM B. *If* $n \not\equiv 1 (\mathrm{mod}\, 4)$, *then*

$$B_n \cong \hat{\Omega}_n \equiv \mathrm{Ker}(\mathrm{sign}\colon \Omega_n \to \boldsymbol{Z}) ,$$

where $\mathrm{sign}(M)$ *means the signature of* M, *and*

$$B_{4k+1} \cong \Omega_{4k+1} \oplus \boldsymbol{Z}_2 .$$

In particular, for $[(M, \mathscr{F})] \in B_n$, *we have* $(M, \mathscr{F}) \sim 0$ *if and only if either* $\dim M = n \neq 4k+1$ *and* $M \sim 0$ *in* Ω_n, *or* $\dim M = 4k+1$ *and* $\chi(W)$ *is even, where* W *gives* $M \sim 0$ *in* Ω_{4k+1}.

If $(M, \mathscr{F})$ is defined by a non-vanishing smooth closed 1-form, then the following two cases occur (see Imanishi [6]):

(i) All leaves of $\mathscr{F}$ are dense in M.

(ii) $\mathscr{F}$ is induced from a fibration $p\colon M \to S^1$, i.e., $\mathscr{F}$ is a bundle foliation.

First we consider the case (i).

LEMMA 1. *Let* $f \in \mathrm{FD}$ *and* f *be homotopic to* id_M. *If* $\mathscr{F}$ *is defined by a closed 1-form* ω, *then* $f^*\omega = \omega$.

PROOF. $f \cong \mathrm{id}_M$ implies $[f^*\omega] = [\omega] \in H^1_{\mathrm{DR}}(M)$, i.e.,

$$(1) \qquad f^*\omega = \omega + dh, \text{ for some } h \in C^\infty(M) .$$

For each $x \in M$ and $v \in T_x\mathscr{F}$, $f \in \mathrm{FD}$ implies $f_*v \in T_{f(x)}\mathscr{F}$. Thus we have $f^*\omega(v) = \omega(f_*v) = 0$. From (1) follows $f^*\omega(v) = \omega(v) + dh(v) = v(h)$, i.e., $v(h) = 0$ for each $x \in M$ and $v \in T_x\mathscr{F}$. This means that $h|L$ is constant for each $L \in \mathscr{F}$. On the other hand, we have $\bar{L} = M$ by the hypothesis. Thus it is seen that h is a constant function and (1) becomes $f^*\omega = \omega$. q.e.d.

PROOF OF THEOREM 1: *The case* (i). Let $\pi: M \times [0, 1] \to M$ be the projection to the first factor. Then $(M, \mathscr{F}) \times [0, 1]$ is defined by the closed 1-form $\pi^*\omega$, and by Lemma 1, $\pi^*\omega$ is preserved by the identification $(x, 1) \sim (f(x), 0)$. Thus $\Sigma_f(M, \mathscr{F})$ is also defined by a closed 1-form. By the hypothesis $f \cong \mathrm{id}_M$, $\Sigma_f M$ is diffeomorphic to $M \times S^1$ and $\partial(M \times D^2) = M \times S^1$ with $\chi(M \times D^2) = \chi(M) = 0$. Hence we have $\Sigma_f(M, \mathscr{F}) \sim 0$ by Theorem B above, regardless of the dimension of M. q.e.d.

Next we consider the case (ii). Note that in this case $\Sigma_f(M, \mathscr{F})$ cannot always be defined by a closed 1-form.

Let $(M, \mathscr{F})$ be defined by a fibration $p: M \to S^1$ with the fiber F. Then $M = F \times [0, 1]/(x, 1) \sim (\varphi(x), 0)$ for some $\varphi \in \mathrm{Diff}^\infty_+(F)$. We set $\mathrm{FD}(M, \mathscr{F}) \cap \mathrm{Diff}^\infty_+(M)_0 = G$ and $\mathrm{Diff}^\infty_+(S^1) = \Gamma$. As $f \in G$ is a bundle map, we can define a homomorphism $\alpha: G \to \mathrm{Diff}^\infty(S^1)$ so that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{f} & M \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ S^1 & \xrightarrow{\alpha(f)} & S^1 . \end{array}$$

LEMMA 2. *$\alpha(f)$ is orientation preserving, i.e., $\alpha(f) \in \Gamma$.*

PROOF. As $\alpha(f)$ is a diffeomorphism of S^1, we have only to show that $[\alpha(f)^*d\theta] = [d\theta]$ in $H^1_{\mathrm{DR}}(S^1)$, where $d\theta$ is a volume element of S^1. As the above diagram commutes, we have $p^*\alpha(f)^*d\theta = f^*p^*d\theta$. By the hypothesis $f \cong \mathrm{id}_M$ we have $p^*[\alpha(f)^*d\theta] = p^*[d\theta]$ in $H^1_{\mathrm{DR}}(M)$. On the other hand, by considering a cross-section $s: S^1 \to M$, we can show that p^* is injective. This completes the proof.

LEMMA 3. *For each $f \in \mathrm{Ker}(\alpha)$, we have $\Sigma_f(M, \mathscr{F}) \sim 0$.*

PROOF. By the hypothesis $f \cong \mathrm{id}_M$ we have $\Sigma_f M = M \times S^1$. Thus, if we show that $\Sigma_f(M, \mathscr{F})$ is a bundle foliation, we can prove this lemma by the same argument as in the case (i). From the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & M \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ S^1 & \xrightarrow{\mathrm{id}} & S^1 , \end{array}$$

we have the following commutative diagram:

$$\begin{array}{ccccc} M \times [0, 1] & \xrightarrow{\pi} & M & \xrightarrow{p} & S^1 \\ \downarrow{\scriptstyle /f} & & & & \| \\ \Sigma_f M & & \xrightarrow{q} & & S^1 . \end{array}$$

This diagram gives a fibration of $\Sigma_f M$ over S^1. A leaf of $\Sigma_f(M, \mathscr{F})$ is of the form $\cdots \cup p^{-1}(\alpha(f)^{-1}(\theta)) \times [0, 1] \cup p^{-1}(\theta) \times [0, 1] \cup p^{-1}(\alpha(f)(\theta)) \times [0, 1] \cup \cdots / \sim\ = p^{-1}(\theta) \times S^1 = q^{-1}(\theta)$, with $\theta \in S^1$. Here we also use $\alpha(f) = \mathrm{id}_{S^1}$. This completes the proof.

LEMMA 4. *The map α is surjective. Further, let φ be the map mentioned above and ψ be the diffeomorphism of M induced by $\varphi \times \mathrm{id}_I$: $F \times I \to F \times I$. Then we can define a homomorphism β: $\Gamma \to G/\langle\psi\rangle$ with $\bar{\alpha} \circ \beta = \mathrm{id}$ on Γ, where $\langle\psi\rangle$ is the normal subgroup of G generated by ψ and $\bar{\alpha}$ is the map $G/\langle\psi\rangle \to \Gamma$ induced by α.*

PROOF. We consider the universal covering $p: \boldsymbol{R} \to S^1 = \boldsymbol{R}/\boldsymbol{Z}$ of S^1. For $g \in \Gamma$, we lift it onto $\tilde{g}: \boldsymbol{R} \to \boldsymbol{R}$. Then we have $\tilde{g}(t + n) = \tilde{g}(t) + n$, for $t \in \boldsymbol{R}$ and $n \in \boldsymbol{Z}$. Consider $M = F \times \boldsymbol{R}/\sim$, where $(\varphi^n(x), t) \sim (x, t + n)$. Define $g' \in G$ by $g'(x, t) = (x, \tilde{g}(t))$. The well-definedness follows from the following diagram:

$$\begin{array}{ccc} (\varphi^n(x), t) & \sim & (x, t + n) \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g'} \\ (\varphi^n(x), \tilde{g}(t)) & \sim (x, \tilde{g}(t) + n) = & (x, \tilde{g}(t + n)) . \end{array}$$

It is easy to see that $g' \in G$. The surjectivity of α follows from this. Also it is easy to see that the above construction gives a homomorphism $\beta: \Gamma \to G$ up to $\langle\psi\rangle$, i.e., $\beta(g) = [g']$. q.e.d.

PROOF OF THEOREM 1: *The case* (ii). Note that $\langle\psi\rangle \subset \mathrm{Ker}(\alpha)$. Then by the above lemmas we have a split exact sequence

$$1 \to \mathrm{Ker}(\alpha)/\langle\psi\rangle \hookrightarrow G/\langle\psi\rangle \underset{\beta}{\overset{\bar{\alpha}}{\rightleftarrows}} \Gamma \to 1 .$$

By Lemma 3, we have $\Sigma_f(M, \mathscr{F}) \sim 0$ for $f \in \mathrm{Ker}(\alpha)$. Thus we have only to show that $\Sigma_f(M, \mathscr{F}) \sim 0$ for each $f \in \beta(\Gamma)$. On the other hand, we know that Γ is perfect, i.e., $\Gamma = [\Gamma, \Gamma]$ (see Herman [4]), and that $\Sigma([g, h]) \sim 0$, where $[g, h] = g \circ h \circ g^{-1} \circ h^{-1}$. Hence $\beta(\Gamma) = [\beta(\Gamma), \beta(\Gamma)]$ implies $\Sigma_f(M, \mathscr{F}) \sim 0$ for $f \in \beta(\Gamma)$. This completes the proof.

3. Proof of Theorem 2. First recall some definitions (cf. Oshikiri [14]).

Let $(M^{n+1}, \mathscr{F})$ be a codimension 1 foliation and $\varphi\colon S^1 \to M^{n+1}$ be an imbedding transverse to $\mathscr{F}$. We assume that $\varphi(S^1)$ has the trivial tubular neighborhood $S^1 \times D^n$. Then the *σ-modification* $\sigma_\varphi(\mathscr{F})$ of $\mathscr{F}$ along φ is defined as follows. Wind the leaves of $\mathscr{F}$ along $S^1 \times \partial D(1/2)^n \subset S^1 \times D^n$, where $D(1/2)^n$ is the n-dimensional disk with radius 1/2 and is concentric with the unit disk D^n. Then we have a foliation on $M^{n+1} - \mathrm{int}(S^1 \times D(1/2)^n)$ with $S^1 \times \partial D(1/2)^n$ as a leaf. Consider a Reeb component on $S^1 \times D(1/2)^n$. These foliations give a foliation on M^{n+1}. We denote this by $\sigma_\varphi(\mathscr{F})$. It is clear that $(M^{n+1}, \mathscr{F})$ is concordant to $(M^{n+1}, \sigma_\varphi(\mathscr{F}))$. Moreover, if R is a rotation along the S^1-factor of this new Reeb component $(S^1 \times D(1/2)^n, \mathscr{F}_R)$, then R can be extended to this concordance so that the extension gives no effect on $(M^{n+1}, \mathscr{F})$. The inverse of the σ-modification is called the σ^*-modification.

Let $\mathscr{F}_0$ and $\mathscr{F}_1$ be the foliations on $S^1 \times [0, 1]$ defined by the following 1-forms:

$$\mathscr{F}_0;\ \omega_0 = (1 - 2r)dr + h(r)d\theta\ ,$$
$$\mathscr{F}_1;\ \omega_1 = dr + h(r)d\theta\ ,$$

where $(\theta, r) \in S^1 \times [0, 1]$ and h is a smooth function of $[0, 1]$ into $\boldsymbol{R}$ such that $h(0) = h(1) = 0$ and $h(r) > 0$ for $0 < r < 1$. Here we choose $h(r)$ so that the holonomy groups of the leaves $S^1 \times \partial[0, 1]$ in $\mathscr{F}_0$ and $\mathscr{F}_1$ are infinitely tangent to the identity map and that $\mathscr{F}_0$ and $\mathscr{F}_1$ are invariant under rotations along the S^1-factor. To perform the *r_t-surgery* on $(M^{n+1}, \mathscr{F})$ we assume that there exists an imbedding $\varphi\colon S^1 \times S^{n-1} \times [0, 1] \to M^{n+1}$ satisfying $\mathscr{F}|_{\mathrm{Im}(\varphi)} = S^{n-1} \times (S^1 \times [0, 1], \mathscr{F}_t)$, for $t = 0$ or 1. We define $r(M^{n+1})$ to be the $(n + 1)$-manifold obtained from $(M^{n+1} - \mathrm{int}\,\varphi(S^1 \times S^{n-1} \times [0, 1])) \cup S^1 \times D^n \times S^0$ by the canonical identification of $\varphi(S^1 \times S^{n-1} \times S^0)$ with $S^1 \times \partial D^n \times S^0$. We consider foliations $\mathscr{F} \,|\, M^{n+1} - \mathrm{int}\,\varphi(S^1 \times S^{n-1} \times [0, 1])$ and $(S^1 \times D^n, \mathrm{Reeb}) \times S^0$, where oriented circles of two Reeb components coincide if $t = 0$ and are opposite if $t = 1$. This gives a foliation on $r(M^{n+1})$. We denote this foliation by $r_t(\mathscr{F})$. We say that the resulting foliated manifold $(r(M), r_t(\mathscr{F}))$ is obtained by the r_t-surgery. In the following, we also use the inverse of this surgery. We call it the r_t^*-surgery.

LEMMA 5. *$(M^{n+1}, \mathscr{F}) \sim (r(M^{n+1}), r_t(\mathscr{F}))$ for $t = 0, 1$. Moreover, we can extend rotations of $S^{n-1} \times (S^1 \times [0, 1], \mathscr{F}_t)$ along the S^1-factor to the foliated cobordism between $(M, \mathscr{F})$ and $(r(M), r_t(\mathscr{F}))$.*

PROOF. See [14]. The second part is clear by the construction of the foliated cobordism. q.e.d.

PROOF OF THEOREM 2: *The case $(S^3, \mathscr{F}_R)$.* The following theorem of Fukui-Ushiki [3] is essential to our proof.

THEOREM (Fukui-Ushiki [3]). *The sequence*

$$1 \to \mathrm{LD}\,(S^3, \mathscr{F}_R)_0 \to \mathrm{FD}(S^3, \mathscr{F}_R)_0 \underset{\gamma}{\rightleftarrows} S^1 \times S^1 \to 1$$

is split exact, where we regard $S^1 \times S^1$ as rotations of the Reeb component $(S^1 \times D^2$, Reeb$)$ along $S^1 \times \{0\} \subset S^1 \times D^2$.

REMARK. In the proof of [3, Theorem], the authors proved $\mathrm{LD} = \mathrm{LD}_0$. Thus [3, Lemma 1] can be stated as above.

Using this theorem and Proposition, we have only to show that $\Sigma(\gamma(S^1 \times S^1)) \sim 0$. Note that even the rotations cannot be extended to the foliated null-cobordism of $(S^3, \mathscr{F}_R)$ constructed by Mizutani and Sergeraert [17]. We denote $f \in \gamma(S^1 \times S^1)$ by (a, b), where a and $b \in S^1$. By Proposition, we have $\Sigma(f) = \Sigma(a, b) \sim \Sigma(a, 0) + \Sigma(0, b)$. We will show that $\Sigma(0, b) \sim 0$. The same proof gives $\Sigma(a, 0) \sim 0$. Let $(S^1 \times [0, 2], \mathscr{F}')$ be the foliation obtained from $(S^1 \times [0, 1], \mathscr{F}_1) \cup (S^1 \times [0, 1], \mathscr{F}_1)$ by the canonical identification of $S^1 \times \{0\}$ with $S^1 \times \{1\}$. We can regard $(S^1 \times [0, 2], \mathscr{F}')$ as a foliation on $S^1 \times [0, 1]$ by $S^1 \times [0, 2] \ni (x, 2t) \leftrightarrow (x, t) \in S^1 \times [0, 1]$. We denote this foliation by $\mathscr{F}_2$.

LEMMA 6. *$(S^1 \times [0, 1], \mathscr{F}_0) \times S^1$ is foliated cobordant to $(S^1 \times [0, 1], \mathscr{F}_2) \times S^1$ relative to the boundary. Moreover, we can extend rotations along the first S^1-factor to this cobordism.*

PROOF. By Lemma 5, we have only to show that the foliated cobordism is obtained by r_t-surgeries and σ-modifications. Proceed as follows with attention to the orientations of Reeb components. First note that $(S^1 \times I, \mathscr{F}_0) \times S^1 \sim (S^1 \times I, \sigma_\varphi(\mathscr{F}_0)) \times S^1$, where $I = [0, 1]$ and φ: $S^1 \to S^1 \times I$ is given by $\varphi(t) = \{t\} \times \{1/2\}$. Performing the r_0-surgery on the second foliation, we have $(S^1 \times D^2, \sigma_\varphi(\mathrm{Reeb})) \cup (S^1 \times D^2, \sigma_\psi(\mathrm{Reeb}))$, where φ and ψ: $S^1 \to S^1 \times D^2$ are given by $\varphi(t) = \{t\} \times \{0\}$, $\psi(t) = \{t\} \times \{0\}$ and $(S^1 \times D^2$, Reeb$)$ means a Reeb component. Consider a foliation $(S^1 \times S^2, \mathscr{F}') = (S^1 \times D^2, \mathrm{Reeb}) \cup (S^1 \times D^2, \mathrm{Reeb})$. This is clearly null-cobordant.

Performing σ-modifications along two $S^1 \times \{0\}$'s, we have a foliation $(S^1 \times S^2, \mathscr{F}'')$. This foliation has two Reeb components. Take one of them and perform the r_0^*-surgery using this Reeb component and that of $(S^1 \times D^2, \sigma_\varphi(\text{Reeb}))$. Again perform the r_0^*-surgery on the rest of Reeb components. Then we have a foliation on $S^1 \times I \times S^1$, which is clearly equivalent to $\sigma_{\varphi'}(\sigma_\varphi(\mathscr{F}_2)) \times S^1$, where $\varphi: S^1 \to S^1 \times I$ is given by $\varphi(t) = \{t\} \times \{1/4\}$ and $\varphi': S^1 \to S^1 \times I$ is given by $\varphi'(t) = \{t\} \times \{3/4\}$. q.e.d.

Let $\mathscr{F}$ be a foliation on T^{n+1} transverse to the fibers of the canonical fibration $S^1 \to T^{n+1} = S^1 \times T^n \to T^n$. Then we can construct $\mathscr{F}$ from mutually commuting diffeomorphisms $f_1, \cdots, f_n$ of S^1 (cf. Herman [5]). Thus we may write $(T^{n+1}, \mathscr{F})$ as $\mathscr{F}(f_1, \cdots, f_n)$. Set $\operatorname{supp}(f_i) = s_i \subset S^1$, where $\operatorname{supp}(f)$ means the support of f. Then we have the following

LEMMA 7 (see also [23]). *If there is an $i \in \{1, \cdots, n\}$ such that $\operatorname{int}(s_i) \cap \operatorname{int}(s_j) = \emptyset$ for $j \neq i$ and that the number of the connected components of s_i is finite, then $\mathscr{F}(f_1, \cdots, f_n) \sim 0$.*

PROOF. The case $n = 1$ was dealt with in Example 1(a) in §1. If $n \geqq 2$, we may assume $i = 1$. If $s_1 = S^1$, then by the hypothesis we have $f_2 = \mathrm{id}_{S^1}$, and $\mathscr{F}(f_1, \mathrm{id}_{S^1}, \cdots) \sim 0$ is clear. Thus we may also assume $s_1 \neq S^1$. Further, we may assume that the number of the connected components of s_1 is one (repeat the following argument N-times if s_1 has N components). Note that f_1 on ∂s_1 is infinitely tangent to the identity, i.e., $j^\infty(f_1) = j^\infty(\mathrm{id})$ at ∂s_1(see Sergeraert [17]), so we can consider $f_1 | s_1 \in \mathrm{Diff}_\infty^\infty[0, 1]$ by regarding s_1 as $[0, 1]$. If we identify $g \in \mathrm{Diff}_\infty^\infty[0, 1] = \mathrm{Diff}_\infty^\infty s_1$ with its obvious extension $\tilde{g} \in \mathrm{Diff}(S^1)$, then by the hypothesis we can construct $\mathscr{F}(g, f_2, \cdots, f_n)$ for any $g \in \mathrm{Diff}_\infty^\infty[0, 1]$. By the result of Sergeraert [17], we can represent $f_1 = [g_1, h_1] \cdots [g_k, h_k]$ for some g_i and $h_j \in \mathrm{Diff}_\infty^\infty[0, 1]$. Hence we have $\mathscr{F}(f_1, \cdots, f_n) \sim \sum_{m=1}^k \mathscr{F}([g_m, h_m], f_2, \cdots, f_n) \sim 0$. q.e.d.

LEMMA 8. *Let R be a rotation of $(S^1 \times [0, 1], \mathscr{F}_1)$ along the S^1-factor. Then $\Sigma(R)$ is equivalent to $\Sigma(\mathrm{id} \times f)$ for some f in $\mathrm{Diff}_\infty^\infty[0, 1]$.*

PROOF. We can construct $(S^1 \times [0, 1], \mathscr{F}_1)$ from a suitable g in $\mathrm{Diff}_\infty^\infty[0, 1]$. By the assumption that $\mathscr{F}_1$ is invariant under rotations along the S^1-factor, g can be imbedded into a 1-parameter family $\{g_t\}$ in $\mathrm{Diff}_\infty^\infty[0, 1]$. It is clear that the rotation R corresponds to some g_t, and this gives the desired f. q.e.d.

LEMMA 9. *$\Sigma(0, 2b)$ is foliated cobordant to the following foliation on $T^4 = S^1 \times I \times S^1 \times S^1 \cup S^1 \times I \times S^1 \times S^1$: We consider $(S^1 \times I, \mathscr{F}_1) \times$*

$S^1 \times S^1$ on the first $S^1 \times I \times S^1 \times S^1$ and $\Sigma_b S^1 \times (I \times S^1, \mathscr{F}_0)$ on the second $S^1 \times I \times S^1 \times S^1$, where b is a rotation of $(I \times S^1, \mathscr{F}_0)$ along the S^1-factor and the last S^1-factor of the second $S^1 \times I \times S^1 \times S^1$ comes from the identification of $S^1 \times (I \times S^1, \mathscr{F}_0) \times [0, 1]$ by b.

PROOF. By Proposition, we can regard $\Sigma(0, 2b)$ as $\Sigma(0, b) + \Sigma(0, b)$. First perform the r_0^*-surgery on $(S^1 \times D^2, \text{Reeb}) \cup (S^1 \times D^2, \text{Reeb}) \subset (S^3, \mathscr{F}_R) \cup (S^3, \mathscr{F}_R)$, which are rotated by b in the construction of $\Sigma(0, b)$'s. Next we perform the r_1^*-surgery on the rest of $(S^1 \times D^2, \text{Reeb})$'s. By Lemma 5, we can extend this foliated cobordisms to the foliated cobordism between $\Sigma(0, 2b)$ and the foliation on T^4 stated in this lemma. q.e.d.

PROOF OF THEOREM 2: *The case $(S^3, \mathscr{F}_R)$ continued.* By Lemmas 6, 8 and 9, we have a foliated cobordism between $\Sigma(0, 2b)$ and $\mathscr{F}(f_1, f_2, f_3)$ on T^4. It is easy to see that this foliation $\mathscr{F}(f_1, f_2, f_3)$ satisfies the assumption in Lemma 7. Hence we have $\Sigma(0, 2b) \sim 0$. This completes the proof.

PROOF OF THEOREM 2: *The case $(S^3, \bar{\mathscr{F}}_R)$.* The proof proceeds in the same way, only Theorem (Fukui-Ushiki [3]) being replaced by the following.

THEOREM (cf. [3]). *The sequence*

$$1 \to \mathrm{LD}(S^3, \bar{\mathscr{F}}_R)_0 \to \mathrm{FD}(S^3, \bar{\mathscr{F}}_R)_0 \to S^1 \times S^1 \times \mathrm{Diff}^\infty_\infty [0, 1] \to 1$$

is split exact.

Note that $\mathrm{Diff}^\infty_\infty [0, 1]$ is perfect (see Sergeraert [17]). From these we have the desired result. q.e.d.

4. Proof of Theorem 3. The following lemma shows that the elements in $\mathrm{FD}(M, \mathscr{F})_0$ are considerably restricted.

LEMMA 10. *Let f_t, $t \in [0, 1]$, be a path in $\mathrm{FD}(M, \mathscr{F})_0$ connecting f and id_M. If there is a $t' \in [0, 1]$ such that $f_{t'}(L) \neq L$, then the holonomy $H(L)$ of L is trivial.*

PROOF. This follows easily from the fact that $H(f_t(L)) \cong H(L)$ for any t. q.e.d.

For the detailed construction of $(S^3, \mathscr{F}_a)$, we refer the reader to Tamura [19, Chap. 8]. In the following we assume that the toral leaves of Reeb components are discrete, since the same method gives the proof in the case where some toral leaves of Reeb components are of the form $T^2 \times \{t\}$, $t \in [0, 1]$. Excepting this assumption, we use only the following two properties of $(S^3, \mathscr{F}_a)$ which seem to be well-known (cf. [20]).

(P. 1) Set $E = S^3 -$ "all $\operatorname{int}(S^1 \times D^2, \text{Reeb})$'s". Then the union of leaves L with $H(L) \neq 0$ is dense in E.

(P. 2) On the neighborhood U of ∂E in E, where $U \cong \bigcup_{i=1}^{k} T_i^2 \times [0, 2)$, there are smooth non-vanishing closed 1-forms $\omega_1, \cdots, \omega_k$ on $T_1^2, \cdots, T_k^2$, respectively, and a smooth function $h(t)$ on $[0, 2)$ with $h(t) = 1$ for $t \geqq 1$, $dh/dt > 0$ on $(0, 1)$, and $h(t) = \exp(-1/t^2)$ on a neighborhood of 0 in $[0, 2)$, such that $\mathscr{F}_a | U$ is equivalent to the foliation given by the 1-form $h(t)\omega_i + (1 - h(t))dt$ on $T_i^2 \times [0, 2)$ for each $i \in \{1, \cdots, k\}$, where we consider ω_i to be lifted canonically onto $T_i^2 \times [0, 2)$.

LEMMA 11. *For any $f \in \mathrm{FD}_0$, there is an $f' \in \mathrm{FD}_0$ such that $f' | E = \mathrm{id}_E$ and $f' \circ f^{-1} \in \mathrm{LD}_0$.*

PROOF. By Lemma 10 and (P. 1), we have $f | E \in \mathrm{LD}_0 | E$. By a result of [2] and [3], $f | (S^1 \times D^2, \text{Reeb})$ is isotopic in LD_0 to a diffeomorphism g satisfying $g | \partial(S^1 \times D^2) = \mathrm{id}$. Define f' by

$$f' = \begin{cases} \mathrm{id}_E & \text{on} \quad E \\ g & \text{on} \quad M - \operatorname{int} E \,. \end{cases}$$

Then f' clearly satisfies the required conditions.

By Lemma 11 and Proposition, we may assume that f is the identity map except on one Reeb component $(S^1 \times D^2, \text{Reeb})$. On this $(S^1 \times D^2, \text{Reeb})$ we have $f | T^2 = \mathrm{id}_{T^2}$, where $T^2 = S^1 \times \partial D^2$. Now we fix one Reeb component and consider f restricted to it. We also denote this restriction of f by the same letter. Further, we fix a foliation-preserving flow $\{\varphi_t; t \in \boldsymbol{R}\}$ transverse to leaves in $\operatorname{int}(S^1 \times D^2)$ and stational on T^2 (cf. Fukui [2]). We also assume that φ_t has a closed orbit $S^1 \times \{0\} \subset S^1 \times D^2$ and that $\varphi_t | S^1 \times \{0\}$ is a rotation by an angle t for each t, where we identify S^1 with $\boldsymbol{R}/\boldsymbol{Z}$.

First we show that f can be isotoped to φ_t for some t by elements in $\mathrm{LD}(S^1 \times D^2, \text{Reeb})_0$, which are the identity on T^2. In the following we denote $\mathrm{FD}(S^1 \times D^2, \text{Reeb})_0$ by $\mathrm{FD}(R)_0$, $\mathrm{LD}(S^1 \times D^2, \text{Reeb})$ by $\mathrm{LD}(R)$ and so on. Set $\mathrm{FD}(\partial) = \{g \in \mathrm{FD}(R)_0 \colon g | T^2 = \mathrm{id}_{T^2}\}$. Then f belongs to $\mathrm{FD}(\partial)$. Also set $\mathrm{LD}(\partial)_0 = \mathrm{FD}(\partial) \cap \mathrm{LD}(R)_0$.

LEMMA 12. *The following sequence is exact and the homomorphism α defined by $\alpha(f) = f | T^2$ is a locally trivial fibration:*

$$1 \to \mathrm{FD}(\partial) \to \mathrm{FD}(R)_0 \xrightarrow{\alpha} \mathrm{Diff}_{+}^{\infty}(T^2)_0 \to 1 \,.$$

PROOF. The exactness follows at once from the definition and the second statement is proved by the same argument as in [2] and [3].

LEMMA 13. $\mathrm{FD}(\partial)$ *is contractible, hence is connected.*

PROOF. By Lemma 12, we have the homotopy exact sequence:

$$\begin{array}{ccccccc} \longrightarrow \pi_1(\mathrm{FD}(R)_0) & \xrightarrow[\#]{} & \pi_1(\mathrm{Diff}^\infty_+(T^2)_0) & \longrightarrow & \pi_0(\mathrm{FD}(\partial)) & \longrightarrow & 0\,. \\ \wr\| & & \wr\| & & & & \\ Z\oplus Z & & Z\oplus Z & & & & \end{array}$$

By [2] and [3], $\pi_1(\mathrm{FD}(R)_0)$ is isomorphic to $\boldsymbol{Z}\oplus\boldsymbol{Z}$ and it is easy to see that the map $\#$ is an isomorphism. Thus we have $\pi_0(\mathrm{FD}(\partial))=0$. The contractibility easily follows from this and the fact that $\pi_i(\mathrm{FD}(R)_0)=\pi_i(\mathrm{Diff}^\infty_+(T^2)_0)=0$ for $i\geqq 2$. q.e.d.

We identify $\{\varphi_t\}\subset\mathrm{FD}(\partial)$ with $\boldsymbol{R}$, and $\{\varphi_n\}_{n\in Z}$ with $\boldsymbol{Z}$.

LEMMA 14. *The following diagram is commutative and exact for horizontal lines, and β and γ are locally trivial fibrations*:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & \mathrm{FD}(\partial) & \xrightarrow{\beta} & S^1 & \longrightarrow & 1 \\ & & \uparrow\!\!\cup & & \uparrow\!\!\cup & & \| & & \\ 0 & \longrightarrow & \boldsymbol{Z} & \longrightarrow & \boldsymbol{R} & \xrightarrow{\gamma} & S^1 & \longrightarrow & 1\,, \end{array}$$

where $\hookrightarrow$ *means the canonical inclusion, $\beta(f)$ is a rotation* $\in\mathrm{SO}(2)$ *on the leaf space of* $(S^1\times\mathrm{int}(D^2),\ \mathrm{Reeb})$ *which is diffeomorphic to S^1, γ is the restriction of β to $\boldsymbol{R}$ and* $L=\mathrm{Ker}(\beta)$.

PROOF. Similar to that of Lemma 12 (cf. [2], [3]).

LEMMA 15. *In Lemma* 14, $\boldsymbol{Z}\hookrightarrow L$ *is a homotopy equivalence.*

PROOF. We call a small neighborhood V of T^2 in $S^1\times D^2$ nice if we can identify V with $T^2\times[0,1)$ by using flow φ_t. Note that any g in $\mathrm{Diff}^\infty_+(T^2)_0$ can be lifted to a diffeomorphism of a nice neighborhood. Let $\mathscr{L}$ be the set of all $f\in L$ such that the restriction of f to some nice neighborhood is a lifting of id_{T^2}. We know that $\mathscr{L}\hookrightarrow L$ is a homotopy equivalence (see [2], [3]). Thus we have only to show that $\boldsymbol{Z}\hookrightarrow\mathscr{L}$ is a homotopy equivalence. On the other hand, it is easy to show that $\mathscr{L}$ is homotopy equivalent to $\boldsymbol{Z}\times L^s(S^2, D^2_+)$, where $L^s(S^2, D^2_+)$ is the smooth loop space of $\mathrm{Diff}^\infty_+(S^2, D^2_+)$, and we know that $\mathrm{Diff}^\infty_+(S^2, D^2_+)$ is contractible (see [2], [3]). Hence we have the desired conclusion.

Let $f\in\mathrm{FD}(\partial)$. Take φ_t with $\beta(f)=\gamma(\varphi_t)$. Then $\varphi_t^{-1}\circ f\in L$ and L is homotopy equivalent to $\boldsymbol{Z}$. Hence we have $\varphi_t^{-1}\circ f=\varphi_n$ up to $\mathrm{LD}(\partial)_0$, i.e., $f=\varphi_t$ for some t up to $\mathrm{LD}(\partial)_0$. Thus we have proved

LEMMA 16. *For any* $f \in \mathrm{FD}(S^3, \mathscr{F}_a)_0$ *we can choose* $f' \in \mathrm{FD}(S^3, \mathscr{F}_a)_0$ *so that* $f \cong f'$ *in* $\mathrm{LD}(S^3, \mathscr{F}_a)_0$, $f'|E = \mathrm{id}_E$ *and that* $f'|(S^1 \times D^2, \mathrm{Reeb})$ *is given by some* φ_t *for each* $(S^1 \times D^2, \mathrm{Reeb})$.

PROOF OF THEOREM 3. To prove Theorem 3, we have only to show that $\Sigma(f') \sim 0$ for f' in Lemma 16. Note that we may assume that f' is the identity outside one Reeb component and that f' is a φ_t on this Reeb component. We denote f' by φ_t or f.

First we cut off the Reeb component from $(S^3, \mathscr{F}_a)$. For this purpose we will define a foliation $\mathscr{F}'$ on S^3. Set $X = S^3 - \mathrm{int}(S^1 \times D^2, \mathrm{Reeb})$. Then, by hypothesis, $f|X = \mathrm{id}_X$. We fix a neighborhood $V \cong T^2 \times [0, 6)$ of $\partial X = T^2$ in X. We may suppose that $\mathscr{F}_a|V$ is given by the 1-form $k(t)\omega + (1 - k(t))dt$, where ω is the closed 1-form on T^2 given in (P. 2), and $k(t)$ is the smooth function on $[0, 6)$, which is the trivial extension of $h(t)$ in (P. 2), i.e., $k(t) = 1$ for $t \in [1, 6)$. Hereafter we use the same letter ω for the canonical lifting of ω onto $T^2 \times [0, 6)$. We may assume that the foliation on T^2 given by ω is not the inverse images of the trivial fibrations $p_i\colon T^2 \to S^1$, where p_i is the natural projection onto the i-th factor for $i = 1, 2$ (see Remark below). Now we define $\mathscr{F}'$ on $S^3 = S^1 \times D^2 \cup T^2 \times [0, 3] \cup S^1 \times S^1 \times [3, 4] \cup T^2 \times [4, 6) \cup X^*$, where $X^* = X - V$, as follows. On $S^1 \times D^2$ we consider the Reeb component. On $T^2 \times [0, 3]$ we consider the foliation defined by the 1-form $n(t)\omega + (1 - n(t))dt$, where $n(t) = h(t)$ on $[0, 2]$ and $n(t) = h(3 - t)$ on $[2, 3]$ (see (P. 2) for $h(t)$). On $S^1 \times S^1 \times [3, 4] \cong S^1 \times S^1 \times [0, 1]$, we consider $S^1 \times (S^1 \times [0, 1], \mathscr{F}_0)$, where the S^1-factor of $\mathscr{F}_0$ is the same as that of $S^1 \times D^2$. On $T^2 \times [4, 6) \cong T^2 \times [0, 2)$ we consider the foliation given by the 1-form $h(t)\omega + (1 - h(t))dt$ (see (P.2)). On X^* we consider $\mathscr{F}_a|X^*$. These define a foliation $\mathscr{F}'$ on S^3. It is easy to show that $(S^3, \mathscr{F}_a)$ is concordant to $(S^3, \mathscr{F}')$. Now we can perform the r_0-surgery on $S^1 \times S^1 \times [3, 4]$, which gives the foliated cobordism between $(S^3, \mathscr{F}_a)$ and $(S^1 \times S^2, \mathscr{F}^*) + (S^3, \mathscr{F}_a)$, where $(S^1 \times S^2, \mathscr{F}^*)$ is given as follows. Decompose $S^1 \times S^2$ into $S^1 \times D^2 \cup S^1 \times S^1 \times [0, 3] \cup S^1 \times D^2$. On $S^1 \times D^2$'s we consider the corresponding Reeb foliations, and on $S^1 \times S^1 \times [0, 3]$ we consider the foliation given by the above 1-form $n(t)\omega + (1 - n(t))dt$. Then φ_t acts on $(S^1 \times S^2, \mathscr{F}^*)$ as a rotation on one of the Reeb components and acts trivially on $(S^3. \mathscr{F}_a)$. Thus we have only to show that $\Sigma_{\varphi_t}(S^1 \times S^2, \mathscr{F}^*) \sim 0$.

Regarding t as $2t'$, we will show that $\Sigma(\varphi_{t'}) + \Sigma(\varphi_{t'}) \sim 0$. We perform the r_0^*-surgery on the Reeb components where $\varphi_{t'}$ acts nontrivially. Then by Lemma 5, we may only consider the suspended foliation of the resulting foliation $\mathscr{F}^{**}$ on $S^1 \times S^2$, which is given as follows. Decompose $S^1 \times S^2$ into $S^1 \times D^2 \cup T^2 \times [0, 3] \cup T^2 \times [3, 4] \cup T^2 \times [4, 7] \cup S^1 \times D^2$. On

$S^1 \times D^2$'s we consider the Reeb components, on $T^2 \times [0, 3]$ the foliation defined by $n(t)\omega + (1 - n(t))dt$, on $T^2 \times [4, 7] \cong T^2 \times [0, 3]$ we also consider the same foliation and on $T^2 \times [3, 4] \cong S^1 \times S^1 \times [0, 1]$ we consider $S^1 \times (S^1 \times [0, 1], \mathscr{F}_0)$, where the S^1-factor of $\mathscr{F}_0$ is the same as that of $S^1 \times D^2$. Then $\varphi_{t'}$ acts on $(S^1 \times [0, 1], \mathscr{F}_0)$ fixing the boundary. We define a foliation $\mathscr{F}''$ on $S^1 \times S^2$ as follows. Decompose $S^1 \times S^2$ into $S^1 \times D^2 \cup T^2 \times [0, 7] \cup S^1 \times D^2$. On $S^1 \times D^2$'s we consider the Reeb components, and on $T^2 \times [0, 7]$ the foliation defined by the 1-form $m(t)\omega + (1 - m(t))dt$, where $m(t) = h(t)$ on $[0, 2)$, $m(t) = h(7 - t)$ on $[6, 7]$ and $m(t) = 1$ on $[2, 6]$. It is easy to show that $(S^1 \times S^2, \mathscr{F}^{**})$ is concordant to $(S^1 \times S^2, \mathscr{F}'')$. Indeed, let $\varphi\colon T^2 \to S^1 \times S^2$ be an imbedding transverse to $\mathscr{F}''$, whose image is $T^2 \times \{7/2\} \subset T^2 \times [0, 7] \subset S^1 \times S^2$. Then the same technique as the σ-modification gives the desired concordance (cf. §3). As $\varphi_{t'}$ fixes the boundary of $S^1 \times (S^1 \times [0, 1], \mathscr{F}_0)$, we can naturally extend $\varphi_{t'}$ to the foliated manifold which gives the concordance between $(S^1 \times S^2, \mathscr{F}^{**})$ and $(S^1 \times S^2, \mathscr{F}'')$, so that $\varphi_{t'}$ acts trivially on $(S^1 \times S^2, \mathscr{F}'')$ (cf. §3). Hence we have $\Sigma(\varphi_t) \sim \Sigma(\varphi_{t'}) + \Sigma(\varphi_{t'}) \sim 0$. q.e.d.

REMARK. In case $\omega = p_i^* d\theta$, where $d\theta$ is a volume element of S^1, we must consider the following two situations. The first is as in $(S^3, \mathscr{F}_R)$ and this case is reduced to Theorem 2. The second is as in $(S^1 \times S^2 = S^1 \times D^2 \cup S^1 \times D^2$, two Reeb components). This foliation is concordant to $(S^1 \times S^2, \{\{x\} \times S^2\}_{x \in S^1})$ and φ_t can be extended to the concordance with the trivial action on the last foliation. Thus these cases give no exceptions.

5. Proof of Theorems 4 and 5 and Corollary. First recall the construction of $(s(M^3), s_\alpha(\mathscr{F}))$ from $(M^3, \mathscr{F})$ (cf. Oshikiri [14]). Let S^1 be a closed curve in M^3, which is transverse to $\mathscr{F}$, with a trivial tubular neighborhood $S^1 \times D^2$. Along $S^1 \times \partial D(1/2)^2$, where $D(1/2)^2$ is the 2-dimensional disk with radius 1/2, we wind the leaves of $\mathscr{F}$. Then we have a foliation on $M^3 - \operatorname{int}(S^1 \times D(1/2)^2)$. On $S^1 \times D(1/2)^2$ we consider the foliation $(S^1 \times D(1/2)^2, \mathscr{F}_\alpha) = (S^3, \mathscr{F}^*) - \operatorname{int}(S^1 \times D^2, \text{Reeb})$, where $\mathscr{F}^*$ is one of $\mathscr{F}_R$, $\bar{\mathscr{F}}_R$ and $\mathscr{F}_\alpha$. Thus we have a foliation $s_\alpha(\mathscr{F})$ on $s(M^3) = (M^3 - \operatorname{int}(S^1 \times D(1/2)^2)) \cup (S^1 \times D(1/2)^2)$, where we identify $\partial(M^3 - \operatorname{int}(S^1 \times D(1/2)^2)) = S^1 \times \partial D(1/2)^2$ with $\partial D(1/2)^2 \times S^1$ canonically.

PROOF OF THEOREM 4. We give only a sketch of the proof, because all the techniques involved are found in [2], [3] or in the preceding proofs.

Set $E = S^1 \times D^2 - \operatorname{int}(S^1 \times D(1/2)^2)$, where $S^1 \times D^2$ is the tubular neighborhood of the closed curve S^1 along which the surgery is performed (see the above construction). Then we can regard E as a subset of both

M^3 and $s(M^3)$. Note that $E \cong S^1 \times [1/2, 1] \times S^1$ and $s_\alpha(\mathscr{F}) \mid E = (S^1 \times [1/2, 1], \mathscr{H}) \times S^1$, where $\mathscr{H}$ is the half part of $\mathscr{F}_0$ on $S^1 \times [0, 1]$. Set $\mathrm{FD}(E) = \{f \in \mathrm{FD}(s(M^3), s_\alpha(\mathscr{F}))_0 \colon f(E) = E\}$. Then it is easy to show that the inclusion $\mathrm{FD}(E) \hookrightarrow \mathrm{FD}(s(M^3), s_\alpha(\mathscr{F}))_0$ is a weak homotopy equivalence. Also set $\mathrm{FD}(\mathscr{E}) = \{f \in \mathrm{FD}(E) \colon f \mid E$ is a rotation of $E \cong (S^1 \times [1/2, 1]) \times S^1$ along the first S^1-factor$\}$. Note that rotations of E along the last S^1-factor can be extended to elements in $\mathrm{LD}(s(M^3), s_\alpha(\mathscr{F}))_0$.

Let $\mathrm{FD}(\mathscr{E})_0$ be the identity component of $\mathrm{FD}(\mathscr{E})$. Then the theorem holds for elements in $\mathrm{FD}(\mathscr{E})_0$. Indeed, let $f \in \mathrm{FD}(\mathscr{E})_0$. Then the following construction gives the desired $g \in \mathrm{FD}(M^3, \mathscr{F})_0$: As $f \mid E$ is a rotation of E, we can cut off new $(S^1 \times D(1/2)^2, \mathscr{F}_\alpha)$ from $s(M^3)$ along $\partial(S^1 \times D^2) \subset \partial E$ by the same method as in the proof of Theorem 3. Define $g \in \mathrm{FD}(M^3, \mathscr{F})$ by $g \mid s(M^3) - \mathrm{int}(S^1 \times D^2) = f \mid s(M^3) - \mathrm{int}(S^1 \times D^2)$ and $g \mid S^1 \times D^2 =$ the rotation given by f. Obviously g is well-defined. It is also clear that $g \in \mathrm{FD}(M^3, \mathscr{F})_0$. Indeed, the isotopy $f_t \colon f \cong \mathrm{id}_{s(M)}$ belonging to $\mathrm{FD}(\mathscr{E})_0$ can easily be deformed to the isotopy $g_t \colon g \cong \mathrm{id}_M$ in $\mathrm{FD}(M, \mathscr{F})_0$. The same method also gives $h \in \mathrm{FD}(S^3, \mathscr{F}^*)_0$. It is clear that $\Sigma_f(s(M^3), s_\alpha(\mathscr{F})) \sim \Sigma_g(M^3, \mathscr{F}) + \Sigma_h(S^3, \mathscr{F}^*)$. By Theorems 2 and 3, $\Sigma_h(S^3, \mathscr{F}^*) \sim 0$. Thus the theorem follows.

Let $f \in \mathrm{FD}(s(M^3), s_\alpha(\mathscr{F}))_0$. Then it is easy to show that f can be deformed into $f' \in \mathrm{FD}(E)$ by elements in $\mathrm{LD}(s(M^3), s_\alpha(\mathscr{F}))_0$. By Proposition, $\Sigma(f) \sim \Sigma(f')$. Let $\{f_t\}$ be an isotopy between $f' = f_1$ and $\mathrm{id}_{s(M)} = f_0$. Then there is an isotopy $\{h_t\}$ in $\mathrm{FD}(E)$ between $f' = h_1$ and $\mathrm{id}_{s(M)} = h_0$, because $\mathrm{FD}(E) \hookrightarrow \mathrm{FD}(s(M), s_\alpha(\mathscr{F}))_0$ is a weak homotopy equivalence.

Let $g \in \mathrm{FD}(E)$. Then the same argument as in [2, Lemma 1.9] shows that $g \mid E$ is a rotation along the S^1-factor of $S^1 \times [1/2, 1]$ up to $\mathrm{LD}(s(M^3), s_\alpha(\mathscr{F}))_0$. Thus, considering the motion of E by $\{h_t\}$, we get a path $\{g_t\}$ in $\mathrm{LD}(s(M^3), s_\alpha(\mathscr{F}))_0$ such that the path $\{g_t^{-1} \circ h_t\}$ belongs to $\mathrm{FD}(\mathscr{E})_0$, i.e., $g_1 \in \mathrm{LD}(s(M^3), s_\alpha(\mathscr{F}))_0$ and $g_1^{-1} \circ h_1 = g_1^{-1} \circ f' \in \mathrm{FD}(\mathscr{E})_0$.

As we have proved Theorem 4 for $\mathrm{FD}(\mathscr{E})_0$, we have a $g \in \mathrm{FD}(M^3, \mathscr{F})_0$ such that $\Sigma(g_1^{-1} \circ f') \sim \Sigma(g)$. On the other hand, $\Sigma(g_1^{-1} \circ f') \sim \Sigma(f') - \Sigma(g_1) \sim \Sigma(f)$, because $g_1 \in \mathrm{LD}(s(M^3), s_\alpha(\mathscr{F}))_0$ and $\Sigma(f') \sim \Sigma(f)$. This completes the proof.

REMARK. The same argument gives the proof of the following statement (see §3 for $\sigma_\varphi(\mathscr{F})$): For each $f \in \mathrm{FD}(M^3, \sigma_\varphi(\mathscr{F}))_0$, there exists a $g \in \mathrm{FD}(M^3, \mathscr{F})_0$ such that $\Sigma_f(M^3, \sigma_\varphi(\mathscr{F})) \sim \Sigma_g(M^3, \mathscr{F})$.

PROOF OF COROLLARY. By Theorem 4 and Proposition, we have $\Sigma_f(s(M^3), s_\alpha(\mathscr{F})) \sim \Sigma_g(M^3, \mathscr{F}) \sim 0$ q.e.d.

PROOF OF THEOREM 5. As $(M^3, \mathscr{F}_S)$ is obtained from some bundle foliation by performing surgeries with $(S^3, \mathscr{F}^*) = (S^3, \mathscr{F}_R)$ (cf. [2], [14]), we have $\Sigma_f(M^3, \mathscr{F}_S) \sim \Sigma_g(N^3$, Bundle foliation$) \sim 0$ for $f \in \mathrm{FD}(M^3, \mathscr{F}_S)_0$ by Theorems 1 and 4. q.e.d.

6. Concluding remarks. First we give the promised examples. Let $n = 4k + 2$ with $k \geqq 2$. Then there exist a codimension 1 foliation $(M^n, \mathscr{F})$, which is null-cobordant in $\mathscr{F}\Omega_1(n)$, and an $f \in \mathrm{LD}(M^n, \mathscr{F})$ such that $\Sigma_f(M, \mathscr{F})$ is of infinite order in $\mathscr{F}\Omega_1(n+1)$. Indeed, let $(S^3, \mathscr{F}_a)$ be the foliation with $gv(\mathscr{F}_a) = a \neq 0$ constructed by Thurston [20]. We use the following theorem.

THEOREM (Neumann [13]). *Let M be a closed oriented smooth manifold. Then M is oriented cobordant to a manifold which is the total space of a fiber bundle over S^1 if and only if the signature of M is zero. Moreover, we can make the fiber null-cobordant.*

For each $k \geqq 2$, Ker(sign: $\Omega_{4k} \to \boldsymbol{Z}$) contains elements of infinite order. By the above theorem, we can represent such an element by M^{4k} which is the total space of a fiber bundle over S^1 with null-cobordant fiber F. Then $M = F \times [0, 1]/(x, 1) \sim (h(x), 0)$ for some $h \in \mathrm{Diff}^\infty_+(F)$. It is clear that $F \times (S^3, \mathscr{F}_a) \sim 0$ and $M \times (S^3, \mathscr{F}_a) = \Sigma_{h\times \mathrm{id}}(F \times S^3, F \times \mathscr{F}_a)$. We show that $M \times (S^3, \mathscr{F}_a)$ is an element of infinite order. Note that $\Omega_*(B\Gamma_1) \otimes \boldsymbol{Q} \cong H_*(B\Gamma_1; \boldsymbol{Q}) \otimes_{\boldsymbol{Q}} (\Omega_* \otimes \boldsymbol{Q})$ (see Stong [18]). If $g: S^3 \to B\Gamma_1$ classifies the foliation $(S^3, \mathscr{F}_a)$, then the element on the right hand side of the above formula, corresponding to $M \times (S^3, \mathscr{F}_a)$, is of the form $\cdots \oplus g_*[S^3] \otimes [M]_{\boldsymbol{Q}} \oplus \cdots$, where "$\oplus$" means the direct sum. As $[M]_{\boldsymbol{Q}} = [M] \otimes 1$ and $g_*[S^3]$ are elements of infinite order, we have a desired example.

Next we give some properties of $\mathscr{F}(f_1, \cdots, f_n)$, whose proofs are omitted because they are easily proved. In the following, we assume that all the f_i's in $\mathscr{F}(f_1, \cdots, f_n)$ commute with each other.

(a) $\mathscr{F}(f_1, \cdots, f_i \circ f_i', \cdots, f_n) \sim \mathscr{F}(f_1, \cdots, f_i, \cdots, f_n) + \mathscr{F}(f_1, \cdots, f_i', \cdots, f_n)$.

(b) $\mathscr{F}(f_1, \cdots, f_i, \cdots, f_j, \cdots, f_n) = -\mathscr{F}(f_1, \cdots, f_j, \cdots, f_i, \cdots, f_n)$.

(c) $\mathscr{F}(f_1, \cdots, f_n) = \mathscr{F}(h \circ f_1 \circ h^{-1}, \cdots, h \circ f_n \circ h^{-1})$ for $h \in \mathrm{Diff}^\infty_+(S^1)$.

(d) $\mathscr{F}(f_1, \cdots, f_i, \cdots, f_n) \sim \mathscr{F}(f_1, \cdots, f_1^{-1} \circ f_i, \cdots, f_n)$.

(e) In particular, $\mathscr{F}(f_1, \cdots, f, \cdots, f, \cdots, f_n) \sim 0$. Note that (b) means $\mathscr{F}(f_1, \cdots, f, \cdots, f, \cdots, f_n)$ only to be of 2-torsion.

We also give some information on $\mathscr{F}(f, g)$ on T^3 (cf. Tsuboi [23]).

PROPOSITION. *Set $N = \{f \in \mathrm{Diff}^\infty_+(S^1): \mathscr{F}(f, g) \sim 0$ for all $g \in c(f)\}$, where $c(f) = \{g \in \mathrm{Diff}^\infty_+(S^1): f \circ g = g \circ f\}$. Then we have:*

(1) *$f \in N$ implies $f^{-1} \in N$.*
(2) *$gNg^{-1} = N$ for $g \in \mathrm{Diff}_+^\infty(S^1)$.*
(3) $\mathrm{SO}(2) \subset N$.
(4) *N contains an open dense set.*

PROOF. (1) and (2) are clear. (3) follows at once from Theorem 2. (4) follows from the property (e) above and the result of Kopell [7] to the effect that the set $\{f \in \mathrm{Diff}_+^\infty(S^1): c(f) = \{f^m: m \in \boldsymbol{Z}\}\}$ is open dense.
q.e.d.

Finally, we pose some questions.

(1) *Do we have $\Sigma_f(M^n, \mathscr{F}) \sim 0$ for $f \in \mathrm{FD}_0$?*

In particular,

(2) *Do we have $\Sigma_f(T^2, \mathscr{F}) \sim 0$ for $f \in \mathrm{FD}(T^2, \mathscr{F})_0$?*

This is related to [16, Question 3]. This is also related to:

(3) *Is $\mathscr{F}(f, g)$ null-cobordant?*

In general,

(4) *Is $\mathscr{F}(f_1, \cdots, f_n)$ null-cobordant?*
(5) *If $f \in \mathrm{FD}(M^n, \mathscr{F}) \cap \mathrm{Diff}_+^\infty(M^n)_0$, then do we have $\Sigma_f(M^n, \mathscr{F}) \sim 0$?*

REFERENCES

[1] R. BURDICK, Oriented manifolds fibered over the circle, Proc. Amer. Math. Soc. 17 (1966), 449–452.
[2] K. FUKUI, On the homotopy type of some subgroups of Diff(M^3), Japan J. Math. 2 (1976), 249–267.
[3] K. FUKUI AND S. USHIKI, On the homotopy type of $F\,\mathrm{Diff}(S^3, \mathscr{F}_R)$, J. Math. Kyoto Univ. 15 (1975), 201–210.
[4] M. HERMAN, Simplicité du groupe des difféomorphismes de class C^∞, isotope à l'identité, du tore de dimension n, C. R. Acad. Sci. Paris 273 (1971), 232–234.
[5] M. HERMAN, The Godbillon-Vey invariant of foliations by planes of T^3, in Geometry and Topology, Rio de Janeiro 1976, Lecture Notes in Math. 597, Springer-Verlag, Berlin-Heidelberg-New York, 1977.
[6] H. IMANISHI, On the theorem of Denjoy-Sacksteder for codimension 1 foliations without holonomy, J. Math. Kyoto Univ. 14 (1974), 607–634.
[7] N. KOPELL, Commuting diffeomorphisms, in Global Analysis, Proc. Symp. in Pure Math. XIV, Amer. Math. Soc. 1970, 165–184.
[8] U. KOSCHORKE, Concordance and bordism of line fields, Inventiones Math. 24 (1974), 241–268.
[9] H. LAWSON, Foliations, Bull. Amer. Math. Soc. 80 (1974), 369–418.
[10] J. LESLIE, Two classes of classical subgroups of Diff(M), J. Diff. Geom. 5 (1971), 427–435.
[11] J. LESLIE, A remark on the group of automorphisms of a foliation having a dense leaf, J. Diff. Geom. 7 (1972), 597–601.
[12] J. MATHER, On Haefliger's classifying space I, Bull. Amer. Math. Soc. 77 (1971), 1111–1115.
[13] W. NEUMANN, Fibering over the circle within a bordism class, Math. Ann. 192 (1971), 191–192.

[14] G. OSHIKIRI, The surgery of codimension-one foliations, Tôhoku Math. J. 31 (1979), 63–70.

[15] B. REINHART, Cobordism and the Euler number, Topology 2 (1963), 173–177.

[16] H. ROSENBERG AND W. THURSTON, Some remarks on foliations, in Dynamical Systems, Acad. Press, New York and London, 1973, 463–478.

[17] F. SERGERAERT, Feuilletages et difféomorphismes infiniment tangents à l'identité, Inventiones Math. 39 (1977), 253–275.

[18] R. STONG, Notes on cobordism theory, Princeton Math. Notes, 1968.

[19] I. TAMURA, Topology of foliations (in Japanese), Iwanami Shoten, Tokyo, 1976.

[20] W. THURSTON, Non-cobordant foliations of S^3, Bull. Amer. Math. Soc. 78 (1972), 511–514.

[21] W. THURSTON, Existence of codimension one foliations, Ann. of Math. 104 (1976), 249–268.

[22] S. MORITA AND T. TSUBOI, The Godbillon-Vey class of codimension one foliations without holonomy, Topology 19 (1980), 43–49.

[23] T. TSUBOI, On 2-cycles of $B\mathrm{Diff}\, S^1$ which are represented by foliated S^1-bundles over T^2, in preparation.

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