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ENTROPY AND ALMOST EVERYWHERE CONVERGENCE OF FOURIER SERIES

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1. Introduction. Fefferman [3] has developed a theory of Fourier analysis using the notion of entropy. He showed in particular that entropy arguments are useful to investigate several problems in Fourier analysis which occur near L^1-space, e.g., the Zygmund class $L\log^+ L$.

In this note we shall prove that if the entropy of f in $[-\pi, \pi]$ is finite, then the partial sums of the Fourier series of f converge a.e. Our proof depends essentially on Carleson's famous theorem in [2]. On the other hand, we shall give a function whose entropy is finite but which does not belong to $L\log^+ L\log^+\log^+ L$ or the L^1-Dini class. This fact will be interesting if we recall Sjölin's theorem which implies that if $f \in L\log^+ L\log^+\log^+ L$, then the Fourier series of f converges a.e.

Our result will be extended to the Riesz-Bochner means of the Fourier series of a function of several variables. Let $Q = \{x = (x_1, \cdots, x_d) \mid -\pi < x_i \leqq \pi\}$ be the fundamental cube in $\boldsymbol{R}^d$. For a set $S \subset Q$ the entropy $E(S)$ of S is defined by

$$E(S) = \inf_{S \subset \cup Q_k} \sum_{k \geqq 1} |Q_k| \log |Q_k|^{-1} ,$$

where Q_k are subcubes of Q, and the entropy $J(f)$ of a nonnegative function f is defined by

$$J(f) = \int_0^\infty E(\{x \in Q \mid f(x) > \lambda\}) d\lambda .$$

For these definitions and basic properties of $E(S)$ and $J(f)$ we refer to Fefferman [3]. Furthermore we define the L^1-Dini class as the class of functions which have the finite L^1-Dini norm $\|f\|_{D^1}$ defined by

$$\|f\|_{D^1} = \|f\|_{L^1} + \iint_{Q^2} \frac{|f(x) - f(y)|}{|x - y|^d} dx dy .$$

Fefferman [3] has proved that if f belongs to the L^1-Dini class, then $J(f)$ is finite, and if $J(f)$ is finite, then f is in the class $L\log^+ L(Q)$.

2. Theorem. Let $d \geqq 1$ and let f be an integrable function on Q. The spherical Riesz-Bochner mean of order δ of f is defined by

$$S_R^\delta(x, f) = \sum_{|n|<R} (1 - |n|^2 R^{-2})^\delta a_n e^{inx} ,$$

where $a_n = (2\pi)^{-d} \int_Q f(x) e^{-inx} dx$. If $d = 1$, then $S_R^0(x, f)$ coincides with the partial sum of the Fourier series of f.

THEOREM 1. *Let $\alpha = (d-1)/2$ ($d \geqq 1$) be the critical index. If $J(f)$ is finite, then*

$$\lim_{R\to\infty} S_R^\alpha(x, f) = f(x) \tag{2.1}$$

holds a.e.

To prove Theorem 1 we need the following lemmas which are given in [2], [3], [4], [5], and [7].

LEMMA 1 (Fefferman [3]). *If $J(f)$ is finite, then f belongs to $L \log^+ L$.*

LEMMA 2. (1) *If f is essentially bounded, then the relation in* (2.1) *holds a.e.*

(2) *For f in $L \log^+ L(Q)$ we have*

$$\lim_{R\to\infty} \int_Q |S_R^\alpha(x, f) - f(x)| dx = 0 .$$

See Stein [4] when $d \geqq 2$ and Carleson [2], Zygmund [7] when $d = 1$.

LEMMA 3 (Stein [5]). *For f in $L \log^+ L(Q)$, let $g(x) = f(x)$ for $x \in Q$ and $g(x) = 0$ otherwise. Let*

$$\sigma_R(x, g) = (2\pi)^{-d} \int_{R^d} H_R(y) g(x - y) dy ,$$

where $H_R(y) = c_0 R^{1/2} J_{d-1/2}(R|y|)|y|^{-d+1/2}$ with $c_0 = 2^{(d-1)/2}\Gamma((d+1)/2)(2\pi)^{d/2}$. Then $\lim_{R\to\infty} S_R^\alpha(x, f) - \sigma_R(x, g) = 0$, uniformly in $x \in G$, where G is any closed subset in the interior of Q.

PROOF OF THEOREM 1. We assume $\int f dx = 1$ and $f \geqq 0$. By the definition of entropy we can select cubes Q_k^n in such a way that

$$\{x \in Q \mid 2^n \leqq f(x) < 2^{n+1}\} \subset \bigcup_{k\geqq 1} Q_k^n ,$$

and

$$\sum_{k\geqq 1} |Q_k^n| \log |Q_k^n|^{-1} \leqq E(\{x \in Q \mid 2^n \leqq f(x) < 2^{n+1}\} + 2^{-2n} .$$

From this we have

$$\sum_{n\geqq 1} \sum_{k\geqq 1} 2^n |Q_k^n| \log |Q_k^n|^{-1} \leqq A J(f) . \tag{2.2}$$

Set $E_\lambda = \{x \in Q \mid \limsup_{T,R\to\infty} |S_R^\alpha(x, f) - S_T^\alpha(x, f)| > \lambda\}$. We shall show

$|E_\lambda| = 0$ for any positive λ. Let $f^N(x) = f(x)$ if $f(x) \geqq 2^N$ and $f^N(x) = 0$ otherwise. Let $f_N = f - f^N$. Then

$$S_R^\alpha(x, f) = S_R^\alpha(x, f_N) + \{S_R^\alpha(x, f^N) - \sigma_R(x, g^N)\} + \sigma_R(x, g^N) ,$$

where $g^N(x) = f^N(x)$ for $x \in Q$ and $g^N(x) = 0$, otherwise. By (1) of Lemma 2, $\limsup_{R\to\infty} S_R^\alpha(x, f_N) = \liminf_{R\to\infty} S_R^\alpha(x, f_N)$ a.e. Since f^N belongs to $L\log^+L$ by Lemma 1, we also have $\lim_{R\to\infty}\{S_R^\alpha(x, f^N) - \sigma_R(x, g^N)\} = 0$ a.e. by Lemma 3. Let $F_N = \bigcup_{n\geqq N}\bigcup_{k\geqq 1} 8Q_k^n$. Since we can make $|F_N|$ as small as we wish, it suffices to consider the measure of $\widetilde{E}_\lambda = \{x \in Q - F_N \,|\, \limsup_{T,R\to\infty}|\sigma_R(x, g^N) - \sigma_T(x, g^N)| > \lambda\}$ for a large but fixed N. By a basic property of Bessel functions, we have $|H_R(y)| \leqq c/|y|^d$. Therefore it follows that

$$\sup_{R>0}|\sigma_R(x, g^N)| \leqq c\int \sum_{n\geqq N}\sum_{k\geqq 1} 2^n \chi_{Q_k^n}(y)\frac{dy}{|x - y|^d} .$$

Thus

$$\begin{aligned}\int_{Q-F_N}\sup_R|\sigma_R(x, g^N)|dx &\leqq \sum_{n\geqq N}\sum_{k\geqq 1} c2^n\int_{x\in Q-F_N}\int_{y\in Q_k^n}\frac{dxdy}{|x - y|^d} \\ &\leqq c\sum_{n\geqq N}\sum_{k\geqq 1} 2^n\int_{x\in Q_k^n}\int_{y\in Q-8Q_k^n}\frac{dxdy}{|x - y|^d} \\ &\leqq c\sum_{n\geqq N}\sum_{k\geqq 1} 2^n|Q_k^n|\log|Q_k^n|^{-1} .\end{aligned}$$

Therefore $|\widetilde{E}_\lambda| \leqq (c/\lambda)\sum_{n\geqq N}\sum_{k\geqq 1} 2^n|Q_k^n|\log|Q_k^n|^{-1}$. Thus we conclude $|E_\lambda| = 0$ by (2.2). This means the relation in (2.1) holds a.e. by Lemma 1 and (2) of Lemma 2.

COROLLARY 1. *If f is in the L^1-Dini class, the relation in* (2.1) *holds a.e.*

This corollary follows from a theorem in Stein [5] which depends on a theorem of Bochner in [1]. We get this result from Theorem 1 and Lemma 4 below, which can be shown similarly as in [3], without referring to a result in [1].

LEMMA 4 ([3]). $J(f) \leqq c\|f\|_{D^1}$.

3. **Remark.** In this section we shall construct a function f on $[-\pi, \pi]$ which has finite entropy, but which deos not belong to both the class $L\log^+L\log^+\log^+L$ and the L^1-Dini class. Thus our theorem is not included in the theorem of Sjölin cited in Introduction. See [6].

LEMMA 5. *Let f be a non-negative function of the form $f = \sum_k a_k\chi_{E_k}$, $\bigcup_k E_k = [-\pi, \pi]$, $E_k \cap E_j = \emptyset$ if $k \neq j$. If there exists a constant $c_1 > 0$*

such that $|f(x)-f(y)| \geqq c_1 f(x)$ *for* $x \in E_j$ $y \in E_k$, *with* $j \neq k$. *Then it follows that*

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{|f(x)-f(y)|}{|x-y|}dxdy \geqq (c_1/2)\sum_k a_k \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|\chi_{E_k}(x)-\chi_{E_k}(y)|\frac{dxdy}{|x-y|}\ .$$

PROOF. The left hand side equals

$$\sum_{k\neq j}\int_{x\in E_k}\int_{y\in E_j}\frac{|f(x)-f(y)|}{|x-y|}dxdy = \sum_k \int_{x\in E_k}\int_{y\in E_k^c}\frac{|f(x)-f(y)|}{|x-y|}dxdy\ .$$

By our assumption the last term is not smaller than

$$c_1\sum_k \int_{x\in E_k}\int_{y\in E_k^c}\frac{|f(x)|}{|x-y|}dxdy = (c_1/2)\sum a_k\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|\chi_{E_k}(x)-\chi_{E_k}(y)|\frac{dxdy}{|x-y|}\ .$$

Let J be an interval $[0, a]$ and

$$S_\delta = \bigcup_{m=0}^{[a/(100\delta)]-1}[100m\delta, (100m+1)\delta]\ , \quad \text{where} \quad a, \delta > 0\ .$$

Then we have $|S_\delta| \sim |J| = a$ and

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|\chi_{S_\delta}(x)-\chi_{S_\delta}(y)|\frac{dxdy}{|x-y|} \geqq ca\log(a/\delta)\ .$$

Let J_k and S_{δ_k} be the sets J and S_δ with $a = 2^{-k}k^{-2}(\log k)^{-2}$ and $\delta = a2^{-2^k}$. Furthermore, we translate J_k and S_{δ_k} so that $J_j \cap J_k = \varnothing$ $(j \neq k)$. Put $f = \sum_{k\geqq 20} 2^k \chi_{S_{\delta_k}}$. Then we have that $\|f\|_{D^1} = \infty$ by Lemma 5 with $c_1 = 1/2$. By a direct computation

$$\int f\log^+ f\log^+\log^+ f dx = \infty\ .$$

On the other hand, the finiteness of the entropy of f is obvious.

REFERENCES

[1] S. BOCHNER, Summation of multiple Fourier series by spherical means, Trans. Amer. Math. Soc. 40 (1936), 175-207.

[2] L. CARLESON, On convergence and growth of partial sums of Fourier series, Acta Math. 116 (1966), 135-157.

[3] R. FEFFERMAN, A theory of entropy in Fourier analysis, Advances in Math. 30 (1978), 171-201.

[4] E. M. STEIN, Localization and summability of multiple Fourier series, Acta Math. 100 (1958), 93-147.

[5] E. M. STEIN, On certain exponential sums arising in mulitple Fourier series, Ann. Math. 73 (1961), 87-109.

[6] P. SJÖLIN, An inequality of Paley and convergence a.e. of Walsh-Fourier series, Ark. Mat. 7 (1967), 551-570.

[7] A. ZYGMUND, Trigonometric Series, I & II, Cambridge 1959.

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