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THE INNER RADII OF FINITE-DIMENSIONAL TEICHMÜLLER SPACES

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Introduction. Let Γ be a Fuchsian group leaving the upper half plane $\boldsymbol{U}$ and hence also the lower half plane $\boldsymbol{L}$ invariant. By means of the Bers embedding ([2]) the Teichmüller space $\boldsymbol{T}(\Gamma)$ of Γ is identified with a bounded domain in the space $B(\boldsymbol{L}, \Gamma)$ of bounded quadratic differentials for Γ. The inner radius $i(\Gamma)$ of $\boldsymbol{T}(\Gamma)$ is the supremum of radii of balls in $B(\boldsymbol{L}, \Gamma)$ centered at the origin which are contained in $\boldsymbol{T}(\Gamma)$. The inequality $i(\Gamma) \geqq 2$ obtained by Ahlfors and Weill ([1]) is well known. If, in addition, Γ is finitely generated and of the first kind, then the strict inequality $i(\Gamma) > 2$ holds (see §2). Our main objective of this paper is to prove the following theorem:

THEOREM. *Let $\sigma = (g; \nu_1, \cdots, \nu_n)$ be a signature different from $(0; \nu_1, \nu_2, \nu_3)$. Then*

$$I(\sigma) = \inf\{i(\Gamma);\ \textit{Fuchsian groups } \Gamma \textit{ with signature } \sigma\} = 2\,.$$

For the definition of signature, see 1.2. The Teichmüller space of a Fuchsian group with signature $(0; \nu_1, \nu_2, \nu_3)$ or a triangle group is a single point and its inner radius is zero.

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1. Preliminaries.

Our basic references in the theory of Fuchsian groups and Teichmüller spaces are [7] and [8].

1.1. We denote by **Möb** the group of all Möbius transformations of the Riemann sphere $\hat{\boldsymbol{C}} = \boldsymbol{C} \cup \{\infty\}$ and $\mathbf{M\ddot{o}b}_{\boldsymbol{U}}$ the subgroup of **Möb** whose transformations leave $\boldsymbol{U}$ and hence $\boldsymbol{L}$ invariant. Then $\mathbf{M\ddot{o}b}_{\boldsymbol{U}}$ is also the group of orientation-preserving isometries of the hyperbolic plane $\boldsymbol{U}$ (and $\boldsymbol{L}$) with the metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}\,, \qquad z = x + iy \in \boldsymbol{U} \quad (\text{or } \boldsymbol{L})\,. \tag{1.1}$$

Geodesics with respect to this metric are circular arcs and straight lines orthogonal to the real line.

Let Γ be a Fuchsian group in $\mathbf{M\ddot{o}b}_{\boldsymbol{U}}$. We consider the action of Γ on $\boldsymbol{U}$. The quotient space $\boldsymbol{R}_\Gamma = \boldsymbol{U}/\Gamma$ is a Reimann surface and the canonical projection $\pi_\Gamma\colon \boldsymbol{U} \to \boldsymbol{R}_\Gamma$ is a ramified universal covering. The metric (1.1) induces a metric on $\boldsymbol{R}_\Gamma$ which is referred to as the hyperbolic metric on $\boldsymbol{R}_\Gamma$ in this paper. For a set $D \subset \hat{\boldsymbol{C}}$, the *stabilizer* of D in

Γ is $\mathrm{Stab}(D, \Gamma) = \{\gamma \in \Gamma; \gamma(D) = D\}$. As a subgroup of Γ, $\mathrm{Stab}(D, \Gamma)$ is also a Fuchsian group.

Let γ be a hyperbolic transformation of Γ. Then geodesic A_γ connecting the fixed points of γ is called the axis of γ. Then γ or more precisely the conjugacy class $[\gamma] = \{\delta\gamma\delta^{-1}; \delta \in \Gamma\}$ degermines a geodesic curve $\pi_\Gamma(A_\gamma)$ on R_Γ. Let l_γ be the positive value determined by $|\mathrm{tr}\,\gamma| = 2\cosh(l_\gamma/2)$, where $\mathrm{tr}\,\gamma$ is the trace of γ represented as a matrix in $SL(2; \boldsymbol{R})$. We say that γ is primitive if $\gamma = \delta^n$ holds for some $\delta \in \Gamma$ and some integer n if and only if $n = \pm 1$. If γ is primitive and $\mathrm{Stab}(A_\gamma, \Gamma)$ contains no elliptic transformations, $g = \pi_\Gamma(A_\gamma)$ is a closed geodesic and l_γ is the length of g.

1.2. Let Γ be a finitely generated Fuchsian group. Suppose that R_Γ has genus g and k boundary curves and m punctures. Suppose also that R_Γ has ramification points $P_1, \cdots, P_l$ with orders $\nu_1, \cdots, \nu_l$, respectively. By reordering we may assume that $\nu_1 \leqq \cdots \leqq \nu_l$. Set $n = l + m$ and $\nu_{l+1} = \cdots = \nu_n = \infty$. We call the ordered sets $(g, n+k)$ and $(g; \nu_1, \cdots, \nu_n; k)$ the *type* and the *signature* of Γ, respectively. If, in particular, Γ is of the first kind, $k = 0$. In this case we abbreviate $(g; \nu_1, \cdots, \nu_n; 0)$ to $(g; \nu_1, \cdots, \nu_n)$.

1.3. Let W be a connected subset of R_Γ and $\tilde{W}$ be a lift of W, that is, a component of $\pi_\Gamma^{-1}(W)$. If $\mathrm{Stab}(\tilde{W}, \Gamma)$ is of type $(0, 3)$, then we also say that W is a set of type $(0, 3)$.

Now we assume that Γ is of the first kind with signature $\sigma = (g; \nu_1, \cdots, \nu_n)$. If $\sigma \neq (0; \nu_1, \nu_2, \nu_3)$, then except in the cases in $(*)$ below there exists a system $\mathscr{G} = \{[\gamma_1], \cdots, [\gamma_S]\}$ of conjugacy classes of $S = 3g - 3 + n$ primitive hyperbolic elements in Γ with the following properties:

(a) The classes $[\gamma_1], \cdots, [\gamma_S]$ determine pairwise disjoint simple closed geodesics $g_1, \cdots, g_S$, respectively;

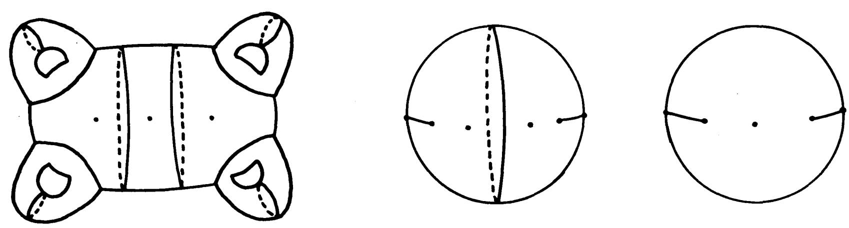
(b) Each component W of $R_\Gamma - \bigcup_{s=1}^{S} g_s$ is of type $(0, 3)$.

$(*)$ The exceptions are the signatures (i) $g = 0$, $n \geqq 4$, $\nu_1 = \cdots = \nu_{n-1} = 2$ and $\nu_n \geqq 3$ and (ii) $g = 0$, $n \geqq 5$ and $\nu_1 = \cdots = \nu_n = 2$. In these cases set $S = n - 3$. Then there is a system $\mathscr{G} = \{[\gamma_1], \cdots, [\gamma_S]\}$ of conjugacy classes of primitive hyperbolic elements in Γ satisfying the following property (a′) and also (b) above with $g_1, \cdots, g_S$ replaced by those in (a′):

(a′) The class $[\gamma_1]$ (and $[\gamma_S]$ for the case (ii)) determines a simple geodesic segment g_1 (and g_S) connecting two ramification points of order 2 and other classes $[\gamma_s]$ determine simple closed geodesics g_s. Moreover $g_1, \cdots, g_S$ are pairwise disjoint.

What we have described above is the so-called pants decomposition of R_Γ (see Figure). A more detailed description can be found in [12, pp. 154–156], but in this paper additional g closed geodesics are needed to cut the hundles.

Let γ be an element of a conjugacy class $[\gamma_s]$ in $\mathscr{G}$. We denote by $C(\omega, \gamma)$ the ω-neighborhood of A_γ with respect to the hyperbolic distance. If A_γ happens to coincide with the positive imaginary axis I, then $C(\omega, \gamma)$ is the set



$(4; \nu_1, \nu_2, \nu_3)$ $(0; 2, 2, 2, 2, \nu_1, \nu_2)$ $(0; 2, 2, 2, 2, 2)$

FIGURE

$$(1.2) \qquad C(\omega)=\{z\in U;\ \theta<\arg z<\pi-\theta\}, \qquad \text{where} \quad \omega=\log\cot\theta/2 .$$

$C(\omega, \gamma)$ is called the *collar* of width ω about A_γ if $C(\omega,\gamma)\cap\delta C(\omega,\gamma)=\varnothing$ for $\delta\in\Gamma-\mathrm{Stab}(A_\gamma, \Gamma)$. Let $\mathscr{P}(\Gamma)$ be the set of parabolic fixed points of Γ. If $\mathscr{P}(\Gamma)\neq\varnothing$, for $p\in\mathscr{P}(\Gamma)$, let D_p be the horodisc based at p with $\mathrm{area}(D_p/\mathrm{Stab}(p,\Gamma))=1$. For a positive number l, let $\omega(l)$ be the value determined by $2\sinh\omega(l)=(\sinh l/2)^{-1}$.

LEMMA 1.1. (The collar lemma. cf. [4], [10, Theorem 4.2]). *For any* $\gamma\in\bigcup_{s=1}^{S}[\gamma_s]$, $C(\omega(l_\gamma), \gamma)$ *is a collar about* A_γ. *Moreover,*

(i) *If* $\gamma\in[\gamma_s]$ *and* $\delta\in[\gamma_t]$ $(1\leqq s, t\leqq S)$ *are distinct, then* $C(\omega(l_\gamma),\gamma)\cap C(\omega(l_\delta),\delta)=\varnothing$.

(ii) *If* $\gamma\in[\gamma_s]$ $(1\leqq s\leqq S)$ *and* $p\in\mathscr{P}(\Gamma)$, *then* $C(\omega(l_\gamma),\gamma)\cap D_p=\varnothing$.

For $(\omega_1, \cdots, \omega_S)\in\boldsymbol{R}_+^S$ with $\omega_s\leqq\omega(l_{\gamma_s})$ $(1\leqq s\leqq S)$ define $\Omega_\Gamma(\omega_s)=\Omega_\Gamma(\omega_1, \cdots, \omega_S)$ to be

$$(1.3) \qquad \Omega_\Gamma(\omega_s)=U-\mathrm{cl}\Big(\bigcup_{p\in\mathscr{P}(\Gamma)}D_p\cup\bigcup_{s=1}^{S}\bigcup_{\gamma\in[\gamma_s]}C(\omega_s,\gamma)\Big)$$

(here $\mathrm{cl}\,B$ means the closure of a set B). By definition of the collar, $C(\omega(l_\gamma), \gamma)$ contains no elliptic fixed points of $\Gamma-\mathrm{Stab}(A_\gamma, \Gamma)$. Then we see without difficulty that, for each component W of $\Omega_\Gamma(\omega_s)$, $\mathrm{Stab}(W, \Gamma)$ is a Fuchsian group of type (0, 3).

1.4. The space $B(\boldsymbol{L}, \Gamma)$ of bounded quadratic differentials for Γ consists of holomorphic functions ϕ in $\boldsymbol{L}$ such that $\phi(z)=\phi(\gamma(z))\gamma'(z)^2$ for $\gamma\in\Gamma$ and $z\in\boldsymbol{L}$ and that $\|\phi\|=\sup_{z\in\boldsymbol{L}}4(\mathrm{Im}\,z)^2|\phi(z)|<\infty$. If Γ is finitely generated and of the first kind, then $B(\boldsymbol{L}, \Gamma)$ is a finite-dimensional space and if Γ is of type (g, n), the $\dim_{\boldsymbol{C}}B(\boldsymbol{L}, \Gamma)=3g-3+n$. Let $Q(\Gamma)$ be the set of conformal mappings f in $\boldsymbol{L}$ such that f admit quasiconformal extensions $\hat{f}$ to $\hat{\boldsymbol{C}}$ with $\hat{f}\Gamma\hat{f}^{-1}=\{\hat{f}\gamma\hat{f}^{-1};\ \gamma\in\Gamma\}\subset\textbf{Möb}$. If $f\in Q(\Gamma)$, its Schwarzian derivative $\{f, z\}=((f''/f')'-(1/2)(f''/f')^2)(z)$ belongs to $B(\boldsymbol{L}, \Gamma)$. The Teichmüller space $\boldsymbol{T}(\Gamma)$ of Γ is the set of all Schwarzian derivatives of functions in $Q(\Gamma)$. The *inner radius* $i(\Gamma)$ of $\boldsymbol{T}(\Gamma)$ is defined to be

$$\sup\{r;\ \phi\in B(\boldsymbol{L}, \Gamma) \text{ and } \|\phi\|<r \text{ imply } \phi\in\boldsymbol{T}(\Gamma)\}$$

and satisfies $i(\Gamma)\geqq 2$ (cf. [1]).

1.5. Let $\boldsymbol{I}$ be the positive imaginary axis and $\mathbf{M\ddot{o}b}_{\boldsymbol{I}}=\{\gamma\in\mathbf{M\ddot{o}b};\ \gamma(\boldsymbol{I})=\boldsymbol{I}\}$ consisting of the transformations of the form:

$$\gamma(z)=\lambda z \quad (\lambda>0) \qquad \text{or} \quad \gamma(z)=\lambda z^{-1} \quad (\lambda<0)\,. \tag{1.4}$$

An element of $\mathbf{M\ddot{o}b}_{\boldsymbol{I}}$ is either hyperbolic or elliptic of order 2 or the identity.

We review Kalme's paper [6]. We consider a holomorphic function $\phi_a(z)=az^{-2}$ in $\boldsymbol{L}$ with a complex parameter a. Let $a=(1-\delta^2)/2$. Then the equation $\{g, z\}=\phi_a(z)$ has a solution expressed by

$$g_a(z)=\begin{cases} z^{\delta} & \text{if} \quad \delta\neq 0\,, \\ \log z & \text{if} \quad \delta=0 \end{cases}$$

(we consider single-valued branches of the functions defined in the simply connected region $\boldsymbol{L}$). If $a\in\Lambda=\{a=(1-re^{2i\theta})/2;\ 0<r<4\cos^2\theta,\ 0\leqq|\theta|<\pi/2\}$, then the solution $\delta=\delta(a)$ of $a=(1-\delta^2)/2$ with $\operatorname{Re}\delta>0$ satisfies $|\delta-1|<1$. In this case, by setting $g_a(z)=z\bar{z}^{\delta-1}$ for $z\in\hat{\boldsymbol{C}}-\boldsymbol{L}$, we can extend g_a to a quasiconformal automorphism of $\hat{\boldsymbol{C}}$. The Beltrami coefficient of g_a is

$$\beta_a(z)=\begin{cases} (\delta-1)z/\bar{z} & \text{for} \quad z\in\hat{\boldsymbol{C}}-\boldsymbol{L}\,, \\ 0 & \text{for} \quad z\in\boldsymbol{L}\,. \end{cases}$$

Let $C(\omega)$ be the subregion of $\boldsymbol{U}$ defined in (1.2). If $a\in\Lambda$, let $\beta_{a,\omega}$, $\omega>0$, be the function defined by $\beta_{a,\omega}(z)=\beta_a(z)$ for $z\in C(\omega)$ and $\beta_{a,\omega}(z)=0$ for $z\in\hat{\boldsymbol{C}}-C(\omega)$. Then $\|\beta_{a,\omega}\|_\infty=|\delta-1|<1$. By a direct computation using (1.4) we see that

$$\beta_{a,\omega}(z)=\beta_{a,v}(\gamma(z))\overline{\gamma'(z)}/\gamma'(z) \qquad \text{for} \quad \gamma\in\mathbf{M\ddot{o}b}_{\boldsymbol{I}}\,. \tag{1.5}$$

Let ε be real. We define $\hat{\beta}_a(z)$ to be the limit of $\beta_a(\gamma_\varepsilon(z))\overline{\gamma'_\varepsilon(z)}/\gamma'_\varepsilon(z)$ as $\varepsilon\to 0$, where $\gamma_\varepsilon(z)=(z-\varepsilon)/(z+\varepsilon)$. Then,

$$\hat{\beta}_a(z)=\begin{cases} (\delta-1)z^2/\bar{z}^2 & \text{for} \quad z\in\hat{\boldsymbol{C}}-\boldsymbol{L}\,, \\ 0 & \text{for} \quad z\in\boldsymbol{L}\,. \end{cases}$$

Set $h_a(z)=\delta|z|^2((\delta-1)z+\bar{z})^{-1}$ for $z\in\hat{\boldsymbol{C}}-\boldsymbol{L}$ and $h_a(z)=z$ for $z\in\boldsymbol{L}$. Then h_a is a quasiconformal automorphism of $\hat{\boldsymbol{C}}$ with the Beltrami coefficient $\hat{\beta}_a$. Obviously $\|\hat{\beta}_a\|_\infty=|\delta-1|$ and $\{h_a, z\}=0$ for $z\in\boldsymbol{L}$.

2. The inequality $i(\Gamma)>2$ for finitely genrated Fuchsian groups Γ of the first kind. If Γ is a finitely generated Fuchsian group of the first kind, then $i(\Gamma)>2$. To see this, assume that $i(\Gamma)=2$. Then $\|\phi\|=2$ for a boundary point ϕ of $\boldsymbol{T}(\Gamma)$ in $B(\boldsymbol{L}, \Gamma)$. Let W_ϕ be a meromorphic function in $\boldsymbol{L}$ satisfying $\{W_\phi, z\}=\phi(z)$. Then it is known that W_ϕ is univalent and $\Gamma^\phi=W_\phi\Gamma W_\phi^{-1}$ is a Kleinian group with precisely one invariant component $W_\phi(\boldsymbol{L})$ ([2, p. 593]). The limit set of Γ_ϕ, which coincides with $\partial W_\phi(\boldsymbol{L})$, cannot be a Jordan closed curve. Then, since $\|\phi\|=2$, a result by Gehring and Pommerenke ([3, Theorem 1]) implies that $W_\phi(L)$ is the parallel slit $\{z;\ -\pi/2<\operatorname{Im} z<\pi/2\}$, if we replace W_ϕ by δW_ϕ for some $\delta\in\mathbf{M\ddot{o}b}$. However this is impossible, because any loxodromic element of Γ^ϕ cannot leave the parallel slit invariant. Hence we can conclude that $i(\Gamma)>2$.

The author learned the above result from Professors H. Shiga and H. Sekigawa.

3. **Proof of the theorem (1).** Our proof of the theorem is somewhat lengthy. So we divide it into three parts. We shall complete the proof in § 5.

3.1. Let $\sigma=(g; \nu_1, \cdots, \nu_n)$ be a signature different from $(0; \nu_1, \nu_2, \nu_3)$. We fix a Fuchsian group Γ_0 with signature σ and a system $\mathscr{G}_0=\{[\gamma_{0,1}], \cdots, [\gamma_{0,S}]\}$ of conjugacy classes of hyperbolic elements in Γ_0 as in 1.3. We shall retain the notations used in § 1. For a subset D of U, we denote by D^L the image of D under the reflection $z\to\bar z$ with respect to the real line. Choose a number ω for which $0<\omega<\omega(l_{\gamma_{0,s}})$ for all s, $1\leqq s\leqq S$. There exists a sequence of quasiconformal automorphisms $\{f_n\}$, $n\in N=\{1, 2, \cdots\}$, of $\hat{C}$ satisfying the following properties: (1) $f_n(\bar z)=\overline{f_n(z)}$ *and* f_n leaves U invariant; (2) f_n takes Γ_0 into a Fuchsian group $\Gamma_n=f_n\Gamma_0 f_n^{-1}$; (3) $l_{\gamma_{n,s}}$ converges to 0 as $n\to\infty$, where $\gamma_{n,s}=f_n\gamma_{0,s}f_n^{-1}$; and (4) $\operatorname{supp}(f_n)_{\bar z}\subset\bigcup_{s=1}^{S}\bigcup_{\gamma\in[\gamma_{0,s}]}(C(\omega,\gamma)\cup C(\omega,\gamma)^L)$. Actually we can construct f_n by pinching simple closed curves freely homotopic to $\pi_{\Gamma_0}(A_{\gamma_{0,s}})$ (see the proof of Theorem 11 in [2]). We set $R_n=R_{\Gamma_n}$ and $\pi_n=\pi_{\Gamma_n}$ the canonical projection. Moreover we set $\mathscr{G}_n=\{[\gamma_{n,1}], \cdots, [\gamma_{n,S}]\}$ and $\mathscr{P}_n=\mathscr{P}(\Gamma_n)$.

Let $\Omega=\Omega_{\Gamma_0}(\omega, \cdots, \omega)$ be the set defined in (1.3). Let $\hat V_{0,1}, \cdots, \hat V_{0,T}$ be the components of $\pi_{\Gamma_0}(\Omega)$, all of which are of type (0, 3). We remove a small neighborhood of $\partial\hat V_{0,t}$ from $\hat V_{0,t}$ to obtain a subregion $V_{0,t}$ of type (0, 3) such that $\operatorname{cl} V_{0,t}\subset\hat V_{0,t}$. The mapping f_n induces a homeomorphism $F_n\colon R_0\to R_n$ between the surfaces. Since $(f_n)_{\bar z}=0$ in Ω, f_n is conformal in Ω and hence F_n is conformal in each $\hat V_{0,t}$. Let $V_{n,t}=F_n(V_{0,t})$. For each t $(1\leqq t\leqq T)$, choose a lift $\tilde V_{0,t}$ of $V_{0,t}$. Let $\tilde d_t(\,,)$ be the distance defined by the hyperbolic metric on $\tilde V_{0,t}$ of constant curvature -1. Since $f_n|_{\tilde V_{0,t}}$ is conformal, the Ahlfors-Schwarz lemma yields

$$d(f_n(z), f_n(w))<\tilde d_t(z, w) \qquad \text{for} \quad z, w\in\tilde V_{0,t}\,, \tag{3.1}$$

where $d(\,,)$ is the hyperbolic distance of U. Let $\hat d_t(\,,)$ denote the distance of $\hat V_{0,t}$ induced by $\tilde d_t(\,,)$. Since $\operatorname{cl} V_{0,t}\subset\hat V_{0,t}$, $\sup_{p,q\in V_{0,t}}\hat d_t(p, q)$ are finite for all t. Let M_1 be the largest among them. Let $d_n(\,,)$ be the hyperbolic distance of R_n. Then, by (3.1) we obtain

$$\operatorname{diam} V_{n,t}=\sup_{p,q\in V_{n,t}} d_n(p, q)<M_1\,, \quad 1\leqq t\leqq T\,. \tag{3.2}$$

Note that M_1 is independent of n.

Let $\hat\Omega_n=\Omega_{\Gamma_n}(\omega(l_{\gamma_{n,s}}))$ and $\hat W_{n,1}, \cdots, \hat W_{n,T}$ be the components of $\pi_n(\hat\Omega_n)$. Here $\hat W_{n,t}$ is given the subscript t so that $\hat W_{n,t}$ is deformable to $V_{n,t}$ by an isotopy on R_n which fixes each ramification point. For a given $\varepsilon>0$, $l_{\gamma_{n,s}}<\varepsilon$ $(1\leqq s\leqq S)$ except for finitely many n. By applying a result by Matelski (the boundedness of the reduced diameter [11, sec. 8.8]) we can find a constant M'_2 independent of n such that:

(3.3) $$\operatorname{diam} \hat{W}_{n,t} = \sup_{p,q \in \hat{W}_{n,t}} d_n(p, q) < M_2', \quad 1 \leqq t \leqq T .$$

Since $F_n: R_0 \to R_n$ is a homeomorphism preserving ramification points and since $V_{n,t}$ is of type (0, 3) in R_n, $V_{n,t}$ meets $\hat{W}_{n,t}$. Hence by (3.2) $V_{n,t}$ is included in the M_1-neighborhood $W_{n,t}$ of $\hat{W}_{n,t}$ with respect to the distance $d_n(\,,\,)$. We set $M_2 = 2M_1 + M_2'$, $\omega_{n,s} = \omega(l_{\gamma_{n,s}}) - M_1$ and $\Omega_n = \Omega_{\Gamma_n}(\omega_{n,s})$. Then we obtain:

LEMMA 3.1. (1) *Each component $W_{n,t}$ of $\pi_n(\Omega_n)$ contains $V_{n,t}$ which is the image of $V_{0,t}$ under the conformal mapping $F_n|_{V_{0,t}}$*; (2) *The diameter of $W_{n,t}$ is less than a constant M_2 independent of n*; *and* (3) *$\omega_{n,s} \to \infty$ as $n \to \infty$.*

4. Proof of the theorem (2).

4.1. Let Γ_n, $\mathscr{G}_n = \{[\gamma_{n,1}], \cdots, [\gamma_{n,S}]\}$ and $\omega_{n,s}$ be as above. For each s $(1 \leqq s \leqq S)$ choose a $\theta = \theta_{n,s} \in \mathbf{M\ddot{o}b}_U$ which sends the axis $A_{\gamma_{n,s}}$ to the positive imaginary axis $\boldsymbol{I}$. Let $\beta_{a,\omega_{n,s}}$, $a \in A$, be the function as in 1.5 defined for $\omega = \omega_{n,s}$. We set $\beta_{a,n,s} = (\beta_{a,\omega_{n,s}} \circ \theta)\overline{\theta'}/\theta'$. Note that $\operatorname{supp} \beta_{a,n,s} \subset \operatorname{cl} C(\omega_{n,s}, \gamma_{n,s})$. Let $\Gamma_{n,s} = \operatorname{Stab}(A_{\gamma_{n,s}}, \Gamma_n)$. Since $\{\theta\eta\theta^{-1};\, \eta \in \Gamma_{n,s}\} \subset \mathbf{M\ddot{o}b}_I$, by (1.5) it holds that

(4.1) $$\beta_{a,n,s}(z) = \beta_{a,n,s}(\eta(z))\overline{\eta'(z)}/\eta'(z) \qquad \text{for} \quad \eta \in \Gamma_{n,s} .$$

Let $\Gamma_n \backslash \Gamma_{n,s}$ denote a system of representatives of the right cosets. We define

$$\mu_{a,n}(z) = \sum_{s=1}^{S} \sum_{\gamma \in \Gamma_n \backslash \Gamma_{n,s}} \beta_{a,n,s}(\gamma(z))\overline{\gamma'(z)}/\gamma'(z) .$$

By (4.1) we see that $\mu_{a,n}$ is independent of the choice of representatives of $\Gamma_n \backslash \Gamma_{n,s}$. Since $\operatorname{supp} \beta_{a,n,s} \circ \gamma \subset C(\omega_{n,s}, \gamma^{-1}\gamma_{n,s}\gamma)$, by Lemma 1.1 the terms of the above sum have disjoint supports. When γ runs over all cosets of $\Gamma_n \backslash \Gamma_{n,s}$, so does $\gamma\eta$ for each $\eta \in \Gamma_n$. Hence $\mu_{a,n}$ is a Beltrami differential for Γ_n with $\|\mu_{a,n}\|_\infty = |\delta(a) - 1| < 1$. We remark that, if $A_\gamma = I$ for some $\gamma \in [\gamma_{n,s}]$, then $\mu_{a,n} = \beta_a$ in $C(\omega_{n,s}) \cup \boldsymbol{L}$, where $C(\omega_{n,s})$ is given in (1.2). Let $g_{a,n}$ be a homeomorphic solution of the equation $g_{\bar{z}} = \mu_{a,n} g_z$ and $\phi_{a,n}(z) = \{g_{a,n}, z\}$ for $z \in \boldsymbol{L}$. Then $g_{a,n}$ is a quasiconformal automorphism of $\hat{\boldsymbol{C}}$ with delatation $K(a) = (1 + |\delta(a) - 1|)/(1 - |\delta(a) - 1|)$ and $\phi_{a,n}$ belongs to $\boldsymbol{T}(\Gamma_n)$.

4.2. In the sequel, when we say that we replace Γ_n by a conjugation $\eta^{-1}\Gamma_n\eta$ for an $\eta \in \mathbf{M\ddot{o}b}_U$, we also mean that we also replace $\mu_{a,n}$, $g_{a,n}$ and $\phi_{a,n}$ by $(\mu_{a,n} \circ \eta)\overline{\eta'}/\eta'$, $g_{a,n} \circ \eta$ and $(\phi_{a,n} \circ \eta)(\eta')^2$, respectively. We employ freely these replacement because of the equation:

(4.2) $$4(\operatorname{Im} z)^2 |\phi_{a,n}(\eta(z))\eta'(z)^2| = 4(\operatorname{Im} \eta(z))^2 |\phi_{a,n}(\eta(z))| .$$

We shall estimate $4(\operatorname{Im} z)^2 |\phi_{a,n}(z)|$ for z near the axes A_γ, $\gamma \in [\gamma_{n,s}]$. The same argument as in [11] applies in this case. We replace Γ_n by a conjugation of Γ_n in $\mathbf{M\ddot{o}b}_U$ so that $A_{\gamma_{n,s}} = I$. We can impose the condition $g_{a,n}(-ci) = (-ci)^{\delta(a)}$ for $c = 1, 2, 3$, because otherwise we need only to replace $g_{a,n}$ by $\eta g_{a,n}$ for some $\eta \in \mathbf{Möb}$. Then the $K(a)$-quasiconformal automorphisms $g_{a,n}$ of $\hat{\mathbf{C}}$ form a normal family and a limit function g_a^* is also a $K(a)$-quasiconformal automorphism of $\hat{\mathbf{C}}$ ([9, Sec. II 5]). Replace $\{g_{a,n}\}$ by a convergent subsequence to g_a^*. By Lemma 3.1 (3), the area of $\hat{\mathbf{C}}-(C(\omega_{n,s})\cup \mathbf{L})$ as a subset of the Euclidean sphere $\mathbf{S}=\hat{\mathbf{C}}$ decreases to 0. Since $\beta_a=\mu_{a,n}$ in $C(\omega_{n,s})\cup \mathbf{L}$, a subsequence of $\{\mu_{a,n}\}$, which is denoted again by $\{\mu_{a,n}\}$, converges to β_a almost everywhere in $\mathbf{S}$. Then we have $g_a^*=g_a$ the function given in 1.5, because both functions have the Beltrami coefficient β_a ([9, Theorem IV 5.2]) and take the same values at $-i$, $-2i$ and $-3i$. It follows that $\phi_{a,n}(z)$ converges to $\phi_a(z)=az^{-2}$ uniformly in every compact subset of $\mathbf{L}$. For positive numbers τ, l, let $K=\mathrm{cl}\, C(\tau)^{\mathbf{L}}\cap\{re^{i\theta}\in \mathbf{L};\ 1\leqq r\leqq e^l\}$. Then $\phi_{a,n}\to\phi_a$ uniformly in K. For large n, $l_{\gamma_{n,s}}<l$ and every $z\in C(\tau)^{\mathbf{L}}$ is equivalent to a point in K under $\{\gamma_{n,s}^{\nu};\ \nu\in \mathbf{Z}\}$. Hence for any $\varepsilon>0$, if $n_{1,s}(\tau,\varepsilon)$ is chosen to be sufficiently large, then $4(\operatorname{Im} z)^2|\phi_{a,n}(z)|<4|a|+\varepsilon$ for $z\in C(\tau)^{\mathbf{L}}$ if $n>n_{1,s}(\tau,\varepsilon)$. We determine $n_{1,s}(\tau,\varepsilon)$ for each s and set $n_1(\tau,\varepsilon)=\max_{1\leqq s\leqq S} n_{1,s}(\tau,\varepsilon)$. Then by using (4.2) we obtain the inequality

$$4(\operatorname{Im} z)^2|\phi_{a,n}(z)|<4|a|+\varepsilon\,, \tag{4.3}$$

which holds for $z\in\bigcup_{s=1}^{S}\bigcup_{\gamma\in[\gamma_{n,s}]} C(\tau,\gamma)^{\mathbf{L}}$ and $n>n_1(\tau,\varepsilon)$.

4.3. In the next section we shall show that the inequality (4.3) holds for all $z\in \mathbf{L}$ and $n>n_\varepsilon$ with n_ε sufficiently large. At present we assume this and prove first the desired estimate $I(\sigma)=2$. Since ε is arbitrary, the inequality (4.3) (established for all $z\in\mathbf{L}$) implies that $\lim_{n\to\infty}\|\phi_{a,n}\|\leqq 4|a|$. Suppose on the contrary that $I(\sigma)=2+2\rho>2$. Again we assume that $A_{\gamma_{n,1}}=I$. Substitute $1/4\in\Lambda$ for a and write ϕ_n instead of $\phi_{1/4,n}$. Then $\lim_{n\to\infty}\|\phi_n\|\leqq 1$ and by assumption $(2+\rho)\phi_n$ belongs to $\mathbf{T}(\Gamma_n)$ for all large n. Thus the equation $\{w,z\}=(2+\rho)\phi_n$ has univalent solutions. Let w_n be one of the solutions which sends $-i$, $-2i$, $-3i$ to 0, 1, ∞ in this order. Then $\{w_n\}$ is a normal family and a limit function w is also univalent in $\mathbf{L}$. As we have seen in 4.2, $(2+\rho)\phi_n(z)$ converges to $((2+\rho)/4)z^{-2}$ uniformy in every compact subset of $\mathbf{L}$. Consequently $\{w,z\}=((2+\rho)/4)z^{-2}$ and $\eta w(z)=z^{\sqrt{-\rho/2}}$ for some $\eta\in\mathbf{Möb}$. However, since $\sqrt{-\rho/2}$ is purely imaginary, w cannot be univalent. This contradiction yields $I(\sigma)=2$.

5. Proof of the theorem (conclusion). Now we show the inequality (4.3) $\|\phi_{a,n}\|<4|a|+\varepsilon$ for sufficiently large n.

5.1. In 3.1 we have chosen a lift $\tilde{V}_{0,t}$ of $\hat{V}_{0,t}$. We replace it by a lift of $V_{0,t}$ contained in $\tilde{V}_{0,t}$. Then by Lemma 3.1 (1) we can find a lift of $\tilde{W}_{n,t}$ of $W_{n,t}$ such that $\tilde{V}_{n,t}=f_n(\tilde{V}_{0,t})\subset\tilde{W}_{n,t}$. Note that the Γ_n-orbits of $\tilde{W}_{n,1},\cdots,\tilde{W}_{T,n}$ cover Ω_n.

We fix a t. Let $H_0=\mathrm{Stab}(V_{0,t},\Gamma_0)$ and $H_n=f_nH_0f_n^{-1}=\mathrm{Stab}(\tilde{V}_{n,t},\Gamma_n)$. These are Fuchsian groups of type (0, 3). Let $\chi_n: H_0\to H_n$ be the isomorphism between the groups defined by $\chi_n\eta=f_n\eta f_n^{-1}$ for $\eta\in H_0$. We fix a point $w\in\tilde{V}_{0,t}$. We may replace Γ_n by a conjugation of Γ_n in $\mathbf{Möb}_U$ so that $f_n(w)=i\in\tilde{V}_{n,t}$. Then from (3.1) $d(i,\chi_n\eta(i))<\tilde{d}_t(w,\eta(w))$ for $\eta\in H_0$. Hence there exists a subsequence of $\{\chi_n\}$, which is denoted again by $\{\chi_n\}$, such that $\chi_n\eta$ converges to a transformation $\chi_\infty\eta$ of $\mathbf{M\ddot{o}b}_U$ for each $\eta\in H_0$. Actually we need only to choose a subsequence so that χ_n converges on the set of two generators of H_0. By the convergence theorem ([5, Theorem 1]) $H_\infty=\{\chi_\infty\eta;\ \eta\in H_0\}$ is a Fuchsian group and $\chi_\infty: H_0\to H_\infty$ is an isomorphism. A hyperbolic element γ of H_0 determined by a boundary curve of $V_{0,t}$ belongs to $[\gamma_{0,s}]$ for some s. Then $\chi_n\gamma\in[\gamma_{n,s}]$ and $|\operatorname{tr}\chi_n\gamma|\to 2=|\operatorname{tr}\chi_\infty\gamma|$ as $n\to\infty$, for $l_{\gamma_{n,s}}\to 0$ as $n\to\infty$. Since χ_∞ is an isomorphism, χ_∞ is parabolic. It follows from this that H_∞ is a triangle group.

The Kraus-Nehari inequality ([8, Theorem II 1.3]) yields $|\phi_{a,n}(z)|\leqq(3/2)(\operatorname{Im}z)^{-2}$. Hence $\phi_{a,n}$ are locally uniformly bounded. By replacing $\{\phi_{a,n}\}$ by an appropriate subsequence, we may assume that $\phi_{a,n}$ converges to a holomorphic function ψ_a uniformly in every compact subset of $\boldsymbol{L}$. Then for each $z\in\boldsymbol{L}$ and each $\eta\in H_0$,

$$\psi_a(z)=\lim_{n\to\infty}\phi_{a,n}(z)=\lim_{n\to\infty}\phi_{a,n}(\chi_n\eta(z))(\chi_n\eta)'(z)^2=\psi_2(\chi_\infty\eta(z))(\chi_\infty\eta)'(z)^2\,.$$

Thus ψ_a is a quadratic differential for the triangle group H_∞ and hence identically zero. Let $B(M)$ denote the disc in $\boldsymbol{U}$ of hyperbolic center i and radius M. Then, for any $\varepsilon>0$, if $n_{2,t}(M,\varepsilon)$ is taken to be sufficiently large, then $4(\operatorname{Im}z)^2|\phi_{a,n}(z)|<\varepsilon$ for $z\in B(M)^{\boldsymbol{L}}$ and $n>n_{2,t}(M,\varepsilon)$. By Lemma 3.1 (2) $\pi_n(B(M_2))$ covers $W_{n,t}$. Hence the Γ_n-orbits of $B(M_2+\tau)^{\boldsymbol{L}}$ cover the hyperbolic τ-neighborhood of the Γ_n-orbits of $\tilde{W}^{\boldsymbol{L}}_{n,t}$. We set $n_2(\tau,\varepsilon)=\max_{1\leqq t\leqq T}n_{2,t}(M_2+\tau,\varepsilon)$ and denote by $\mathscr{N}_\tau(\Omega_n)$ the hyperbolic τ-neighborhood of Ω_n. Then by using (4.2) we can conclude that $4(\operatorname{Im}z)^2|\phi_{a,n}(z))|<\varepsilon$ for $z\in\mathscr{N}_\tau(\Omega_n)^{\boldsymbol{L}}$ and $n>n_2(\tau,\varepsilon)$.

5.2. Choose an arbitrary parabolic fixed point $p\in\mathscr{P}_0=\mathscr{P}(\Gamma_0)$. Then $p_n=f_n(p)\in\mathscr{P}_n$. We replace Γ_n by a conjugation of Γ_n in $\mathbf{M\ddot{o}b}_U$ so that $(\Gamma_n)_{p_n}=\operatorname{Stab}(p_n,\Gamma_n)$ is generated by $z\to z+1$. In this case, $D_n=D^{\boldsymbol{L}}_{p_n}=\{z;\ \operatorname{Im}z<-1\}$. We can identify $\boldsymbol{L}/(\Gamma_n)_{p_n}$ with the punctured disc $\Delta=\{z;\ 0<|z|<1\}$. Let $\pi_\Delta(z)=e^{-2\pi iz}$ be the canonical projection $\boldsymbol{L}\to\Delta$. The density of the hyperbolic metric on Δ is $\rho(z)=(-|z|\log|z|)^{-1}$. Since $\phi_{a,n}(z)=\phi_{a,n}(z+1)$, $\phi_{a,n}$ defines a function $\tilde{\phi}_{a,n}$ in Δ such that $(\tilde{\phi}_{a,n}\circ\pi_\Delta)(\pi'_\Delta)^2=\phi_{a,n}$. Let $\Delta_1=\pi_\Delta(D_n)=\{z;\ 0<|z|<e^{-2\pi}\}$. By the Kraus-Nehari inequality we have

$$(5.1)\qquad |\tilde{\phi}_{a,n}(\zeta)|\leqq c_1=(3/2)(e^{-2\pi})^2\qquad\text{for}\quad \zeta\in\partial\Delta_1\,.$$

Since $\tilde{\phi}_{a,n}(\zeta)\to 0$ as $\zeta\to 0$ ([7, p. 111]), (5.1) holds for all $\zeta\in\Delta_1$ by the maximum principle. Thus, $4(\operatorname{Im}z)^2|\phi_{a,n}(z)|=4\rho(\zeta)^{-2}|\tilde{\phi}_{a,n}(\zeta)|<4c_1(-|\zeta|\log|\zeta|)^2$ for $z\in D_n$ with $\zeta=\pi_\Delta(z)$. For a given $\varepsilon>0$, choose r (>1) to be so large that $4c_1(2\pi re^{-2\pi r})^2<\varepsilon$. Then $4(\operatorname{Im}z)^2|\phi_{a,n}(z)|<\varepsilon$ for z with $\operatorname{Im}z<-r$. Since ∂D_n lies in the boundary of $\Omega^{\boldsymbol{L}}_n$, the part $\{z;\ -r\leqq\operatorname{Im}z<-1\}$ is contained in $\mathscr{N}_\tau(\Omega_n)^{\boldsymbol{L}}$ with $\tau>\log r$. Hence $4(\operatorname{Im}z)^2|\phi_{a,n}(z)|\to 0$ for $z\in D_n$ and $n>n_2(\tau,\varepsilon)$ with $\tau>\log r$.

5.3. Choose a point z_n for which $4(\operatorname{Im}z)|\phi_{a,n}(z_n)|=\|\phi_{a,n}\|$. Since R_n is a compact surface with a finite number of punctures and since $4(\operatorname{Im}z)^2|\phi_{a,n}(z)|\to 0$ as $\pi_n(z)$ approaches to a puncture, such a point z_n certainly exists. We assume that there are infinitely many n for which $\|\phi_{a,n}\|>4|a|+\varepsilon$ and consider only those n in the sequel.

Let $n(\tau, \varepsilon)=\max\{n_1(\tau, \varepsilon), n_2(\tau, \varepsilon)\}$ and fix a τ_0 $(>\log r)$. From the argument in 5.1–5.3 it follows that z_n belongs to $(C(\omega_{n,s}, \gamma_{n,s})-C(\tau_0, \gamma_{n,s}))^{\boldsymbol{L}}$ for some $s=s_n$ if $n>n(\tau_0, \varepsilon)$, if we replace z_n by $\eta(z_n)$ for some $\eta\in\Gamma_n$. For the sake of simplicity we assume that $z_n\in C(\omega_{n,1}, \gamma_{n,1})^{\boldsymbol{L}}$ for all n without loss of generality. We write γ_n, ω_n instead of $\gamma_{n,1}$, $\omega_{n,1}$. Again we assume that $A_{\gamma_n}=I$. Moreover, without losing the property that $\mu_{a,n}=\beta_a$ in $C(\omega_n)\cup\boldsymbol{L}$ we may assume that $|z_n|=1$ and $\operatorname{Re} z_n>0$, because otherwise we need only to take a conjugation of Γ_n with respect to an element of $\mathbf{M\ddot{o}b}_{\boldsymbol{I}}$. For τ $(>\tau_0)$, if $n>n_2(\tau, \varepsilon)$, the hyperbolic distance from z_n to $\partial C(\omega_n)^{\boldsymbol{L}}\subset\partial\Omega_n^{\boldsymbol{L}}$ is larger than τ. Hence the disc $\mathrm{B}(\overline{z_n}, \tau)\subset\boldsymbol{U}$ of hyperbolic center $\overline{z_n}$ and radius τ is contained in $C(\omega_n)$. The transformation $\xi_n(z)=\gamma_{\varepsilon_n}(z)=(z-\varepsilon_n)/(z+\varepsilon_n)$ with $\varepsilon_n=-\tan\arg(z_n/2)$ sends the disc $B(\tau)$ of hyperbolic center i and radius τ onto $B(\overline{z_n}, \tau)$. Hence two functions $\hat{\mu}_{a,n}=(\mu_{a,n}\circ\xi_n)\overline{\xi_n'}/\xi_n'$ and $(\beta_a\circ\xi_n)\overline{\xi_n'}/\xi_n'$ coincide with each other in $B(\tau)\cup\boldsymbol{L}$. Since the hyperbolic distance from $\overline{z_n}$ to $A_{\gamma_n}=I$ is larger than τ for $n>n_1(\tau, \varepsilon)$ and since n eventually exceeds $n(\tau, \varepsilon)$ for any τ as $n\to\infty$, we see that $\varepsilon_n\to 0$ as $n\to\infty$. Hence a subsequence of $\{\hat{\mu}_{a,n}\}$, which is denoted again by $\{\hat{\mu}_{a,n}\}$, converges to $\hat{\beta}_a$ given in 1.5 almost everywhere in $\boldsymbol{S}$. Let $h_{a,n}=\eta_n\circ g_{a,n}\circ\xi_n$, where $\eta_n\in\mathbf{M\ddot{o}b}$ is so chosen that $h_{a,n}(-ci)=-ci$ for $c=1, 2, 3$. Then $h_{a,n}$ satisfies $(h_{a,n})_{\bar{z}}=\hat{\mu}_{a,n}(h_{a,n})_z$. By proceeding as in 4.2 we see that $h_{a,n}$ converges uniformly to h_a. Consequently $\phi_{a,n}(\xi_n(z))\xi_n'(z)^2=\{h_{a,n}, z\}$ converges to $\{h_a, z\}=0$ uniformly in every compact subset of $\boldsymbol{L}$. In particular, $(\operatorname{Im} z)^2|\phi_{a,n}(z_n)|=|\{h_{a,n}, -i\}|\to 0$ as $n\to\infty$. This contradicts the assumption $\|\phi_{a,n}\|>4|a|+\varepsilon$.

Now the inequality (4.3) is established for all $z\in\boldsymbol{L}$ and for all large n. Then, as we have seen in 4.3, the desired estimate of the inner radii $I(\sigma)=2$ is obtained.

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