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NORMS OF HANKEL OPERATORS AND UNIFORM ALGEBRAS, II

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Abstract. Let H^∞ be an abstract Hardy space associated with a uniform algebra. Denoting by (f) the coset in $(L^\infty)^{-1}/(H^\infty)^{-1}$ of an f in $(L^\infty)^{-1}$, define $\|(f)\|=\inf\{\|g\|_\infty\|g^{-1}\|_\infty;\ g\in(f)\}$ and $\gamma_0=\sup\{\|(f)\|;\ (f)\in(L^\infty)^{-1}/(H^\infty)^{-1}\}$. If γ_0 is finite, we show that the norms of Hankel operators are equivalent to the dual norms of H^1 or the distances of the symbols of Hankel operators from H^∞. If H^∞ is the algebra of bounded analytic functions on a multiply connected domain, then we show that γ_0 is finite and we determine the essential norms of Hankel operators.

0. Introduction. Let X be a compact Hausdorff space, let $C(X)$ be the algebra of complex-valued continuous functions on X, and let A be a uniform algebra on X. For $\tau\in M_A$, the maximal ideal space of A, set $A_0=\{f\in A;\ \tau(f)=0\}$. Let m be a representing measure for τ on X.

The abstract Hardy space $H^p=H^p(m)$, $1\leqq p\leqq\infty$, determined by A is defined to be the closure of A in $L^p=L^p(m)$ when p is finite and to be the weak*-closure of A in $L^\infty=L^\infty(m)$ when p is infinite. Put $H_0^p=\left\{f\in H^p;\ \int_X f dm=0\right\}$, $K^p=\left\{f\in L^p;\ \int_X fg dm=0 \text{ for all } g\in A_0\right\}$ and $K_0^p=\left\{f\in K^p;\ \int_X f dm=0\right\}$. Then $H_0^p\subset K_0^p$ and $H^p\subset K^p$.

Let $Q^{(1)}$ be the orthogonal projection from L^2 to $(H^2)^\perp=\bar{K}_0^2$ and $Q^{(2)}$ the orthogonal projection from L^2 to $\bar{H}_0^2$. For a function ϕ in L^∞ we denote by M_ϕ the multiplication operator on L^2 determined by ϕ. As in the previous paper [14], two generalizations of the classical Hankel operators are defined as follows. For ϕ in L^∞ and f in H^2

$$H_\phi^{(j)}f=Q^{(j)}M_\phi f \quad (j=1,2)\,.$$

If A is a disc algebra and $\tau(f)=\tilde{f}(0)$, where $\tilde{f}$ denotes the holomorphic extension of f in A, then τ is in M_A. The normalized Lebesgue measure m on the unit circle T is a representing measure for τ. Then H^2 is the classical Hardy space and $H_0^2=K_0^2$. Hence $H_\phi^{(1)}=H_\phi^{(2)}$ and it is the classical Hankel operator H_ϕ. It is well known that

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(a) $\|H_\phi\| = \|\phi + H^\infty\|$

and

(b) $\|H_\phi\|_e = \|\phi + H^\infty + C(T)\|$,

where the essential norm $\|H_\phi\|_e$ of H_ϕ is the distance to the compact operators. (a) is due to Nehari (cf. [16, Theorem 1.3], [15]), while (b) is due to Adamyan, Arov and Krein (cf. [16, p. 6], [2]). (b) yields Hartman's result (cf. [16, Theorem 1.4], [11]) to the effect that

(c) H_ϕ is compact if and only if ϕ is in $H^\infty + C(T)$.

In the previous paper [14] we considered the generalizations of (1). The main idea was to consider Hankel operators on vH^2 for every nonnegative invertible function v in L^∞, avoiding a factorization theorem of H_0^1. Namely, if h is in H_0^1 and $\int_X |h|dm \leqq 1$, then $h = fg$, $f \in H^2$ and $g \in H_0^2$ where $\int_X |f|^2 dm \leqq 1 + \varepsilon$ and $\int_X |g|^2 dm \leqq 1 + \varepsilon$ for some $\varepsilon > 0$. Let v be a nonnegative function in L^∞ with v^{-1} in L^∞. Let $Q_v^{(1)}$ be the orthogonal projection from L^2 onto $(vH^2)^\perp = v^{-1}\bar{K}_0^2$ and $Q_v^{(2)}$ the orthogonal projection from L^2 onto $v^{-1}\bar{H}_0^2$. If v is a constant function, then $Q_v^{(j)} = Q^{(j)}$ $(j = 1, 2)$. For $\phi \in L^\infty$ and $f \in vH^2$, $H_\phi^{(j)v}$ is the operator defined by

$$H_\phi^{(j)v} f = Q_v^{(j)} M_\phi f\ , \quad (j = 1, 2)\ .$$

If v is a nonzero constant, then $H_\phi^{(j)v} = H_\phi^{(j)}$ $(j = 1, 2)$. Put $(L^\infty)_+^{-1} = \{v \in L^\infty \colon v^{-1} \in L^\infty \text{ and } v \geqq 0\}$. The following theorem was shown in the previous paper [14] and gives (a).

Generalized Nehari's Theorem I. *Let ϕ be a function in L^∞, then*

$$\sup\{\|H_\phi^{(2)v}\|; v \in (L^\infty)_+^{-1}\} = \|\phi + K^\infty\|\ .$$

If K^∞ is dense in K^1, then

$$\sup\{\|H_\phi^{(1)v}\|; v \in (L^\infty)_+^{-1}\} = \|\phi + H^\infty\|\ .$$

We now show two lemmas which will be used in later sections. Let P_v be the orthogonal projection from L^2 onto vH^2. If v is a constant function, we shall write $P_v = P$.

Lemma 1. *Let ϕ be a function in L^∞. Then for any v and u in $(L^\infty)_+^{-1}$ and for $j = 1, 2$*

$$\|H_\phi^{(j)v} - H_\phi^{(j)u}\| \leqq \|\phi\|_\infty(\|v^{-1}\|_\infty + \|u^{-1}\|_\infty)\|v - u\|_\infty\ .$$

Proof. Since $\|M_v f\|_2 \geqq \|v^{-1}\|_\infty^{-1}\|f\|_2$ for all $f \in H^2$, by [7, Lemma 1.1.]

$$\|P_v - P_u\| \leqq (\|v^{-1}\|_\infty + \|u^{-1}\|_\infty)\|v - u\|_\infty\ .$$

Similarly, $\|Q_v^{(j)} - Q_u^{(j)}\| \leqq (\|v^{-1}\|_\infty + \|u^{-1}\|_\infty)\|v - u\|_\infty$. Hence for $j = 1, 2$

$$\begin{aligned}\|H_\phi^{(j)v} - H_\phi^{(j)u}\| &= \|Q_v^{(j)}M_\phi P_v - Q_u^{(j)}M_\phi P_v + Q_u^{(j)}M_\phi P_v - Q_u^{(j)}M_\phi P_u\| \\ &\leqq \|Q_v^{(j)} - Q_u^{(j)}\| \, \|M_\phi P_v\| + \|P_v - P_u\| \, \|Q_u^{(j)}M_\phi\| \\ &\leqq \|\phi\|_\infty(\|v^{-1}\|_\infty + \|u^{-1}\|_\infty)\|v-u\|_\infty .\end{aligned}$$

LEMMA 2. *Let ϕ be a function in L^∞. If $H_\phi^{(j)}$ $(j = 1, 2)$ is compact, then $H_\phi^{(j)v}$ $(j = 1, 2)$ is compact for any v in $(L^\infty)_+^{-1}$.*

PROOF. For any $f \in vH^2$ and $g \in v^{-1}\bar{K}_0^2$, $(H_\phi^{(1)v}f, g) = (vH_\phi^{(1)}(v^{-1}f), g)$. Hence $H_\phi^v f = Q^{(1)v}(M_v H_\phi^{(1)} M_{v^{-1}} f)$ for any $f \in vH^2$. The proof for $H_\phi^{(2)v}$ is similar. This implies the lemma.

Let N_τ denote the set of representing measures for τ on X. In this paper we sometimes will impose the following two conditions on τ:

(1) N_τ is finite dimensional and $n = \dim N_\tau$.

(2) m is a core measure of N_τ.

Let N^∞ be the real annihilator of A in L_R^∞. Then $\dim N^\infty = n$ and $A + \bar{A}_0 + N_c^\infty$ is weak*-dense in L^∞, where $N_c^\infty = N^\infty + iN^\infty$ (cf. [10, p. 109]). $L^2 = H^2 \oplus \bar{H}_0^2 \oplus N_c^\infty$. Set $\mathscr{E} = \exp N^\infty$; then $\mathscr{E}$ is a subgroup of $(L^\infty)_+^{-1}$. Moreover, together with (1) we often will make the following stronger conditions (3) on τ instead of (2).

(3) m is a unique logmodular measure of N_τ.

Then the linear span of $N^\infty \cap \log|(H^\infty)^{-1}|$ is N^∞ (cf. [10, p. 114]). Choose $h_1, \cdots, h_n \in (H^\infty)^{-1}$ so that $\{\log|h_j|\}_{j=1}^n$ is a basis for N^∞. Put $u_j = \log|h_j|$ $(1 \leqq j \leqq n)$ and $\mathscr{E}_0 = \{\exp(\sum_{j=1}^n s_j u_j); 0 \leqq s_j \leqq 1\}$. Then $\mathscr{E}_0 \subset \mathscr{E}$. The following theorem was shown in the previous paper [14].

GENERALIZED NEHARI'S THEOREM II. *Assume the assumptions* (1) *and* (3) *on τ. Let ϕ be a function in L^∞, then*

$$\sup_{v \in \mathscr{E}_0} \|H_\phi^{(2)v}\| = \|\phi + H^\infty + N_c^\infty\|$$

and

$$\sup_{v \in \mathscr{E}_0} \|H_\phi^{(1)v}\| = \|\phi + H^\infty\| .$$

Moreover the supremums in both equalities are attained.

In Section 1, γ_0, which is defined in Abstract, is studied. Under the assumptions (1) and (2) we determine when γ_0 is finite. In Section 2, we give examples of concrete uniform algebras to which results in this paper can apply. Moreover γ_0 is calculated in some examples. In Section 3, if γ_0 is finite, we show that $\|H_\phi^{(1)}\|$ (resp. $\|H_\phi^{(2)}\|$) is equivalent to $\|\phi + H^\infty\|$ (resp. $\|\phi + K^\infty\|$). In Section 4 we give applications of results in Section 3 to weighted norm inequalities for conjugation operators and invertible Toeplitz operators in uniform algebras. In Section 5 we determine the essential norms of Hankel operators in the case of (I) in Section 2. In Section 6 we consider the relationship between γ_0 and the factorization theorem of H_0^1. In Section 7 we consider the relationship between generalized Nehari's Theorem and Arveson's distance formula for nest algebras.

1. Quotient group and a constant. Denoting by (f) the coset in $(L^\infty)^{-1}/(H^\infty)^{-1}$ of an f in $(L^\infty)^{-1}$, define

$$\|(f)\| = \inf\{\|g\|_\infty\|g^{-1}\|_\infty;\ g \in (f)\}$$

and

$$\gamma_0 = \sup\{\|(f)\|;\ (f) \in (L^\infty)^{-1}/(H^\infty)^{-1}\}\ .$$

Then $\|(f)(h)\| \leqq \|(f)\|\,\|(h)\|$ and, in general, γ_0 can be finite or infinite. Let L_R^∞ be the space of real-valued functions in L^∞. Let $\log|(H^\infty)^{-1}|$ be the lattice in L_R^∞ consisting of the elements of the form $\log|f|$, $f \in (H^\infty)^{-1}$. There is a natural map of $(L^\infty)^{-1}/(H^\infty)^{-1}$ onto $L_R^\infty/\log|(H^\infty)^{-1}|$ which sends (f) to $(\log|f|)$. Define $|||(\log|f|)||| = \inf\{\|\log|f| + \log|g|\|_\infty;\ g \in (H^\infty)^{-1}\}$ and $\gamma_1 = \sup\{|||(\log|f|)|||;\ (\log|f|) \in L_R^\infty/\log|(H^\infty)^{-1}|\}$.

PROPOSITION 1. *$\|(f)\| = \exp 2|||(\log|f|)|||$ and $\gamma_0 = \exp 2\gamma_1$.*

PROOF. It suffices to show that $\|(f)\| = \exp 2|||(\log|f|)|||$ for all $f \in (L^\infty)^{-1}$. Pick such an f.

$$\begin{aligned}\|(f)\| &= \inf\{\|fg\|_\infty\|f^{-1}g^{-1}\|_\infty;\ g \in (H^\infty)^{-1}\}\\ &= \exp\inf\{\text{ess.}\sup(\log|f| + \log|g|) - \text{ess.}\inf(\log|f| + \log|g|)\}\ .\end{aligned}$$

Since the constants are in $(H^\infty)^{-1}$, this last quantity can be rewritten as

$$= \exp 2\inf\{\text{ess.}\sup|\log|f| + \log|g||\} = \exp 2|||(\log|f|)|||\ .$$

The proof of Proposition 1 is parallel to that of Proposition 2.2 in [17]. Rochberg [17] considered $(H^\infty)^{-1}/\exp H^\infty$ instead of $(L^\infty)^{-1}/(H^\infty)^{-1}$. If A is a disc algebra, then $\|(f)\| = 1$ for any $(f) \in (L^\infty)^{-1}/(H^\infty)^{-1}$ and so $\gamma_0 = 1$ because of Proposition 1 and $L_R^\infty = \log|(H^\infty)^{-1}|$. Let $\operatorname{crls}\log|(H^\infty)^{-1}|$ denote the closed real linear span of $\log|(H^\infty)^{-1}|$.

LEMMA 3. (1) *If $v = \sum_{j=1}^n t_j \log|h_j|$ with $0 \leqq t_j \leqq 1$ and $h_j \in (H^\infty)^{-1}$ $(1 \leqq j \leqq n)$, then $\sup\{|||(tv)|||;\ -\infty < t < \infty\} < \infty$.* (2) *If $v \in L_R^\infty$ is not in $\operatorname{crls}\log|(H^\infty)^{-1}|$, then $\sup\{|||tv|||;\ -\infty < t < \infty\} = \infty$.*

PROOF. (1) $|||(tv)||| = |||(\sum_{j=1}^n tt_j\log|h_j|)||| = |||(\sum_{j=1}^n (tt_j - [tt_j])\log|h_j|)|||$, where $[\cdot]$ is the greatest integer function. Hence $\sup|||(tv)||| < \infty$.

(2) There exists a positive constant ε such that $\|tv + \operatorname{crls}\log|(H^\infty)^{-1}|\|_\infty \geqq \varepsilon\|tv\|_\infty$ for any t. Hence $\sup|||(tv)||| = \infty$.

THEOREM 2. *Suppose τ satisfies the conditions* (1) *and* (2). *Then*

m is a unique logmodular measure if and only if γ_0 is finite.

PROOF. By Proposition 1 it is sufficient to show that m is a unique logmodular measure if and only if γ_1 is finite. m is a unique logmodular measure if and only if $\operatorname{crls}\log|(H^\infty)^{-1}| = L^\infty_R$ (see [10, p. 114]). By this and (2) in Lemma 3, if m is not a unique logmodular measure, then $\gamma_1 = \infty$. Suppose m is a unique logmodular measure. If $v \in L^\infty_R$, then $v = u_0 + \log|g|$ with $u_0 \in N^\infty$ and $g \in (H^\infty)^{-1}$ (cf. [10, p. 109]). Moreover, we can choose $u_0 \in \log \mathscr{E}_0$ (see Introduction). By (1) in Lemma 3 γ_1 is finite and in fact $\gamma_1 \leqq \sup\{\|\sum_{j=1}^n s_j u_j\|_\infty; 0 \leqq s_j \leqq 1\}$.

2. Concrete uniform algebras. (I) Let Y be a compact subset of the plane, and let $R(Y)$ be the uniform closure of the set of rational functions in $C(Y)$. We regard $R(Y)$ as a uniform algebra on its Shilov boundary, the topological boundary X of Y. Suppose the complement Y^c of Y has a finite number n of components and the interior Y^0 of Y is a nonempty connected set. Let $A = R(Y)|X$; then $M_A = Y$. If $\tau \in M_A$ is in Y^0 and m is a harmonic measure, then m is a unique logmodular measure of N_τ and $\dim N_\tau = n < \infty$ [10, p. 116]. Then $N^\infty \subset C(X)$ By Theorem 2, γ_0 is finite. Let $X = X_0 \cup X_1 \cup \cdots \cup X_n$, where X_0 is the "outside" component of X and $X_1, \cdots, X_n$ are the "inside" components of X. Define $v_j \in L^\infty_R$ to be 1 on X_j and 0 on $X \setminus X_j$ $1 \leqq j \leqq n$. Then $\gamma_1 = \sup\{|||(\sum_{j=1}^n t_j v_j)|||; -\infty < t_j < \infty \text{ and } 1 \leqq j \leqq n\}$.

(II) In (I) let Y be the annulus $\{r \leqq |z| \leqq 1\}$. Then $\gamma_0 = r^{-1/2}$. Since $(L^\infty_R/$ the uniform closure of $\operatorname{Re} H^\infty)$ has dimension one, we get

$$\gamma_1 = \sup_{0\leqq t\leqq 1} \inf\{\|t\log|z| - (\operatorname{Re} f + n\log|z|)\|_\infty; f \in H^\infty \text{ and } n \text{ ranges over all integers}\}.$$

For any integer n

$$\begin{aligned}&\inf\{\|(t-n)\log|z| - \operatorname{Re} f\|_\infty; f \in H^\infty\}\\ &\quad = |t-n|\log r^{-1} \inf\{\|\chi_E - \operatorname{Re} f\|_\infty; f \in H^\infty\} = \frac{1}{2}|t-n|\log r^{-1},\end{aligned}$$

where $\chi_E = 0$ on $|z| = 1$ and $\chi_E = 1$ on $|z| = r$. Thus

$$\gamma_1 = \sup_{0\leqq t\leqq 1} \inf_n \frac{1}{2}|t-n|\log r^{-1} = \frac{1}{4}\log r^{-1}.$$

We shall show that

$$\inf\{\|\chi_E - \operatorname{Re} f\|_\infty; f \in H^\infty\} = 1/2.$$

Choosing $f = 1/2$, the infimum is not greater than 1/2. If the infimum is less than 1/2 then by a theorem of Runge we can show that $\chi_E \in H^\infty$ as in the proof of Theorem in [13, p. 182]. This contradiction implies that the infimum is just 1/2.

(III) Let $\mathscr{A}$ be the disc algebra and let A be a subalgebra of $\mathscr{A}$ which contains the constants and which has finite codimension in $\mathscr{A}$. If $\tau(f) = \tilde{f}(0)$ for $f \in A$ and m is the normalized Lebesgue measure on the circle T, then it is easy to check that m is a core point of N_τ and $N^\infty \subset C(T)$. If $A \neq \mathscr{A}$, then H^∞ is contained properly in the classical Hardy space. Hence H^∞ is not τ-maximal. On the other hand if τ has a unique logmodular measure m, then H^∞ is τ-maximal ([9, Theorem 5.5]). This implies that m is not a unique logmodular measure and hence Theorem 2 implies that γ_0 is infinite.

(IV) The unit polydisc U^n and the torus T^n are cartesian products of n copies of the unit disc U and of the unit circle T, respectively. $A(U^n)$ is the class of all continuous complex functions on the closure $\bar{U}^n$ of U^n with holomorphic restrictions to U^n. Let $A = A(U^n)|X$ and $X = T^n$. This is the so-called polydisc algebra. For simplicity we assume $n = 2$. Let m be the normalized Lebesgue measure; then m is a representing measure for τ on X where $\tau(f) = f(0)$ and $0 \in U^2$. Suppose $1 \leqq p \leqq \infty$ and $\boldsymbol{Z}_+^2 = \{(n, m) \in \boldsymbol{Z}^2; n \geqq 0 \text{ and } m \geqq 0\}$. Then $H^p = \{f \in L^p; \hat{f}(n, m) = 0 \text{ if } (n, m) \notin \boldsymbol{Z}_+^2\}$ and $K^p = \{f \in L^p; \hat{f}(n, m) = 0 \text{ if } (-n, -m) \in \boldsymbol{Z}_+^2\}$. K^∞ is dense in K^p. Unfortunately we do not know whether γ_0 is finite or not.

3. Norms of Hankel operators. Assuming γ_0 is finite, we show that $\|H_\phi^{(1)}\|$ (resp. $\|H_\phi^{(2)}\|$) is equivalent to $\|\phi + H^\infty\|$ (resp. $\|\phi + K^\infty\|$).

THEOREM 3. *Let ϕ be a function in L^∞. Then*

$$\|H_\phi^{(2)}\| \leqq \|\phi + K^\infty\| \leqq \gamma_0 \|H_\phi^{(2)}\| .$$

If K^∞ is dense in K^1, then

$$\|H_\phi^{(1)}\| \leqq \|\phi + H^\infty\| \leqq \gamma_0 \|H_\phi^{(1)}\| .$$

PROOF. Let $v \in (L^\infty)_+^{-1}$. If $\int_X |f|^2 v^2 dm \leqq 1$ and $\int_X |g|^2 v^{-2} dm \leqq 1$, then $\int_X |f|^2 dm \leqq \|v^{-2}\|_\infty$ and $\int_X |g|^2 dm \leqq \|v^2\|_\infty$. Hence

$$\begin{aligned}\|H_\phi^{(1)v}\| &= \sup\left\{\left|\int_X fg\phi dm\right|; f \in H^2,\ g \in K_0^2,\ \int_X |f|^2 v^2 dm \leqq 1 \text{ and } \int_X |g|^2 v^{-2} dm \leqq 1\right\} \\ &\leqq (\|v^{-2}\|_\infty \|v^2\|_\infty)^{1/2} \|H_\phi^{(1)}\| .\end{aligned}$$

If $h \in (H^\infty)^{-1}$, then $v|h|H^2 = b(vH^2)$ and $v^{-1}|h|^{-1}\bar{K}_0^2 = b(v^{-1}\bar{K}_0^2)$ with $b = |h|/h$. Then $Q_{v|h|}^{(1)} = M_b Q_v^{(1)} M_{\bar{b}}$ and so $H_\phi^{(1)v|h|} = M_b Q_v^{(1)} M_{\bar{b}} M_\phi |v|h|H^2$. Hence $\|H_\phi^{(1)v}\| = \|H_\phi^{(1)v|h|}\|$. Thus for any $h \in (H^\infty)^{-1}$

$$\|H_\phi^{(1)v}\| \leqq \|v|h|\,\|_\infty \|v^{-1}|h|^{-1}\|_\infty \|H_\phi^{(1)}\| .$$

It is easy to see that

$$\sup_{v\in(L^\infty)_+^{-1}}\{\inf\{\|v|h|\,\|_\infty\|v^{-1}|h|^{-1}\|_\infty;\, h\in(H^\infty)^{-1}\}\} = \gamma_0\,.$$

Now generalized Nehari's Theorem I shows that $\|H_\phi^{(1)}\| \leqq \|\phi + H^\infty\| \leqq \gamma_0\|H_\phi^{(1)}\|$. Similarly the inequality for $H_\phi^{(2)}$ follows.

4. Applications. In the previous paper [14] we gave applications of generalized Nehari's Theorems I and II to weighted norm inequalities and invertible Toeplitz operators. In this section we shall give applications of Theorem 3.

Recall P is the orthogonal projection from L^2 to H^2. Let $\mathscr{P}^{(1)}$ denote P restricted to $H^\infty + \bar{K}_0^\infty$ and $\mathscr{P}^{(2)}$ denote P restricted to $H^\infty + \bar{H}_0^\infty$. We are interested in knowing when $\mathscr{P}^{(j)}$ $(j = 1, 2)$ is bounded in $L^2(w) = L^2(wdm)$, where w is a nonnegative weight function in L^1. Put

(d) $$\sup\left\{\left|\int_X fgwdm\right|;\, f\in H^\infty,\, g\in K_0^\infty \text{ and } \int_X|f|^2wdm = \int_X|g|^2wdm = 1\right\} = \rho_1$$

and

(e) $$\sup\left\{\left|\int_X fgwdm\right|;\, f\in H^\infty,\, g\in H_0^\infty \text{ and } \int_X|f|^2wdm = \int_X|g|^2wdm = 1\right\} = \rho_2.$$

Then it is easy to see that $\|\mathscr{P}^{(j)}\|^2 \leqq (1-\rho_j^2)^{-1}$. The following lemma is known [19]. We shall give the proof for completeness.

LEMMA 3. $\|\mathscr{P}^{(j)}\|^2 = (1-\rho_j^2)^{-1}$ $(j = 1, 2)$.

PROOF. If $\gamma = \|\mathscr{P}^{(1)}\| < \infty$, then for any real t and for any $f\in H^\infty$ and $g\in K_0^\infty$ we have

$$t^2\frac{\gamma^2-1}{\gamma^2}\int_X|f|^2wdm + \int_X|g|^2wdm + 2t\,\mathrm{Re}\int_X fgwdm \geqq 0\,.$$

Hence

$$\left|\int_X fgwdm\right|^2 \leqq \frac{\gamma^2-1}{\gamma^2}\int_X|f|^2wdm\int_X|g|^2wdm$$

and so $\gamma^2 \geqq (1-\rho_1^2)^{-1}$. We can prove it for $j = 2$ in the same method.

If A is a disc algebra, then $\mathscr{P} = \mathscr{P}^{(1)} = \mathscr{P}^{(2)}$ is bounded in $L^2(w)$ if and only if $w = |h|^2$ for some outer function h in H^2 and $\|\phi + H^\infty\| < 1$ with $\phi = |h|^2/h^2$. This result is called Helson-Szegö's theorem [12]. This was generalized to general uniform algebras by the author [14]. The following generalization seems to be better than the previous one.

COROLLARY 1. *Suppose K^∞ is dense in K^1. Let $w = |h|^2$ for some function h in H^2 such that hH^∞ is dense in H^2 and hK^∞ is dense in K^2.*

Let $\phi = |h|^2/h^2$.

(1) *If* $\|\phi + H^\infty\| = \rho < 1$ *then* $\mathscr{P}^{(1)}$ *is bounded in* $L^2(w)$ *and* $\|\mathscr{P}^{(1)}\| \leqq (1-\rho^2)^{-1/2}$.

(2) *If* $\mathscr{P}^{(1)}$ *is bounded in* $L^2(w)$ *and* $\|\mathscr{P}^{(1)}\| = \gamma$ *then*

$$\|\phi + H^\infty\| < \gamma_0\gamma^{-1}(\gamma^2 - 1)^{1/2}$$

Hence if $\gamma < \gamma_0(\gamma_0^2 - 1)^{1/2}$ *then* $\|\phi + H^\infty\| < 1$.

PROOF. (1) $\rho_1 \leqq \rho$, since

$$\rho = \|\phi + H^\infty\| = \sup\left\{\left|\int_X F\phi dm\right|; F \in K_0^1 \text{ and } \|F\|_1 \leqq 1\right\}$$
$$\geqq \sup\left\{\left|\int_X \phi fg dm\right|; f \in H^2, g \in K_0^2 \text{ and } \int_X |f|^2 dm = \int_X |g|^2 dm = 1\right\} = \rho_1 .$$

In the last equality we used the facts that $w = |h|^2$ and that hH^∞ (resp. hK_0^∞) is dense in H^2 (resp. K_0^2).

(2) Since $\rho_1 = \gamma^{-1}(\gamma^2 - 1)^{1/2}$ by Lemma 3 and $\rho_1 = \|H_\phi^{(1)}\|$ by the proof of (1), Theorem 3 implies (2).

K^∞ is dense in K^2 if we impose the assumptions (1) and (2) or if A is a polydisc algebra. We have a similar result for $\mathscr{P}^{(2)}$ (or $\|\phi + K^\infty\|$) as in Corollary 1.

For ϕ in L^∞ let T_ϕ be the operator on H^2 defined by $T_\phi f = P(M_\phi f)$. The operator T_ϕ will be called a Toeplitz operator. We are interested in knowing when T_ϕ is left invertible. In case A is a disc algebra, Widom [18] showed that T_ϕ is left invertible if and only if $\|\phi + H^\infty\| < 1$. Abrahamse [1] generalized Widom's theorem to the case of (I) in concrete uniform algebras such that ∂Y consists of $n+1$ non-intersecting analytic Jordan curves. The author [14] generalized it to general uniform algebras. However these generalizations are not sufficient because except in the case of a disc algebra we cannot determine ϕ when T_ϕ is left invertible.

COROLLARY 2. *Suppose* K^∞ *is dense in* K^1. *Let* ϕ *be a unimodular function in* L^∞.

(1) *If* $\|\phi + H^\infty\| = \rho < 1$, *then* $\|T_\phi f\|_2 \geqq (1-\rho^2)^{1/2}\ \|f\|_2$ *for any* f *in* H^2.

(2) *If* $\|T_\phi f\|_2 \geqq \varepsilon\|f\|_2$ *for any* f *in* H^2, *then*

$$\|\phi + H^\infty\| \leqq \gamma_0(1-\varepsilon^2)^{1/2} .$$

Hence if $\varepsilon > \gamma_0^{-1}(\gamma_0^2 - 1)^{1/2}$, *then* $\|\phi + H^\infty\| < 1$.

PROOF. Since ϕ is a unimodular function, $\|H_\phi^{(1)} f\|_2^2 + \|T_\phi f\|_2^2 = \|f\|_2^2$ for any $f \in H^2$. Theorem 3 and this imply the corollary.

In the case of (I) for concrete uniform algebras, $\|\phi + H^\infty\| < 1$ may not hold even if T_ϕ is left invertible (cf. [1]).

5. Essential norms of Hankel operators. In this section we shall concentrate on concrete uniform algebras, that is, (I) in Section 2 such that ∂Y consists of $n+1$ non-intersecting analytic Jordan curves. Hence τ satisfies the conditions (1) and (3). Using generalized Nehari's Theorem II we shall generalize (b) in Introduction to this context.

Let $s=(s_1, s_2, \cdots s_n)\in I^n=[0,1]\times\cdots\times[0,1]$. Then the mapping $s\mapsto \exp(\sum_{j=1}^n s_j u_j)$ is continuous, one-to-one and onto from I^n to $\mathscr{E}_0$. Put

$$H_\phi^{(j)s}=H_\phi^{(j)v} \quad (j=1,2)\,,$$

where $v=\exp(\sum_{j=1}^n s_j u_j)$.

LEMMA 4. *Let ϕ be a function in L^∞. Then for $j=1,2$ and for any v and u in $\mathscr{E}_0$*

$$\|H_\phi^{(j)v}-H_\phi^{(j)u}\|\leqq\|\phi\|_\infty(2\sup_{v\in\mathscr{E}_0}\|v^{-1}\|_\infty)\|v-u\|_\infty\,.$$

The proof is clear by Lemma 1.

LEMMA 5. *If ϕ in $H^\infty+C(X)$, then $H_\phi^{(j)v}$ $(j=1,2)$ is compact for any v in $\mathscr{E}_0$.*

PROOF. By Lemma 2 it is sufficient to show that $H_\phi^{(j)}$ is compact for any $\phi\in C(X)$. Let $\phi=(z-a)^{-1}$ for some $a\in Y^0$. Then

$$H_\phi^{(j)}f=Q^{(j)}\left[\frac{f(a)}{z-a}+\frac{f-f(a)}{z-a}\right]=Q^{(j)}\left[\frac{f(a)}{z-a}\right]$$

for any $f\in H^2$ because $\{f\in H^2: f(a)=0\}=(z-a)H^2$. Hence $H_\phi^{(j)}$ has rank one. Similarly if $\phi=(z-a)^{-n}$ for a positive integer n, we can show that $H_\phi^{(j)}$ has rank n. For any $\phi\in C(X)$ we can approximate ϕ by the following functions: $\sum_{j=0}^n b_j(z-a_j)^{-j}$ where $a_j\in Y^0$ and b_j is constant $(0\leqq j\leqq n)$. Since $\|H_\phi^{(1)}\|\leqq\|\phi+H^\infty\|$ and $\|H_\phi^{(2)}\|\leqq\|\phi+H^\infty+N_c^\infty\|$, we can show that $H_\phi^{(j)}$ is compact if $\phi\in C(X)$.

THEOREM 4. *Let ϕ be a function in L^∞. Then*

$$\sup_{v\in\mathscr{E}_0}\|H_\phi^{(1)v}\|_e=\sup_{v\in\mathscr{E}_0}\|H_\phi^{(2)v}\|_e=\|\phi+H^\infty+C(X)\|\,.$$

Moreover, the suprema in both equalities are attained.

PROOF. By Lemma 5 it is clear that $\sup\{\|H_\phi^{(j)v}\|_e;\, v\in\mathscr{E}_0\}\leqq\|\phi+H^\infty+C(X)\|$ for $j=1,2$. We shall show the opposite inequality. Let F be the Ahlfors function for Y^0 and $\tau\in Y^0$. Then $F\in C(X)$ (see [8, p. 114]). For any $v\in\mathscr{E}_0$ with $v=\exp(\sum_{j=1}^n t_j u_j)$ and $t=(t_1, t_2, \cdots, t_n)\in I^n$, put

$$f^{(j)}(t,l)=\|H_{F^l\phi}^{(j)t}\| \quad (l=0,1,2,\cdots; j=1,2)\,.$$

Then $f^{(j)}(t, l) \geqq f^{(j)}(t, l+1)$ and by Lemma 4

$$|f^{(j)}(t, l) - f^{(j)}(s, l)| \leqq \|\phi\|_\infty (2 \sup_{v \in \mathscr{E}_0} \|v^{-1}\|_\infty) \|\exp(\sum_{j=1}^{n} t_j n_j) - \exp(\sum_{j=1}^{n} s_j u_j)\|_\infty .$$

Hence $\{f^{(j)}(t, l)\}_{l=1}^\infty$ is an equicontinuous collection on I^n and uniformly bounded on I^n. By Ascoli's theorem, there exists a subsequence $\{f^{(j)}(t, l_i)\}_{i=1}^\infty$ of $\{f^{(j)}(t, l)\}_{l=1}^\infty$ and a continuous function $f^{(j)}(t)$ on I^n such that

$$\sup_{t \in I^n} |f^{(j)}(t) - f^{(j)}(t, l_i)| \to 0 \quad (\text{as } i \to \infty) .$$

Since $\{f^{(j)}(t, l)\}_{l=1}^\infty$ is a decreasing sequence, this actually converges to $f^{(j)}(t)$ uniformly on I^n. Thus

$$\lim_{l} \sup_{t \in I^n} f^{(j)}(t, l) = \sup_{t \in I^n} f^{(j)}(t) .$$

By generalized Nehari's Theorem II, $\sup\{f^{(1)}(t, l); t \in I^n\} = \|F^l\phi + H^\infty\|$ and $\sup\{f^{(2)}(t, l); t \in I^n\} = \|F^l\phi + H^\infty + N_c^\infty\|$ and so for $j = 1, 2$

$$\sup_{t \in I^n} f^{(j)}(t) \geqq \|\phi + H^\infty + C(X)\| ,$$

because the closure of $\cup_{n=1}^\infty \bar{F}^n H^\infty$ coincides with the closure of $\cup_{n=1}^\infty \bar{F}^n (H^\infty + N_c^\infty)$, which is $H^\infty + C(X)$ (cf. [1, Theorem 1.22]). For any $t \in I^n$, let S_t denote the multiplication by F on vH^2 where $v = \exp(\sum_{j=1}^n t_j u_j)$. Let $S_t^{(1)*}$ be the adjoint of S_t from vH^2 to $v^{-1}\bar{K}_0^2$ and $S_t^{(2)*}$ the adjoint of S_t from vH^2 to $v^{-1}\bar{H}_0^2$. If $K_t^{(1)}$ is any compact operator from vH^2 to $v^{-1}\bar{K}_0^2$ and $K_t^{(2)}$ is any compact operator from vH^2 to $v^{-1}\bar{H}_0^2$, and l is positive integer, then for $j = 1, 2$

$$\|H_\phi^{(j)t} - K_t^{(j)}\| \geqq \|(H_\phi^{(j)t} - K_t^{(j)})S_t^l\| \geqq \|H_\phi^{(j)t} S_t^l\| - \|K_t^{(j)} S_t^l\| .$$

Since $(S_t^{(j)l})^* \to 0$ strongly, we have $\|K_t^{(j)} S_t^l\| \to 0$. Also

$$H_\phi^{(j)t} S_t^l = H_{F^l\phi}^{(j)t} .$$

Hence we can prove that the suprema are attained as in the proof of generalized Nehari's Theorem II.

$$\|H_\phi^{(j)t} - K_t^{(j)}\| \geqq \overline{\lim} \|H_{F^l\phi}^{(j)t}\| = \overline{\lim}\, f^{(j)}(t, l) = f^{(j)}(t) .$$

Thus $\|H_\phi^{(j)t}\|_e \geqq f^{(j)}(t)$ and

$$\sup_{t \in I^n} \|H_\phi^{(j)t}\|_e \geqq \sup_{t \in I^n} f^{(j)}(t) \geqq \|\phi + H^\infty + C(X)\| .$$

The following theorem is another generalization of (b) in Introduction.

THEOREM 5. *Let ϕ be a function in L^∞. Then for $j = 1, 2$*

$$\|H_\phi^{(j)}\|_e \leqq \|\phi + H^\infty + C(X)\| \leqq \gamma_0 \|H_\phi^{(j)}\|_e .$$

The proof follows as in the case of a disc algebra (see [16, Theorem 1.4]) if we use Theorem 3 and the Ahlfors function.

6. Factorization theorems. We say H_0^1 has the weak approximate γ-factorization if H_0^1 satisfies the following property: For any F in H_0^1 and any $\varepsilon > 0$, there exist $\{f_j\}_{j=1}^n$ in H^2 and $\{g_j\}_{j=1}^n$ in H_0^2 such that

$$\sum_{j=1}^n \|f_j\|_2\|g_j\|_2 \leqq \gamma\|F\|_1$$

and

$$\left\|F - \sum_{j=1}^n f_j g_j\right\|_1 < \varepsilon .$$

PROPOSITION 6. *There exists a constant* γ *with* $\gamma \geqq 1$ *such that* $\|\phi + K^\infty\| \leqq \gamma\|H_\phi^{(2)}\|$ *for all* ϕ *in* L^∞ *if and only if* H_0^1 *has the weak approximate* γ*-factorization.*

PROOF. Let V_γ be the closure in L^1 of the following set:

$$\left\{\sum_{j=1}^n f_j g_j; f_j \in H^2,\ g_j \in H_0^2 \text{ and } \sum_{j=1}^n \|f_j\|_2\|g_j\|_2 \leqq \gamma\right\} .$$

Put $V^1 = \{F \in H_0^1; \|F\|_1 \leqq 1\}$. Then V_γ is the closed convex subset in γV^1. If H_0^1 has the weak approximate γ-factorization, then $V^1 \subset V_\gamma$ and so $\|\phi + K^\infty\| \leqq \gamma\|H_\phi^{(2)}\|$, since

$$\left|\int_X\left(\sum_{j=1}^n f_j g_j\right)\phi dm\right| \leqq \|H_\phi^{(2)}\|\sum_{j=1}^n \|f_j\|_2\|g_j\|_2 .$$

Conversely, suppose $\|\phi + K^\infty\| \leqq \gamma\|H_\phi^{(2)}\|$. If H_0^1 does not have the weak approximate γ-factorization, then there exists $F \in V^1$ with $F \notin V_\gamma$. Then by the Hahn-Banach theorem there exists $\phi \in L^\infty$ such that

$$\left|\int_X \phi F dm\right| > \sup\left\{\left|\int_X \phi f dm\right|; f \in V_\gamma\right\}$$

and so $\|\phi + K^\infty\| > \gamma\|H_\phi^{(2)}\|$.

For K_0^1 we can define the weak approximate γ-factorization and prove Proposition 6 with $H_\phi^{(1)}$, H^∞ and K_0^1 instead of $H_\phi^{(2)}$, K^∞ and H_0^1, respectively. In (I) for concrete uniform algebras we have factorization theorems of H_0^1 and K_0^1. M. Hayashi pointed out a factorization theorem of H_0^1. We now give a proof and clarify its relationship with γ_0.

THEOREM 7. *Suppose* A *is a concrete uniform algebra* (I).

(1) *If* f *is in* H_0^1, *then there is a* g *in* H^2 *and an* h *in* H_0^2 *such that* $f = gh$ *and* $\|g\|_2\|h\|_2 \leqq \gamma_2\|f\|_1$, *where* $\gamma_2 = \sup\{\|v^{-1}\|_\infty\|v\|_\infty; v \in \mathscr{E}_0\}$. *In this case* $\gamma_2 \geqq \gamma_0$.

(2) *If* f_1 *is in* K_0^1, *then there is a* g_1 in H^2 *and an* h_1 *in* K_0^2 *such*

that $f_1 = g_1 h_1$ *and* $\|g_1\|_2 \|h_1\|_2 \leqq \gamma_3 \|f_1\|_1$, *where* $\gamma_3 = \gamma_2 \|v_0\|_\infty$ *and* $K_0^1 = v_0 H_0^1$.

PROOF. (1) A function $f \in H_0^1$ is of the form $f = FG^2$ where $F \in H_0^\infty$ with $|F| \in \mathscr{E}$ and $G \in H^2$ [3, p. 138]. If $|F| = \exp(\sum_{j=1}^n t_j u_j)$, let $k = \prod_{j=1}^n (h_j)^{l_j}$ and $l_j = [t_j/2]$. Then $k \in (H^\infty)^{-1}$. Put $s_j = 2(t_j/2 - [t_j/2])$. Then $q = Fk^{-1} \in H_0^\infty$ and $|q| = \exp(\sum_{j=1}^n s_j u_j) \in \mathscr{E}_0^2 = \{v^2; v \in \mathscr{E}_0\}$. Let $g = kG$ and $h = qG$. Then $f = gh$ and

$$\int_X |g|^2 dm \int_X |h|^2 dm = \int_X |q^{-1}|\,|f| dm \int_X |q|\,|f| dm \leqq \|q^{-1}\|_\infty \|q\|_\infty \left[\int_X |f| dm\right]^2$$
$$\leqq \sup\{\|q^{-1}\|_\infty \|q\|_\infty;\ |q| \in \mathscr{E}_0^2\} \left[\int_X |f| dm\right]^2 .$$

If $u \in (L^\infty)_+^{-1}$ then $u = v|g|$ with $v \in \mathscr{E}_0$ and $g \in (H^\infty)^{-1}$. Hence

$$\gamma_2 = \sup\{\|v\|_\infty \|v^{-1}\|_\infty;\ v \in \mathscr{E}_0\} \geqq \sup\{\|(u)\|;\ u \in (L^\infty)_+^{-1}\} = \gamma_0 .$$

(2) A function $f_1 \in K_0^1$ is of the form $f_1 = v_0 f$ for some $f \in H_0^1$. Apply (1) to this f, and let $g_1 = g$ and $h_1 = v_0 h$, then $g_1 \in H^2$ and $h_1 \in K_0^2$. Now (2) follows.

(1) of Theorem 7 gives $\|H_\phi^{(2)}\| \leqq \|\phi + K^\infty\| \leqq \gamma_2 \|H_\phi^{(2)}\|$ in the case of (I) for concrete uniform algebras.

(2) of Theorem 7 gives that $\|H_\phi^{(1)}\| \leqq \|\phi + H^\infty\| \leqq \gamma_3 \|H_\phi^{(1)}\|$. For any uniform algebra with finite γ_0, Theorem 3 and Proposition 6 show that both H_0^1 and K_0^1 have the weak approximate γ_0-factorizations.

7. Arveson's distance formula. Let $\mathscr{A}$ be a (possibly non-self-adjoint) algebra of operators on a Hilbert space $\mathscr{H}$, and let T be an arbitrary bounded operator. Then $d(T, \mathscr{A}) \geqq \sup_P \|(1-P)TP\|$, where $d(T, \mathscr{A})$ is the distance from T to $\mathscr{A}$ and where the supremum is taken over the lattice lat $\mathscr{A}$ of all $\mathscr{A}$-invariant projections. Arveson [5, Theorem 1.1.] showed that if $\mathscr{A}$ is a nest algebra (i.e., lat $\mathscr{A}$ is totally ordered) then the equality holds above. Let $\mathscr{H} = L^2$ and $P_v^{(1)} = 1 - Q_v^{(1)}$. Generalized Nehari's Theorem I implies that if K^∞ is dense in K^1 and lat $\mathscr{A} \ni P_v^{(1)}$ for any v in $(L^\infty)_+^{-1}$, then for any ϕ in L^∞

$$d(M_\phi, \mathscr{A}) = \sup_P \|(I-P)M_\phi P\| .$$

Let $\mathscr{C}(\mathscr{H})$ be the space of all compact operators on $\mathscr{H}$ and $\mathscr{B}$ the the norm closure of $\mathscr{A} + \mathscr{C}(\mathscr{H})$. Then $d(T, \mathscr{B}) \geqq \sup_P \|(I-P)TP\|_e$. Theorem 4 implies that if lat $\mathscr{A} \ni P_v^{(1)}$ for any v in $\mathscr{E}_0$ then

$$d(M_\phi, \mathscr{B}) = \sup_P \|(I-P)M_\phi P\|_e \quad \text{for any } \phi \text{ in } L^\infty.$$

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