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NONLINEAR SEMIGROUP FOR THE UNNORMALIZED CONDITIONAL DENSITY

YASUHIRO FUJITA

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1. Introduction. We are concerned with partially observable control problems. Let X_t be a state process being controlled, Y_t an observation process and U_t an admissible control defined on a probability space (Ω, F, P). The process X_t and Y_t are governed by the following stochastic differential equations:

$$dX_t = b(X_t, U_t)dt + \sigma(X_t)dW_t \quad 0 < t \leqq T , \tag{1.1}$$

$$dY_t = h(X_t)dt + d\tilde{W}_t \quad 0 < t \leqq T , \tag{1.2}$$

where W_t and $\tilde{W}_t$ are independent Wiener processes with values in $\boldsymbol{R}^N$ and $\boldsymbol{R}^M$, respectively (for simplicity, we let $M = 1$ here).

Our object is to minimize

$$J = Ef(X_T) \tag{1.3}$$

by a suitable choice of an admissible control, where f is a given cost function. Define Z_t by

$$Z_t = \exp\left[\int_0^t h(X_s)dY_s - (1/2)\int_0^t |h(X_s)|^2 ds\right].$$

Then, by Girsanov's formula, Y_t and W_t turn out as independent Wiener processes under the new probability measure $\hat{P}$ defined by $d\hat{P} = Z_T^{-1}dP$. In partially observable control problems, an admissible control U_t is usually measurable with respect to $\sigma_t(Y)$ (the σ-field generated by the observation process Y_s for $0 \leqq s \leqq t$). But, in this note we apply the same idea of admissibility as that in Fleming and Pardoux [5], namely we merely require that U_t is independent of W and $Y_r - Y_t$ for $r \geqq t$. Let F_t denote $\sigma_t(Y, U)$ and $L(u)$ be the infinitesimal generator of X_t with a constant control u. Bensoussan [1] and Pardoux [9] showed that the unnormalized conditional probability $P(t, \omega)$, defined by

$$\hat{E}[g(X_t)Z_t|F_t](\omega) = \int_{\boldsymbol{R}^N} g(x)P(t, \omega)(dx)$$

for any bounded Borel function g on $\boldsymbol{R}^N$, has a density $p(t, x, \omega)$ under mild assumptions on b, σ and h. Furthermore, $p(t)$ is regarded as a

Sobolev space $H^2(\boldsymbol{R}^N)$-valued process, which satisfies the following Zakai equation:

(1.4) $$dp(t) = L^*(U_t)p(t)dt + hp(t)dY_t\,, \quad 0 \leqq t \leqq T\,.$$

In § 2, we show some regularity results on $p(t)$. In § 3, we regard $p(t)$ as a state which is governed by the equation (1.4). So, our problem (1.3) turns out to be the minimization of $J = \hat{E}F(p(t))$, and we construct a nonlinear semigroup Q_t associated with the optimal value. In § 4, we look for the generator of Q_t which is related to Mortensen equation.

Our semigroup Q_t is heavily related to the semigroup constructed by Fleming [4]. He regarded an unnormalized conditional distribution itself as state and constructed a nonlinear semigroup on the space of functions of measures on $\boldsymbol{R}^N$. Here, we mainly use the $L^2(\boldsymbol{R}^N)$-theory instead of his method of measure theory.

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2. The control problem for the unnormalized conditional density. Assume the following conditions (A1)~(A5):

(A1) Γ is a convex compact subset of $\boldsymbol{R}^L$.

(A2) $a \in C_b^3(\boldsymbol{R}^N, \boldsymbol{R}^{N^2})$, where $a = (a_{ij}) = \sigma\sigma^*$.

(A3) $b \in C_b(\boldsymbol{R}^N \times \boldsymbol{R}^L, \boldsymbol{R}^N)$ and $b(\cdot, u) \in C_b^2(\boldsymbol{R}^N, \boldsymbol{R}^N)$ for each $u \in \Gamma$.

(A4) $h \in C_b^2(\boldsymbol{R}^N, \boldsymbol{R})$.

(A5) There exists $\alpha > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geqq \alpha|\xi|^2 \qquad x \in \boldsymbol{R}^N \quad \text{and} \quad \xi \in \boldsymbol{R}^N\,,$$

where $C_b^k(\boldsymbol{R}^m, \boldsymbol{R}^n)$ is the space of functions whose partial derivatives up to order k are bounded continuous $\boldsymbol{R}^n$-valued functions on $\boldsymbol{R}^m$.

Choose any $T > 0$ which is fixed throughout this note. For each $t \in [0, T]$, put

$$\Omega_t = \{(Y, U)\colon Y_0 = 0,\ Y \in C([0, T], \boldsymbol{R}^N),\ U \in L^2([0, T], \Gamma)\}$$

and $F_t = \sigma_t(Y, U)$.

DEFINITION. A probability measure π_t on (Ω_t, F_t) is called an admissible control on $[0, t]$, if Y is a $(\pi_t, \{F_s\})$-Wiener process for $0 \leqq s \leqq t$.

Let $\mathscr{A}_t$ denote the set of all admissible controls on $[0, t]$. When $t = T$, we denote Ω_T and $\mathscr{A}_T$ by Ω and $\mathscr{A}$. For simplicity, we use the following notations, $L^2 = L^2(\boldsymbol{R}^N)$, $H^i = H^i(\boldsymbol{R}^N)$ $(i = -1, 1, 2, 3)$, $(\cdot, \cdot) =$ scalar product in L^2, $|\cdot| = L^2$-norm, $\|\cdot\| = H^1$-norm, $\|\cdot\|_2 = H^2$-norm,

$\langle \cdot, \cdot \rangle$ = duality pairing between H^1 and H^{-1}. For each $\pi \in \mathscr{A}$ denote by $M_\pi^2(0, T; L^2)$ the space of L^2-valued measurable processes Φ such that

(i) $\Phi(t)$ is an F_t-adapted process,

(ii) $$E_\pi \int_0^T |\Phi(s)|^2 ds < \infty ,$$

where E_π stands for the expectation with respect to π. We define similarly $M_\pi^2(0, T: X)$ for $X = \boldsymbol{R}^N$ and H^i. For $\Phi \in M_\pi^2(0, T: L^2)$, we define an L^2-valued stochastic process $\int_0^t \Phi(s) dY_s$ by

$$\left(\phi, \int_0^t \Phi(s) dY_s\right) = \int_0^t (\phi, \Phi(s)) dY_s , \quad \phi \in L^2 .$$

Define the operators $L(u)$ and $L^*(u)$ of $L(H^1, H^{-1})$ by

$$\begin{aligned} \text{(2.1)} \qquad \langle L(u)\phi, \psi \rangle &= \langle \phi, L^*(u)\psi \rangle \\ &= -\frac{1}{2} \sum_{i,j=1}^N \int_{\boldsymbol{R}^N} a_{ij}(x) \frac{\partial \phi}{\partial x_i} \frac{\partial \psi}{\partial x_j} dx + \sum_{i=1}^N \int_{\boldsymbol{R}^N} \tilde{b}_i(x, u) \frac{\partial \phi}{\partial x_i} \psi dx \end{aligned}$$

for $\phi, \psi \in H^1$ and $u \in \Gamma$, where $L(H^1, H^{-1})$ is the space of bounded linear operators from H^1 into H^{-1} and

$$\tilde{b}_i(x, u) = b_i(x, u) - \frac{1}{2} \sum_{j=1}^N \frac{\partial a_{ij}}{\partial x_j}(x) .$$

Thanks to (A1)$\sim$(A5), it is easily proved that there exist $\lambda \in \boldsymbol{R}$ and $\alpha > 0$, so that for all $\phi \in H^1$ and $u \in \Gamma$,

$$\text{(2.2)} \qquad -\langle L^*(u)\phi, \phi \rangle + \lambda |\phi|^2 \geqq (\alpha/2) \|\phi\|^2 .$$

We consider the following two Zakai equations (2.3) and (2.4):

$$\text{(2.3)} \qquad \begin{cases} p \in M_\pi^2(0, T: H^1) \\ dp(t) = L^*(U_t)p(t)dt + hp(t)dY_t \\ p(0) = \psi \in L^2 . \end{cases}$$

$$\text{(2.4)} \qquad \begin{cases} p \in M_\pi^2(0, T: H^1) \\ dp(t) = [L^*(U_t)p(t) + f(t)]dt + [hp(t) + g(t)]dY_t \\ p(0) = \psi \in L^2 , \end{cases}$$

where $f \in M_\pi^2(0, T: H^{-1})$ and $g \in M_\pi^2(0, T: L^2)$. We state the following propositions without proof, which are easy variants of the results of Bennsoussan [1] and Pardoux [9].

PROPOSITION 2.1. *For each $\pi \in \mathscr{A}$, the equation* (2.4) *has a unique solution p, which satisfies*

(i) $$p \in L^2(\Omega, d\pi, C([0, T], L^2))$$

(ii) $$|p(t)|^2 = |\psi|^2 + 2\int_0^t \langle L^*(U_s)p(s) + f(s), p(s)\rangle ds + 2\int_0^t (hp(s) + g(s), p(s))dY_s + \int_0^t |hp(s) + g(s)|^2 ds .$$

PROPOSITION 2.2. *Besides* (A1)~(A5), *we assume that* ψ *belongs to* H^2. *Then, for each* $\pi \in \mathscr{A}$, *the equation* (2.3) *has a unique solution* p, *which satisfies* (ii) *for* $f = g = 0$ *and*

(iii) $$p \in M_\pi^2(0, T: H^3) \cap L^2(\Omega, d\pi; C([0, T], H^2)) .$$

Furthermore, $\partial p/\partial x_i$ $(i = 1, \cdots, N)$ *satisfies the equation*

$$\begin{cases} d\left(\dfrac{\partial p}{\partial x_i}\right) = \left[L^*(U_t)\left(\dfrac{\partial p}{\partial x_i}\right) + \tilde{f}(t)\right]dt + \left[h\left(\dfrac{\partial p}{\partial x_i}\right) + \tilde{g}(t)\right]dY_t . \\ \left(\dfrac{\partial p}{\partial x_i}\right)(0) = \dfrac{\partial \psi}{\partial x_i} \end{cases}$$

where $\tilde{f} \in M_\pi^2(0, T: H^{-1})$ *and* $\tilde{g} \in M_\pi^2(0, T: L^2)$ *are defined by*

$$\langle \tilde{f}(t), \phi\rangle = -\frac{1}{2}\sum_{k,l=1}^N \int_{R^N} \frac{\partial a_{kl}}{\partial x_i}(x)\frac{\partial \phi}{\partial x_k}\frac{\partial p}{\partial x_l}(t)dx + \sum_{k=1}^N \int_{R^N} \frac{\partial \tilde{b}_k}{\partial x_i}(x, U_t)\frac{\partial p}{\partial x_k}(t)\phi dx \qquad \text{for} \quad \phi \in H^1$$

and

$$\tilde{g}(t) = (\partial h/\partial x_i)p(t) .$$

LEMMA 2.1. *There exist constants* $K_1, K_2 \geqq 1$, *such that for any* $\psi \in L^2$

$$\sup_{\pi \in \mathscr{A}} E_\pi |p(t)|^2 \leqq K_1|\psi|^2 \quad 0 \leqq t \leqq T , \tag{2.5}$$

$$\sup_{\pi \in \mathscr{A}} E_\pi |p(t)|^4 \leqq K_2|\psi|^4 \quad 0 \leqq t \leqq T . \tag{2.6}$$

PROOF. Using Propostion 2.1 and (2.2), we get

$$|p(t)|^2 \leqq |\psi|^2 + K_3\int_0^t |p(s)|^2 ds + \int_0^t (hp(s), p(s))dY_s , \tag{2.7}$$

where $K_3 = (2\lambda + |h|_\infty^2)$ and $|h|_\infty = \sup_{x\in R^N}|h(x)|$. Taking the expectation of both sides of (2.7) and using Gronwall's inequality, we obtain (2.5) for $K_1 = e^{K_3 T}$.

Next, from (2.7), we get

$$E_\pi|p(t)|^4 \leqq 4|\psi|^4 + 4(K_3^2 T + |h|_\infty^2)E_\pi\int_0^t |p(s)|^4 ds . \tag{2.8}$$

By Gronwall's inequality, we obtain (2.6).

Using the same methods as in Lemma 2.1, we obtain the following.

LEMMA 2.2. *There exist constants* $K_4, K_5 \geqq 1$, *such that for any* $\psi \in H^1$

$$\sup_{\pi \in \mathscr{A}} E_\pi \|p(t)\|^2 \leqq K_4 \|\psi\|^2 \quad 0 \leqq t \leqq T, \tag{2.9}$$

$$\sup_{\pi \in \mathscr{A}} E_\pi \|p(t)\|^4 \leqq K_5 \|\psi\|^4 \quad 0 \leqq t \leqq T. \tag{2.10}$$

3. Nonlinear semigroup. Hereafter, we assume that the initial $p(0) = \psi$ belongs to H^2. Let C denote the space of functionals on H^2 satisfying the following two conditions:

(C1) For any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $\phi, \tilde{\phi} \in H^2$ and $\|\phi - \tilde{\phi}\| < \delta$, then

$$|F(\phi)/(1 + \|\phi\|^2) - F(\tilde{\phi})/(1 + \|\tilde{\phi}\|^2)| < \varepsilon.$$

$$\sup_{\phi \in H^2} [|F(\phi)|/(1 + \|\phi\|^2)] < \infty. \tag{C2}$$

For simplicity, we put $\rho(\phi) = (1 + \|\phi\|^2)$ for $\phi \in H^1$. We define a norm $\|\cdot\|_C$ by

$$\|F\|_C = \sup_{\phi \in H^2} [|F(\phi)|\rho^{-1}(\phi)].$$

Then, C becomes a Banach space.

Define Q_t by

$$Q_t F(\psi) = \inf_{\pi \in \mathscr{A}} E_\pi F(p_\psi(t)),$$

where $p_\psi(t)$ is the solution of (2.3). Then, we have the following theorem.

THEOREM 1. *Q_t maps C into C.*

PROOF. For $F \in C$ and $\psi, \tilde{\psi} \in H^2$, we get

$$\begin{aligned}
&|Q_t F(\psi)\rho^{-1}(\psi) - Q_t F(\tilde{\psi})\rho^{-1}(\tilde{\psi})| \\
&\quad \leqq \sup_{\pi \in \mathscr{A}} [E_\pi |F(p_\psi(t))\rho^{-1}(p_\psi(t)) - F(p_{\tilde{\psi}}(t))\rho^{-1}(p_{\tilde{\psi}}(t))|\, \rho(p_\psi(t))\rho^{-1}(\psi)] \\
&\qquad + \sup_{\pi \in \mathscr{A}} [E_\pi |F(p_{\tilde{\psi}}(t))|\, \rho^{-1}(p_{\tilde{\psi}}(t))\, |\rho(p_\psi(t))\rho^{-1}(\psi) - \rho(p_{\tilde{\psi}}(t))\rho^{-1}(\tilde{\psi})|] \\
&\quad \equiv I_1 + I_2, \quad \text{say}.
\end{aligned}$$

For any $\varepsilon > 0$, we can choose $\delta = \delta(\varepsilon) > 0$ so that

$$|F(\phi)\rho^{-1}(\phi) - F(\tilde{\phi})\rho^{-1}(\tilde{\phi})| < \varepsilon,$$

whenever $\phi, \tilde{\phi} \in H^2$ and $\|\phi - \tilde{\phi}\| < \delta$. Put $A = \{\omega: \|p_\psi(t) - p_{\tilde{\psi}}(t)\| < \delta\}$. Then,

$$I_1 \leqq \varepsilon \sup_{\pi \in \mathscr{A}} E_\pi[1_A \rho(p_\psi(t))\rho^{-1}(\psi)] + 2\|F\|_C \sup_{\pi \in \mathscr{A}} E_\pi[1_{A^c} \rho(p_\psi(t))\rho^{-1}(\psi)] ,$$

where 1_A stands for the characteristic function of A. Using (2.9) and (2.10), we get

$$I_1 \leqq \varepsilon(1 + K_4) + (2/\delta)\|F\|_C[2(1 + K_5)]^{1/2}\|\psi - \tilde{\psi}\| .$$

In the same way, we have

$$I_2 \leqq \|F\|_C(1 + 6K_4)\|\psi - \tilde{\psi}\| .$$

Hence, we get

$$\begin{aligned}&|Q_tF(\psi)\rho^{-1}(\psi) - Q_tF(\tilde{\psi})\rho^{-1}(\tilde{\psi})| \\ &\qquad \leqq \varepsilon(1 + K_4) + \|F\|_C[(2/\delta)\{2(1 + K_5)\}^{1/2} + (1 + 6K_6)]\|\psi - \tilde{\psi}\| .\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, Q_tF satisfies (C1). Appealing to

$$|Q_tF(\psi)| \leqq \|F\|_C(1 + K_4)\rho(\psi) ,$$

we see that Q_tF satisfies (C2). This completes the proof.

THEOREM 2. *$Q_{s+t}F = Q_sQ_tF$ for $F \in C$ and $s, t \geqq 0$, $s + t \leqq T$.*

From Theorems 1 and 2, we see that Q_t is a semigroup on C.

The proof of Theorem 2 is based on two lemmas. We write $p_\psi(t)$ as $p_\psi^{YU}(t)$ to emphasize its dependence on $(Y, U) \in \Omega$. Let us denote

$$Y_t^s = Y_{s+t} - Y_s , \qquad U_t^s = U_{s+t} .$$

Clearly, $(Y, U) \in \Omega$ implies $(Y^s, U^s) \in \Omega_{T-s}$.

LEMMA 3.1. *For each $\pi \in \mathscr{A}$ and $0 < s < T$, the following equation holds as an element of $C([0, T - s], H^2)$ with π_{T-s}-probability 1,*

$$p_\psi^{YU}(s + t) = p_{p^s}^{Y^sU^s}(t) \qquad \text{for} \quad t \in [0, T - s] , \tag{3.1}$$

where $p^s = p_\psi^{YU}(s)$.

PROOF. We have for $0 < t < T - s$, π_{T-s}-a.s.

$$\begin{aligned}p_\psi^{YU}(s + t) &= \psi + \int_0^{s+t} L^*(U_\theta)p(\theta)d\theta + \int_0^{s+t} hp(\theta)dY_\theta \\ &= \psi + \int_0^s L^*(U_\theta)p(\theta)d\theta + \int_0^s hp(\theta)dY_\theta + \int_s^{s+t} L^*(U_\theta)p(\theta)d\theta + \int_s^{s+t} hp(\theta)dY_\theta \\ &= p_\psi^{YU}(s) + \int_0^t L^*(U_\theta^s)p(s + \theta)d\theta + \int_0^t hp(s + \theta)dY_\theta^s .\end{aligned}$$

Since the solution of (2.3) is uniquely determined in $L^2(\Omega_{T-s}, \pi_{T-s}, C([0, T - s]H^2))$, this completes the proof.

Let $\pi_s(Y, U)$ be the regular conditional distribution for (Y^s, U^s) given

F_s. Next lemma is proved by the same method as in Fleming [4].

LEMMA 3.2. *If $\pi \in \mathscr{A}$, then we get for $0 \leqq s, t, s+t \leqq T$,*

(i) $$\pi_s(Y, U) \in \mathscr{A}_{T-s}\,, \qquad \pi_s\text{-a.s.}\,,$$

(ii) $$E_\pi F(p_\psi^{Y,U}(s+t)) = \int_\Omega E_{\pi_s(Y,U)}[F(p_{p^s}^{Y^sU^s}(t))]d\pi_s\,.$$

PROOF OF THEOREM 2. Step 1. From Lemma 3.2 (ii), we see

$$E_\pi F(p_\psi^{YU}(s+t)) \geqq \int_\Omega Q_t F(p_\psi^{YU}(s))d\pi_s \geqq Q_s Q_t F(\psi)\,. \tag{3.2}$$

Since (3.2) holds for all $\pi \in \mathscr{A}$, we get $Q_{s+t}F(\psi) \geqq Q_s Q_t F(\psi)$. Step 2. We prove the opposite inequality. Let $\varepsilon \in (0, 1)$ be arbitrary. Since H^2 is a separable Hilbert space, we can find a sequence of Borel sets $B_1, B_2, \cdots$ such that $B_i \cap B_j = \varnothing$ if $i \neq j$, $\operatorname{diam} B_i < \varepsilon$ and $\bigcup_{i=1}^\infty B_i = H^2$. For any $\psi_i \in B_i$, choose $\pi_i \in \mathscr{A}$ so that

$$Q_t F(\psi_i) + \varepsilon > E_{\pi_i} F(p_{\psi_i}(t))\,. \tag{3.3}$$

On the other hand, recalling the same calculation as for I_1 in the proof of Theorem 1, we get for any $\psi, \tilde{\psi} \in H^2$

$$\begin{aligned} &|Q_t F(\psi) - Q_t F(\tilde{\psi})| \\ &\qquad \leqq K_6 \varepsilon \rho(\psi) + K_6[\rho(\psi) + \|F\|_C(\|\psi\| + \|\tilde{\psi}\|)]\,\|\psi - \tilde{\psi}\|\,, \end{aligned}$$

where $K_6 = \max\{(1 + K_4), 1 + 6K_6 + 2(1 + K_4)^{1/2}\}$. Hence, for each $\psi \in B_i$,

$$\begin{aligned} E_{\pi_i} F(p_\psi(t)) &\leqq Q_t F(\psi_i) + \varepsilon + 4\varepsilon K_6 \rho(\psi) + \varepsilon K_6 \|F\|_C(1 + 2\|\psi\|) \\ &\leqq Q_t F(\psi) + \varepsilon K_7 \rho(\psi)\,, \end{aligned}$$

where K_7 is a suitable positive constant depending only on $\|F\|_C$ and K_6.

Put $\pi_s(Y, U) = \sum_{i=1}^\infty \pi_i 1_{\Omega_i}$, where $\Omega_i = \{\omega\colon p_\psi^{YU}(s, \omega) \in B_i\}$. For a given $\pi_s \in \mathscr{A}_s$, we can find $\pi \in \mathscr{A}$ so that $\pi_s(Y, U)$ is a regular conditional distribution for (Y^s, U^s) given F_s and $\pi | F_s = \pi_s$. By Lemma 3.2 and the above results, we see

$$\begin{aligned} E_\pi F(p_\psi^{YU}(s+t)) &\leqq \sum_{i=1}^\infty \int_{\Omega_i} [Q_t F(p_\psi^{YU}(s)) + \varepsilon K_7 \rho(\psi)]d\pi_s \\ &= E_{\pi_s}[Q_t F(p_\psi^{YU}(s)) + \varepsilon K_7 \rho(\psi)]\,. \end{aligned}$$

Therefore, we obtain

$$Q_{s+t}F(\psi) \leqq E_{\pi_s}[Q_t F(p_\psi^{YU}(s))] + \varepsilon K_7 \rho(\psi)\,.$$

On taking the infimum over $\pi_s \in \mathscr{A}_s$, we have

$$Q_{s+t}F(\psi) \leqq Q_s Q_t F(\psi) + \varepsilon K_7 \rho(\psi)\,.$$

Step 3. From Steps 1 and 2, we see, for any $\varepsilon > 0$

$$\|Q_{s+t}F - Q_s Q_t F\|_C \leqq \varepsilon K_7 .$$

Letting ε tend to 0, we get Theorem 2.

Now, we consider the continuity of Q_t. We put

$$H_r^2 = \{\phi \in H^2 \colon \|\phi\|_2 \leqq r\} \quad (0 < r < \infty) .$$

THEOREM 3. *For each r, $Q_t F(\psi) \to Q_s F(\psi)$ uniformly on H_r^2, as $|t-s| \to 0$.*

For the proof of Theorem 3, we need two more lemmas, of which the first is obvious.

LEMMA 3.3. *There exists a positive constant K_8 so that*

$$\sup_{u \in \Gamma} \|L^*(u)\|_{L(H^1, H^{-1})} \leqq K_8 < \infty .$$

LEMMA 3.4. *There exist positive constants K_9, K_{10} so that for any $\psi \in H^2$*

$$\sup_{\pi \in \mathscr{A}} E_\pi |p_\psi(t) - p_\psi(s)|^2 \leqq K_9 |t-s| \, \|\psi\|^2 , \tag{3.4}$$

$$\sup_{\pi \in \mathscr{A}} E_\pi \|p_\psi(t) - p_\psi(s)\|^2 \leqq K_{10} |t-s| \, \|\psi\|_2^2 . \tag{3.5}$$

PROOF. We prove only (3.4). Using Lemma 2.2, we can prove (3.5) by the same methods as (3.4). For simplicity, put $s=0$. We set $\tilde{p}(t) = p_\psi(t) - \psi$. Then, $\tilde{p}(t)$ satisfies the following equation π-a.s.:

$$d\tilde{p}(t) = [L^*(U_t)\tilde{p}(t) + L^*(U_t)\psi]dt + [h\tilde{p}(t) + h\psi]dY_t ,$$
$$\tilde{p}(0) = 0 .$$

Using Proposition 2.1, Lemma 3.3 and the inequality

$$2|ab| \leqq \mu a^2 + b^2/\mu \quad (a, b \in \boldsymbol{R},\ \mu > 0) ,$$

we get

$$E_\pi |\tilde{p}(t)|^2 + (\alpha - K_8\mu) E_\pi \int_0^t \|\tilde{p}(s)\|^2 ds \leqq 2\lambda E_\pi \int_0^t |\tilde{p}(s)|^2 ds$$
$$+ (K_8 t/\mu) \|\psi\|^2 + K_1 t |h|_\infty^2 |\psi|^2$$

We choose $\mu > 0$ sufficiently small so that $\alpha - K_8\mu > 0$. Then, by Gronwall's inequality, we get for any $\pi \in \mathscr{A}$,

$$E_\pi |\tilde{p}(t)|^2 \leqq (1 + 2\lambda T e^{2\lambda T})[(K_8 t/\mu) \|\psi\|^2 + K_1 t |h|_\infty^2 |\psi|^2] .$$

Put $K_9 = 2(1 + 2\lambda T e^{2\lambda T}) \times [\max\{K_8/\mu, K_1 |h|_\infty^2\}]$. Then, we have

$$\sup_{\pi \in \mathscr{A}} E_\pi |\tilde{p}(t)|^2 \leqq K_9 t \|\psi\|^2 .$$

This completes the proof.

PROOF OF THEOREM 3. For any $\varepsilon > 0$, choose $\delta = \delta(\varepsilon) > 0$ so that if $\phi, \tilde{\phi} \in H^2$ and $\|\phi - \tilde{\phi}\| < \delta$, then

$$|F(\phi)\rho^{-1}(\phi) - F(\tilde{\phi})\rho^{-1}(\tilde{\phi})| < \varepsilon .$$

By (3.5), we get

$$|Q_tF(\psi) - Q_sF(\psi)| \leqq \varepsilon(1 + K_4)\rho(\psi) + 2\|F\|_C\|\psi\| K_{10}^{1/2} \times |t - s|[(1/\delta)\|\varphi\|\{2(1 + K_5\|\psi\|)\}^{1/2} + \|\psi\|_2 K_4^{1/2}] .$$

Hence, choosing a suitable positive constant K_{11}, we get on H_r^2

$$|Q_tF(\psi) - Q_sF(\psi)| \leqq K_{11}(\varepsilon + |t - s|^{1/2})(1 + r^3) .$$

Since $\varepsilon > 0$ is arbitrary, this completes the proof.

4. **The generator of the semigroup Q_t.** Let C^2 denote the totality of $F \in C$ satisfying the following conditions:

(i) F is defined on L^2 and twice continuously Fréchet differentiable on L^2.

(ii) $\phi \in H^1$ implies that the first derivative $DF(\phi)$ is in H^1 and $DF(\phi)/(1 + \|\phi\|)$ is bounded and uniformly continuous on H^1.

(iii) The second derivatine D^2F is bounded and uniformly continuous on L^2.

By Pardoux [9], for $F \in C^2$, we have Ito's formula in infinite dimension for the solution of (2.3) as follows:

$$F(p(t)) = F(\psi) + \int_0^t \langle DF(p(s)), L^*(U_s)p(s)\rangle ds + \int_0^t (DF(p(s)), hp(s))dY_s + (1/2)\int_0^t (D^2F(p(s))hp(s), hp(s))ds . \tag{4.1}$$

We define the operators $\mathscr{L}(u)$ and $\mathscr{L}$ on C^2 by

$$\mathscr{L}(u)F(\psi) = \langle DF(\psi), L^*(u)\psi\rangle + (1/2)(D^2F(\psi)h\psi, h\psi) \quad u \in \Gamma ,$$

and

$$\mathscr{L}F(\psi) = \inf_{u \in \Gamma} \mathscr{L}(u)F(\psi) .$$

We can easily see that $\mathscr{L}F$ belongs to C. Taking the expectation in (4.1), we have for each $\pi \in \mathscr{A}$,

$$E_\pi F(p(t)) - F(\psi) = E_\pi \int_0^t \mathscr{L}(U_s)F(p(s))ds . \tag{4.2}$$

THEOREM 4. *For each $r < \infty$,*

$$\lim_{t\downarrow 0}(1/t)[Q_tF(\psi) - F(\psi)] = \mathscr{L}F(\psi)$$

holds uniformly on H_r^2.

PROOF. Let $\varepsilon > 0$ be arbitrary. We choose $\delta = \delta(\varepsilon) > 0$ so that, if $\|\phi - \tilde{\phi}\| < \delta$, then

$$\|DF(\phi)/(1 + \|\phi\|) - DF(\tilde{\phi})/(1 + \|\tilde{\phi}\|)\| < \varepsilon$$

and

$$\|D^2F(\phi) - D^2F(\tilde{\phi})\|_{L(L^2,L^2)} < \varepsilon .$$

So, we get

$$\begin{aligned}
&\sup_{\pi\in\mathscr{A}} E_\pi\left|\int_0^t \mathscr{L}(U_s)F(p(s))ds - \int_0^t \mathscr{L}(U_s)F(\psi)ds\right| \\
&\quad\leqq \sup_{\pi\in\mathscr{A}} E_\pi\int_0^t |\langle DF(p(s)), L^*(U_s)p(s)\rangle - \langle DF(\psi), L^*(U_s)\psi\rangle|\,ds \\
&\qquad + (1/2)\sup_{\pi\in\mathscr{A}} E_\pi\int_0^t |(D^2F(p(s))hp(s), hp(s)) - (D^2F(\psi)h\psi, h\psi)|\,ds \\
&\quad\equiv J_1 + J_2\,, \quad \text{say} .
\end{aligned}$$

By Lemma 3.3, we get

$$\begin{aligned}
J_1 &\leqq K_8 \sup_{\pi\in\mathscr{A}} E_\pi\int_0^t \|DF(p(s))\|\cdot\|p(s) - \psi\|\,ds \\
&\quad + K_8\|\psi\|\sup_{\pi\in\mathscr{A}} E_\pi\int_0^t \|DF(p(s)) - DF(\psi)\|\,ds .
\end{aligned}$$

Put $\|DF\|_{C^1} = \sup_{\phi\in H^2}[|DF(\phi)|/(1 + \|\phi\|)]$. Then, using Lemmas 2.2 and 3.4 and choosing a suitable positive constant K_{12} depending only on K_4, K_8, K_{10} and $\|DF\|_{C^1}$, we have

$$J_1 \leqq K_{12}t(\varepsilon + t^{1/2})(1 + \|\psi\|_2^3) . \tag{4.3}$$

Put $\|D^2F\|_{C^2} = \sup_{\phi\in L^2}\|D^2F(\phi)\|_{L(L^2,L^2)}$. Then, choosing a suitable positive constant K_{13} depending only on K_1 K_9, $|h|_\infty$ and $\|D^2F\|_{C^2}$, we have

$$J_2 \leqq K_{13}t(\varepsilon + t^{1/2})(1 + \|\psi\|_2^3) . \tag{4.4}$$

Next, we note that

$$\begin{aligned}
\inf_{\pi\in\mathscr{A}} E_\pi\int_0^t \mathscr{L}(U_s)F(\psi)ds &\geqq \inf_{\pi\in\mathscr{A}} E_\pi\int_0^t \mathscr{L}F(\psi)ds = t\mathscr{L}F(\psi) \\
&= \inf_{\pi\in\mathscr{A}}\int_0^t \mathscr{L}(u)F(\psi)ds \geqq \inf_{\pi\in\mathscr{A}} E_\pi\int_0^t \mathscr{L}(U_s)F(\psi)ds .
\end{aligned} \tag{4.3}$$

Hence, all inequalities are replaced by equalities. So, we get from (4.2)$\sim$(4.5),

$$
\begin{aligned}
&|(1/t)[Q_t F(\psi) - F(\psi)] - \mathscr{L}F(\psi)| \\
&\quad \leqq \sup_{\pi \in \mathscr{A}}(1/t)\left| E_\pi F(p(t)) - F(\psi) - E_\pi \int_0^t \mathscr{L}(U_s)F(\psi)ds \right| \\
&\quad \leqq \sup_{\pi \in \mathscr{A}}(1/t)\left| E_\pi \int_0^t \mathscr{L}(U_s)F(p(s)) - E_\pi \int_0^t \mathscr{L}(U_s)F(\psi)ds \right| \\
&\quad \leqq (K_{12} + K_{13})(\varepsilon + t^{1/2})(1 + \|\psi\|_2^3) .
\end{aligned}
$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\lim_{t \downarrow 0}(1/t)[Q_t F(\psi) - F(\psi)] = \mathscr{L}F(\psi)$$

uniformly on H_r^2. This completes the proof.

We denote $Q_t F(\psi)$ by $W(t, \psi)$. By Theorem 4, we expect that $W(t, \psi)$ is a solution of the following equations:

$$
\text{(4.6)} \qquad \begin{cases} dW/dt(t, \psi) = \mathscr{L}W(t, \psi) & \text{in} \quad (0, T) \times H^2 , \\ W(0, \psi) = F(\psi) & \text{on} \quad H^2 . \end{cases}
$$

It is, however, very difficult to derive the regularity of $W(t, \psi)$ with respect to t and ψ. We extend the concept of viscosity solution to infinite dimension and show that $W(t, \psi)$ is a viscosity solution of (4.6) in that sense.

Let G be a continuous functional on $(0, T) \times H^2$ so that $G(t)$ belongs to C for each $t \in (0, T)$. Denote by $E_+(G)$ the set of all $(t_0, \psi_0) \in (0, T) \times H^2$, where $[\max\{G(t, \psi); (t, \psi) \in (0, T) \times H^2\}]$ is attained. Similarly, denote $E_-(G)$ the set of all $(t_0, \psi_0) \in (0, T) \times H^2$, where $[\min\{G(t, \psi); (t, \psi) \in (0, T) \times H^2\}]$ is attained. We remark that if (t_0, ψ_0) belongs to $E_+(G)$ (resp. $E_-(G)$) and $\|\psi_0 - \tilde{\psi}_0\|_2 = 0$, then $(t_0, \tilde{\psi}_0)$ belongs to $E_+(G)$ (resp. $E_-(G)$). The following is due to Lions [6]:

DEFINITION. $W_0 \in C([0, T) \times H^2)$ is said to be a viscosity solution of (4.6), when it has the following properties:

$W_0(0, \psi) = F(\psi)$ and for any $G \in C([0, T] \times H^2)$ we have

$$\text{(4.7)} \qquad dG/dt - \mathscr{L}G \leqq 0 \qquad \text{at} \quad (t_0, \psi_0) \in E_+(W_0 - G)$$

$$\text{(4.8)} \qquad dG/dt - \mathscr{L}G \geqq 0 \qquad \text{at} \quad (t_0, \psi_0) \in E_-(W_0 - G) ,$$

if G is twice differentiable with respect to t, d^2G/dt^2 is bounded on $(0, T) \times L^2$, dG/dt belongs to C and $G(t, \psi)$ belongs to C^2 for each $t \in (0, T)$.

THEOREM 5. *$W(t, \psi)$ is a viscosity solution of* (4.6).

For the proof of Theorem 5, we introduce the following order in C.

DEFINITION. We say that $F \leqq \tilde{F}$ in C, if $F(\phi) \leqq \tilde{F}(\phi)$ for all $\phi \in H^2$.

LEMMA 4.1. *If $F \leqq \tilde{F}$ in C, then $Q_t F \leqq Q_t \tilde{F}$ in C for all $0 \leqq t \leqq T$.*

PROOF. Since $F \leqq \tilde{F}$ in C, we have for all $\psi \in H^2$ $\pi \in \mathscr{A}$ and $0 \leqq t \leqq T$,

$$Q_t F(\psi) \leqq E_\pi F(p_\psi(t)) \leqq E_\pi \tilde{F}(p_\psi(t)) .$$

On taking the infimum over $\pi \in \mathscr{A}$, we get

$$Q_t F(\psi) \leqq Q_t \tilde{F}(\psi) .$$

This completes the proof.

LEMMA 4.2. *Let $F \in C^2$ and $H \in C^\cdot$ Then, for each $\psi \in H^2$,* we have

$$\lim_{\theta \downarrow 0}(1/\theta)[Q_\theta(F + \theta H) - F](\psi) = H(\psi) + \mathscr{L}F(\psi)$$

PROOF. We have

$$\begin{aligned} &|(1/\theta)[Q_\theta(F + \theta H) - F](\psi) - [H(\psi) + \mathscr{L}F(\psi)]| \\ &\quad \leqq |(1/\theta)[Q_\theta(F + \theta H)(\psi) - Q_\theta F(\psi) - \theta H(\psi)]| \\ &\qquad + |(1/\theta)[Q_\theta F(\psi) - F(\psi)] - \mathscr{L}F(\psi)| \\ &\quad \equiv M_1 + M_2 , \quad \text{say} . \end{aligned}$$

Since $\mathscr{L}$ is the infinitesimal generator of the semigroup Q_t, we see that $M_2 \to 0$ as $\theta \downarrow 0$. On the other hand, we have

$$M_1 \leqq \sup_{\pi \in \mathscr{A}} E_\pi |H(p_\psi(\theta)) - H(\psi)| .$$

By (3.4), we have $M_1 \to 0$ as $\theta \downarrow 0$. This completes the proof.

PROOF OF THEOREM 5. By Theorem 3, we see easily $W \in C([0, T] \times H^2)$. Let $(t_0, \psi_0) \in E_+(W - G)$ and $M = (W - G)(t_0, \psi_0)$. Then, considering $G(t, \psi) + M$ instead of $G(t, \psi)$, we may assume $(W - G)(t_0, \psi_0) = 0$ without loss of generality. For $\theta \in (0, t_0)$ we have

$$G(t_0, \psi_0) = W(t_0, \psi_0) = [Q_\theta Q_{t_0-\theta} F](\psi_0) = [Q_\theta W(t_0 - \theta)](\psi_0) .$$

Since $W(t) \leqq G(t)$ in C for $0 \leqq t \leqq T$, using Lemma 4.1 we get

$$[Q_\theta W(t_0 - \theta)](\psi_0) \leqq [Q_\theta G(t_0 - \theta)](\psi_0) .$$

Since d^2G/dt^2 is bounded on $(0, T) \times L^2$, there exists $\varepsilon(\theta) > 0$, so that for all $\phi \in H^2$

$$G(t_0 - \theta, \phi) \leqq G(t_0, \phi) - [dG/dt(t_0, \phi)] + \theta\varepsilon(\theta) ,$$

and

$$\varepsilon(\theta) \to 0 \quad \text{as} \quad \theta \downarrow 0 .$$

Let $\varepsilon_0 > 0$ be arbitrary. We choose $\theta_0(\varepsilon_0) > 0$ in such a way that if $0 < \theta <$

$\theta_0(\varepsilon_0)$, then $0 < \varepsilon(\theta) < \varepsilon_0$. By Lemma 4.1, we have $G(t_0, \psi_0) \leqq Q_\theta[G(t_0) + H](\psi_0)$, where $H = -(dG/dt)(t_0, \cdot) + \varepsilon_0$. Hence, we have $(1/\theta)[Q_\theta(G(t_0) + H) - G(t_0)](\psi_0) \geqq 0$. Since dG/dt belongs to C, using Lemma 4.2 we have $dG/dt(t_0, \psi_0) - \mathscr{L}G(t_0, \psi_0) - \varepsilon_0 \leqq 0$. Since $\varepsilon_0 > 0$ is arbitrary letting ε_0 tend to 0, we get $dG/dt(t_0, \psi_0) - \mathscr{L}G(t_0, \psi_0) \leqq 0$.

The proof is similar, when (t_0, ψ_0) belongs to $E_-(W - G)$. This completes the proof.

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Department of Mathematics and System Fundamentals
Division of System Science
Kobe University
Rokko, Kobe, 657
Japan