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Perfect Graphs and Complex Surface Singularities with Perfect Local Fundamental Group

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Abstract. In this paper we introduce the term "perfect graph" to refer to those graphs which characterize resolutions of certain isolated singular points of complex surfaces. Using techniques for graphical evaluation of determinants, we reduce questions about perfect graphs to problems involving partial fraction representations of positive integers; the solutions to those Diophantine problems thus have interesting geometric interpretations.

1. Introduction and statement of results. In [5] Brieskorn gave the first examples of isolated singularities of complex n-varieties, $n \geq 3$, that are topologically non-singular (locally homeomorphic to the $2n$-ball) but analytically singular. Earlier Mumford [16] had shown that this is impossible in dimension 2. In this paper we pursue the natural analogue of the Brieskorn singularities for complex surfaces, namely those singular points $x \in X$ which are *homologically* non-singular in the sense of being locally homeomorphic to the cone on a homology 3-sphere. (The rational double point E_8 is the most familiar example.) This condition is equivalent to the requirement that the local fundamental group of x in X be a perfect group (cf., for example, [16], [17], and [19], where the topic of classifying isolated two-dimensional singularities by the group-theoretic properties of the local fundamental group is introduced and developed).

Let x be an isolated singularity of a normal complex surface X, and let p: $\tilde{X} \to X$ be the minimal resolution of singularities. We will assume that the exceptional curve $C = p^{-1}(x) = \bigcup_{i=1}^{n} C_i$ is contractible, that each component C_i is non-singular rational, and that the components meet transversally with no triple intersections. In this case the topology of the singularity is completely determined by the weighted dual intersection graph G_p of the exceptional curve. In particular, the local fundamental group $\pi_1(x)$ can be computed directly from G_p in terms of generators and relations, by the technique of Mumford [16]. Using this method it can be shown that $\pi_1(x)$ is perfect exactly when the intersection matrix $(-C_i \cdot C_j)$ has determinant 1. Indeed, the following are necessary and sufficient conditions for a weighted graph G to be the dual graph of the minimal resolution of a normal complex surface singularity whose minimal resolution is normal ("good") and whose local fundamental group is perfect:

(a) G is a tree (a connected graph with no circuits).

(b) Each weight w_i is an integer ≥ 2.

(c) The associated intersection matrix is positive definite with determinant 1.

(Section 1 of [4] gives an elementary expository review of the geometry of complex surface singularities as reflected by their resolutions, graphs, and local fundamental groups; [1] investigates and classifies some of the global settings in which such "perfect" singular points occur.)

These considerations motivate the following:

(1.1) DEFINITIONS. A weighted graph G is *perfect* if it satisfies conditions (a), (b), and (c) above. A graph G is *perfectable* if there exist integer weights w_i for its vertices such that the resulting weighted graph $G(w_1, \cdots, w_n)$ is perfect. Such a set of weights is called a set of *perfect weights*. A *minimal perfectable graph* is a perfectable graph none of whose proper subgraphs is perfectable.

The goal of this paper is to find perfect graphs and to point out connections between perfect graphs and solutions of certain Diophantine equations of interest in number theory. Our results can be summarized as follows:

(1.2) MAIN THEOREM. *Let G be any graph which is not of the form shown in Figure 1 for $n=0$, 1, or 2. Then G is perfectable if and only if G is a tree that contains one of the 25 minimal perfectable graphs listed in Table* I *at the end of this paper.*

In consequence we obtain the following results for particular kinds of graphs.

(1.3) THEOREM. *There is no perfect weighted graph on 7 or fewer vertices. The perfect weighted graphs on 8 vertices appear in Figure 2. Of the 47 trees on 9 vertices exactly 30 are perfectable (most with several sets of perfect weights). In fact, "almost all" trees with sufficiently many vertices are perfectable; that is,*

$$\lim_{n\to\infty} \frac{\textit{number of perfectable trees on } n \textit{ vertices}}{\textit{total number of trees on } n \textit{ vertices}} = 1 .$$

The *distance* between two points of a graph G is the number of edges in the shortest path joining them. The *diameter* of a graph G is the maximum of the distances between

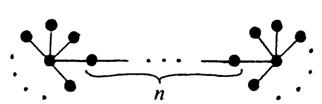


FIGURE 1

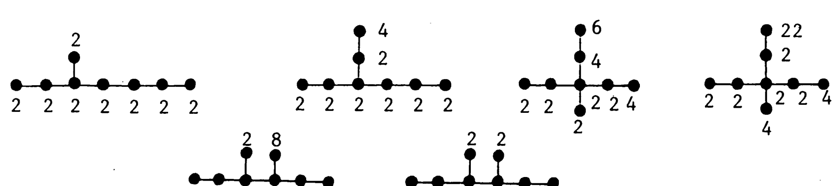


FIGURE 2

pairs of points in G. In the case of a tree it is the number of edges in the longest "chain" (tree without branch points) contained in G.

(1.4) THEOREM. (a) *Among ordinary and extended Dynkin diagrams of types A, D, and E, only E_8 and $\tilde{E}_8$ are perfectable, and each of these graphs has only one set of perfect weights.*

(b) *Every tree of diameter $d \geq 29$ is perfectable except those isomorphic to A_n, D_n, and $\tilde{D}_n$.*

(c) *Every tree of diameter $d \geq 7$ is perfectable except A_n, D_n, $\tilde{D}_n$, and the graphs in Figure 3.*

DEFINITION. Let $p_1 \leq p_2 \leq \cdots \leq p_r$ be positive integers. A graph G is *of type $E_{p_1,\cdots,p_r}$* if G has a vertex v_0 such that $G - \{v_0\}$ is the disjoint union of r graphs of types $A_{p_1,\cdots,p_r}$, each joined to v_0 only at a terminal vertex (cf. Figure 4).

(1.5) THEOREM. (a) *There is a one-to-one correspondence between perfect weighted graphs of type $E_{p_1,\cdots,p_r}$ and solutions in reduced proper fractions s_i/t_i of the equation*

$$\sum_{i=1}^{r} \frac{s_i}{t_i} + \frac{1}{\prod_{i=1}^{r} t_i} = n$$

with n an integer ≥ 2 and s_i, t_i positive integers for $i = 1, \cdots, r$.

(b) *A graph G of type $E_{p_1,\cdots,p_r}$, with $r \geq 3$ and $p_{r-1} \geq 2$, is perfectable if and only if G contains one of the following:*

$$E_{1,2,4}\,;\, E_{2,2,3}\,;\, E_{1,2,2,2}\,;\, E_{1,1,1,2,3}\,;\, E_{1,1,1,1,1,2,2}\,.$$

This last result shows one of the connections of this topic with certain problems of independent interest in number theory. For instance, putting each $s_i = 1$ above leads to the following unresolved question in the theory of Egyptian fractions (Paul Erdös offers \$100 for a solution): Given positive integers $t_1, \cdots, t_k$, relatively prime in pairs and all ≥ 2, do there always exist integers $n, t_{k+1}, \cdots, t_r$, all ≥ 2, such that $n = \sum_{i=1}^{r}(1/t_i) + 1/(\prod_{i=1}^{r} t_i)$? Connections with number theory will be discussed more

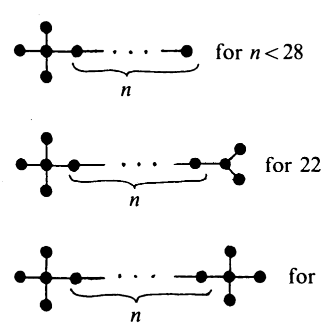


FIGURE 3

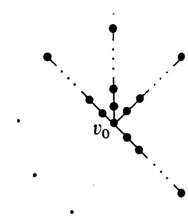


FIGURE 4

fully in Section 4.

As noted above, finding perfect weighted graphs is equivalent to finding symmetric bilinear forms $\phi: \boldsymbol{Z}^n \times \boldsymbol{Z}^n \to \boldsymbol{Z}$ of determinant 1, corresponding to the intersection matrices $(-C_i \cdot C_j)$. In Section 2 we present techniques for quickly calculating the determinant of the intersection matrix associated to a weighted graph directly from the graph. The results are stated in sufficient generality to apply in a wide range of settings.

2. Graphical evaluation of determinants. In this section, we describe methods of evaluating the determinant of a matrix by use of an associated graph. The initial results are useful for general sparse matrices, but do not seem to be well known. The more specialized versions are useful in our classification of perfectable graphs. (See also references [1], [4], [7], and [8].)

Let $M=(m_{ij})$ be an $n \times n$ matrix with entries in a commutative ring A with identity element 1 different from 0. The determinant of M is given by the formula

$$|M| = \sum_{\sigma \in S_n} (\operatorname{sgn} \sigma) m_{1,\sigma(1)} m_{2,\sigma(2)} \cdots m_{n,\sigma(n)}, \tag{2.1}$$

where S_n is the set of all permutations of $\{1, 2, \cdots, n\}$. Let $c=[i_1, \cdots, i_k]$ $(1 \le k \le n)$ denote the k-cycle in S_n that cyclically permutes the distinct indices $i_1, \cdots, i_k$. (When $k=1$, c is the identity permutation of a singleton set.) c is *even* if k is even. The *weight* of c is the ring element $w(c)=m_{i_1,i_2} \cdots m_{i_{k-1},i_k} m_{i_k,i_1}$. ($w(c)=m_{i_1,i_1}$ if $k=1$.) The *signed weight* of c is the ring element $\tilde{w}(c)=(\operatorname{sgn} c)w(c)=(-1)^{k-1}w(c)$. More generally, if $\sigma=c_1 \cdots c_s$ is a product of disjoint cycles, we define $w(\sigma)=w(c_1) \cdots w(c_s)$ and $\tilde{w}(\sigma)=\tilde{w}(c_1) \cdots \tilde{w}(c_s)=(\operatorname{sgn} \sigma)w(\sigma)=(-1)^{e(\sigma)}w(\sigma)$, where $e(\sigma)$ is the number of even cycles among $c_1, \cdots, c_s$. Then (2.1) can be rewritten as

$$|M| = \sum_{\sigma \in S_n} \tilde{w}(\sigma) = \sum_{\sigma \in S_n} (-1)^{e(\sigma)} w(\sigma). \tag{2.2}$$

Define the *associated graph* of M to be the directed graph $G=G(M)$ with n vertices (labelled $1, 2, \cdots, n$) that has a directed edge (i, j) from the vertex i to the vertex j precisely when $m_{ij} \neq 0$. ($i=j$ is allowed.) A *circuit* of length k $(1 \le k \le n)$ in G is a k-cycle $c=[i_1, \cdots, i_k]$ such that (i_r, i_{r+1}) (for $1 \le r < k$) and (i_k, i_1) are directed edges of G. A product $p=c_1 \cdots c_s$ of circuits of G will be called a *circuit partition* of G if the domains of $c_1, \cdots, c_s$ form a partition of $\{1, 2, \cdots, n\}$. The set of all circuit partitions of G will be denoted P. Formula (2.2) implies that

$$|M| = \sum_{p \in P} \tilde{w}(p) = \sum_{p \in P} (-1)^{e(p)} w(p).$$

This means that we can calculate $|M|$ just by looking at the graph G with its directed edges labelled by the ring elements m_{ij}. The labelled graph G uniquely determines the matrix M, so we can write $|G|$ for $|M|$ and obtain the graph-theoretic formula

$$|G| = \sum_{p \in P} \tilde{w}(p) = \sum_{p \in P} (-1)^{e(p)} w(p)\,. \tag{2.3}$$

Decomposition of a determinant relative to components of the associated graph. If G has connected components $G_1, \cdots, G_t$, and if $P_1, \cdots, P_t$ are the corresponding sets of circuit partitions, then $P = P_1 \cdots P_t$, so

$$|G| = \sum \tilde{w}(p_1 \cdots p_t) = \prod_{i=1}^{t} \Big(\sum_{p_i \in P_i} \tilde{w}(p_i) \Big) = \prod_{i=1}^{t} |G_i|\,, \tag{2.4}$$

where the first sum is taken over all $(p_1, \cdots, p_t) \in P_1 \times \cdots \times P_t$. A directed graph is said to be *strongly connected* if there is a directed path from each vertex to every other vertex. Equivalently, each pair of distinct vertices can be joined by a circuit. A *strongly connected component* is a maximal strongly connected subgraph. Each connected component of a directed graph contains one or more strongly connected components. The strongly connected components partition the vertices of the graph, but edges that do not belong to any circuit of the graph are not in any of the strongly connected components. If G has strongly connected components $G'_1, \cdots, G'_s$, and if $P'_1, \cdots, P'_s$ are the corresponding sets of circuit partitions, then $P = P'_1 \cdots P'_s$ as before, so $|G| = |G'_1| \cdots |G'_s|$ by (2.3).

Expansion of a determinant relative to a vertex of the associated graph. Let $C(i)$ denote the collection of all circuits of G passing through vertex i. Since every circuit partition of G must have one factor which is a circuit passing through i, we have $P(G) = \{c \cdot p : c \in C(i), p \in P(G-c)\}$, where $G-c$ is obtained from G by deleting all the vertices of the circuit c and all the edges of G incident with those vertices. By (2.3),

$$|G| = \sum_{c \in C(i)} \sum_{p \in P(G-c)} \tilde{w}(c)\tilde{w}(p) = \sum_{c \in C(i)} \tilde{w}(c) \Bigg[\sum_{p \in P(G-c)} \tilde{w}(p) \Bigg]; \quad \text{i.e.,}$$

$$|G| = \sum_{c \in C(i)} \tilde{w}(c) |G-c|\,. \tag{2.5}$$

Let $C_k(i)$ denote the set of circuits of length k passing through vertex i. Then it follows that

$$|G| = \sum_{1 \le k \le n} (-1)^{k-1} \sum_{c \in C_k(i)} w(c) |G-c| = m_{ii} |G - \{i\}| - \sum_{2 \le k \le n} (-1)^k \sum_{c \in C_k(i)} w(c) |G-c|\,. \tag{2.6}$$

In the important special case where G has no circuits of length >2,

$$|G| = m_{ii} |G - \{i\}| - \sum_{c \in C_2(i)} w(c) |G-c|\,. \tag{2.7}$$

Expansion of a determinant relative to an exclusive circuit of the associated graph. A circuit c of a graph G is *exclusive* if none of its directed edges belongs to another circuit. Let $G-E(c)$ denote the graph obtained from G by deleting the directed edges (but not the vertices) of c. Then $P(G)=c\cdot P(G-c)\cup P(G-E(c))$, so (2.3) implies

$$|G|=\sum_{p\in P(G-c)}\tilde{w}(c)\tilde{w}(p)+\sum_{q\in P(G-E(c))}\tilde{w}(q)=\tilde{w}(c)|G-c|+|G-E(c)|\,. \tag{2.8}$$

When G has more than one exclusive circuit, successive applications of (2.8) with different exclusive circuits can be used to expand $|G|$ in terms of determinants of smaller subgraphs of G.

Associated graphs and determinants for a special class of symmetric matrices. We will need to compute determinants of matrices representing intersection forms, which are symmetric bilinear forms associated to resolutions of singularities of complex surfaces. In our applications, the symmetric matrices for these forms have integral entries greater than 1 along the diagonal, and nothing but 0's and -1's off the diagonal.

Let M be an $n\times n$ symmetric matrix of the type just described. We represent M by a graph on n vertices, each labelled with the corresponding diagonal entry of M. The i-th such entry is the weight of the circuit of length 1 at the i-th vertex, so we denote it w_i and call it the *weight* of the i-th vertex. Note: To simplify the labelling of our graphs, *we omit the label w_i when $w_i=2$.* For distinct i and j in $\{1, 2, \cdots, n\}$, either $m_{ij}=m_{ji}=0$ or $m_{ij}=m_{ji}=-1$. In the second case, we join the i-th and j-th vertices with a single unlabelled, undirected edge. This yields an undirected graph $G=G(M)$, some of whose vertices may be labelled with a positive integer (not 2). Conversely, any such graph G together with an ordering of the vertices uniquely determines a symmetric matrix M with positive integers >1 along its diagonal and with 0's and -1's elsewhere.

For such a graph G, the terms "connected" and "strongly connected" are synonymous. Every circuit of length >1 has signed weight -1, so (2.6) and (2.8) become

$$|G|=w_i|G-\{i\}|-\sum_{k>1}\sum_{c\in C_k(i)}|G-c| \tag{2.9}$$

and

$$|G|=|G-E(c)|-|G-c|\,. \tag{2.10}$$

EXAMPLE. We compute the determinant of the graph G in Figure 5.

Using (2.3) is impractical because the number of circuit partitions is too large. It is better to break up the calculation into several easier calculations by use of (2.9)

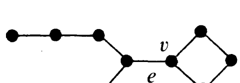
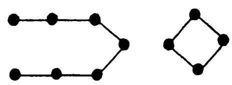
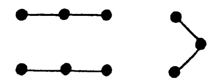


FIGURE 5



$G-E(c)$



$G-c$

FIGURE 6

and/or (2.10) in combination with (2.4).

First note that recursive use of (2.9) shows that a chain of k vertices has determinant $k+1$. Now it follows from (2.9) that the cyclic graph or "necklace" formed by joining a vertex to the two ends of a chain of k vertices (with $k \geq 2$) has determinant 0.

If we apply (2.9) and (2.4) to the original graph G at v, we get $|G| = 2\cdot 8\cdot 4-4\cdot 4\cdot 4-8\cdot 3-8\cdot 3-8-8=-64$. The last two terms arise from deletion of the two circuits of length 4 through v.

A better method is to apply (2.10) to the exclusive circuit of length 2 represented by the undirected edge e of G. There are only two terms, as pictured in Figure 6. Thus $|G|=8\cdot 0-4\cdot 4\cdot 4=-64$.

When G is a tree, (2.9) and (2.10) become

(2.11)
$$|G|=w_i|G-\{i\}|-\sum_{c\in C_2(i)}|G-c|$$

and

(2.12)
$$|G|=|G_1||G_2|-|G_1'||G_2'|,$$

where $G-E(c)=G_1 \cup G_2$ and $G-c=G_1' \cup G_2'$ (canonical disjoint unions).

It is useful to recast (2.11) in notation that emphasizes what remains of the graph rather than what was deleted. For clarity, we now use subscripted v's rather than integers to name vertices.

PROPOSITION. *Let v_0 be a vertex with weight w_0 in a weighted tree $\tilde{G}$, and set $G=\tilde{G}-\{v_0\}$. Let $v_1, \cdots, v_r$ be the vertices joined to v_0 in $\tilde{G}$. For $i=1, \cdots, r$, let G_i denote the component of G that contains v_i, and put $G_i'=G_i-\{v_i\}$. Then*

(2.13)
$$|\tilde{G}|=w_0\prod_{i=1}^{r}|G_i|-\sum_{i=1}^{r}|G_i'|\prod_{j\neq i}|G_j|,$$

so

(2.14)
$$w_0=\sum_{i=1}^{r}\frac{|G_i'|}{|G_i|}+\frac{|\tilde{G}|}{\prod_{i=1}^{r}|G_i|}.$$

(2.15) APPLICATION. Say that v_0 is a *terminal* vertex if exactly one other vertex of $\tilde{G}$ is joined to v_0. In that case $G_1=G$, so we write G' for G_1' and (2.13) becomes

(2.16)
$$|\tilde{G}|=w_0|G|-|G'|.$$

In practice, we start with a tree G and join a new vertex v_0 to a vertex v_i of G to form a larger tree $\tilde{G}$. We try to choose w_0 so that $1 \leq |\tilde{G}| \leq |G|$. Repetition of the process often produces a graph with determinant 1.

We also note for later use that if G is positive definite (that is, if the associated matrix of G is positive definite) and $|\tilde{G}|>0$, then $\tilde{G}$ is positive definite. This follows from the fact [18, p. 250] that a symmetric $n \times n$ matrix M is positive definite if and only if the upper left $k \times k$ submatrix of M has positive determinant for $k=1, 2, \cdots, n$.

3. Perfect graphs. We will now apply these ideas to the special graphs under consideration in this paper.

(3.1) LEMMA. *If G is a perfectable graph, then any tree containing G is also perfectable.*

PROOF. Every tree $\tilde{G}$ containing G can be constructed from G by successively adjoining vertices $v_{n+1}, \cdots, v_m$ to G , each by means of a single edge. In this way we obtain a chain of trees $G=\tilde{G}_n \subset \tilde{G}_{n+1} \subset \cdots \subset \tilde{G}_m = \tilde{G}$. For $k=n+1, \cdots, m$, let u_k be the vertex of $\tilde{G}_{k-1}$, to which v_k was joined in forming $\tilde{G}_k$. Set $\tilde{G}'_k = \tilde{G}_{k-1} - \{u_k\}$.

If G is perfectable, fix a set of perfect weights $w_1, \cdots, w_n$. Assign weights w_k to the vertices v_k $(k=n+1, \cdots, m)$ by defining $w_k = |\tilde{G}'_k| + 1$. Then repeated use of Application (2.15) above shows that each weighted tree $\tilde{G}_k$ is positive definite with determinant 1. Hence each $\tilde{G}_k$ is perfect. In particular, $\tilde{G}=\tilde{G}_m$ is perfect.

In view of Lemma (3.1), finding all perfectable graphs is equivalent to finding all minimal perfectable graphs. Our main result is that the graphs of Table I are minimal perfectable. To show this we must first check that each of these trees G is positive definite and has determinant 1. These properties can easily be verified recursively by use of (2.16). For example, consider entry (13) of Table I. We construct this weighted graph one vertex at a time, using the formula $|\tilde{G}_{k+1}| = w_{k+1}|\tilde{G}_k| - |\tilde{G}_{k-1}|$ to compute the determinants. (See Figure 7.) $\tilde{G}_k$ is the subgraph spanned by $v_1, \cdots, v_k$. The determinants $|\tilde{G}_1|, \cdots, |\tilde{G}_9|$ are 12, 23, 34, 45, 179, 1381, 692, 3, and 1. Since each determinant is positive and the last determinant is 1, the graph is perfect. The other 24 examples are checked similarly.

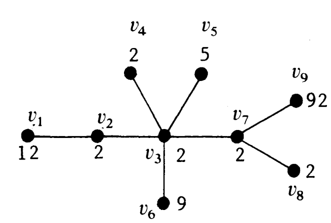


FIGURE 7

To show that each of these perfectable graphs G is *minimally* perfectable we must show that for each proper subgraph G' of G there do not exist weights $w_i \geq 2$ for which G' is perfect. On the integer lattice $\boldsymbol{Z}^n$, write $(x_1, \cdots, x_n) < (y_1, \cdots, y_n)$ if $x_i \leq y_i$ for all i and $x_{i_0} < y_{i_0}$ for at least one index i_0. If G is any graph on vertices $v_1, \cdots, v_n$, we have the mapping $|G| : \boldsymbol{Z}^n \to \boldsymbol{Z}$ whose value at $(w_1, \cdots, w_n)$ is the determinant of the weighted graph $G(w_1, \cdots, w_n)$.

(3.2) LEMMA. *Let G be a tree on vertices $v_1, \cdots, v_n$ and let $\bar{w} = (w_1, \cdots, w_n)$ be a point of $\boldsymbol{Z}^n$ at which $G(w_1, \cdots, w_n)$ is positive definite. Then if $\bar{y} = (y_1, \cdots, y_n) > \bar{w}$, $G(y_1, \cdots, y_n)$ is also positive definite and $|G(y_1, \cdots, y_n)| > |G(w_1, \cdots, w_n)|$.*

PROOF. The first assertion is obvious, since the intersection matrix of $G(\bar{y})$ is the sum of the intersection matrix of $G(\bar{w})$ and the diagonal matrix $D(\bar{y} - \bar{w})$ whose diagonal entries are the non-negative integers $y_i - w_i$ and whose off-diagonal entries are zero. Since $G(\bar{w})$ is positive definite and $D(\bar{y} - \bar{w})$ is positive semi-definite, $G(\bar{y})$ is positive definite.

As for the determinant, (2.13) implies that, for each i,

$$|G|(w_1, \cdots, w_n) = w_i \cdot |G - \{v_i\}|(w_1, \cdots, \hat{w}_i, \cdots, w_n) - (\text{terms that do not involve } w_i),$$

where $\hat{}$ means "omit this entry". Since $G - \{v_i\}$ is positive definite at $(w_1, \cdots, \hat{w}_i, \cdots, w_n)$, $|G|$ is a strictly increasing linear function of w_i. Hence $|G|(\bar{y}) > |G|(\bar{w})$ as claimed.

In fact, more is true.

(3.3) LEMMA. *Let v_0 be a vertex in a weighted tree $\tilde{G}$ and set $G = \tilde{G} - \{v_0\}$. Then the ratio $|G|/|\tilde{G}|$ strictly decreases as $(w_0, w_1, \cdots, w_n)$ increases with respect to $<$ in $\boldsymbol{Z}^{n+1}$.*

PROOF (induction on n). If $n = 0$ the claim is just that $1/w_0 > 1/y_0$ whenever $w_0 < y_0$. Now let $n > 0$ and suppose the assertion to be true for all smaller trees. Let $v_1, \cdots, v_r$ be the vertices joined to v_0 in $\tilde{G}$, and for $i = 1, \cdots, r$, let G_i be the component of G that contains v_i. By (2.14),

$$\frac{|\tilde{G}|}{|G|} = w_0 - \sum_{i=1}^{r} \frac{|G_i'|}{|G_i|},$$

where $G_i' = G_i - \{v_i\}$. $|\tilde{G}|/|G|$ clearly increases as a function of w_0, and by the induction hypothesis, each term $|G_i'|/|G_i|$ strictly decreases as a function of $(w_1, \cdots, w_n)$. Thus $|\tilde{G}|/|G|$ strictly increases as a function of $(w_0, w_1, \cdots, w_n)$.

To determine whether any particular graph is perfectable or not is now a finite calculation (perhaps a lengthy one if G is complicated). If G, v_0, G_i, and G_i' are defined as above, then it is clear from Lemma (3.3) that for each i the function $|G_i'|/|G_i|$ achieves a maximum value M_i on the part of $\boldsymbol{Z}^n$ where G_i is positive definite. Hence we have the bound

$$w_0 = \sum_{i=1}^{r} \frac{|G_i'|}{|G_i|} + \frac{1}{\prod_{i=1}^{r} |G_i|} \leq \sum_{i=1}^{r} M_i + 1$$

for the weight w_0 on v_0. Since there is a similar bound for each weight, only finitely many choices of weights $w_0, \cdots, w_n$ need be checked. In practice it may not be easy to determine the maxima M_i if G_i is a large and complicated graph, but shortcuts of a number-theoretic nature are often available. For example, for most of the graphs in question, judicious use of (2.14) shows that many of the weights must be quite small (often 2 is the only possibility), and must be chosen such that, for all choices of v_0 and of components G_i, G_j of $G - \{v_0\}$, the determinants $|G_i|$ and $|G_j|$ are coprime. If all but two weights have been determined, the last two weights must satisfy a quadratic equation whose coefficients are the determinants of various subgraphs and which has at most finitely many solutions (often none) in integers.

We have carried out these calculations for each of the graphs in Table II at the end of this paper, with this result:

(3.4) PROPOSITION. *None of the graphs in Table* II *is perfectable.*

It is now easy to complete the proof of the minimal perfectability of the graphs in Table I: each proper subgraph of a graph in Table I is contained in one of the non-perfectable graphs in Table II, so by Lemma (3.1) it must also be non-perfectable.

Likewise the proofs of Theorems (1.2), (1.3), (1.4), and part (b) of (1.5) are completed by verifying that every graph described in these theorems either contains a graph from Table I, and so is perfectable, or else is contained in a graph from Table II, and so is not perfectable. In particular, every tree not of the type shown in Figure 1 with $n = 0$, 1, or 2, is accounted for. Also, for any perfectable graph G, a finite computational search suffices to find all sets of perfect weights. This was done, for instance, for the 8-vertex graphs listed in Theorem (1.3). (The first part of Theorem (1.5) will be proved in the next section.)

As for the last assertion of Theorem (1.3), the existence of a single perfectable graph is sufficient to prove that

$$\lim_{n \to \infty} \frac{\text{number of perfectable graphs on } n \text{ vertices}}{\text{total number of trees on } n \text{ vertices}} = 1 ,$$

since for any given tree G_0, almost all trees with sufficiently many vertices contain G_0. Indeed, our results show that for each $n > 31$, all but at most 5 trees on n vertices are

FIGURE 8

perfectable, the 5 exceptions being the Dynkin diagrams A_n and D_n, the extended Dynkin diagram $\tilde{D}_{n-1}$, and possibly one or both of the graphs in Figure 8.

4. Open questions and connections with number theory. The equation $w_0 = \sum(|G_i'|/|G_i|) + 1/(\prod |G_i|)$ makes it clear that finding perfect graphs is equivalent to finding solutions of certain Diophantine equations. As a particularly interesting example, we now give a proof of part (a) of Theorem (1.5).

(4.1) PROPOSITION. *There is a one-to-one correspondence between perfect weighted graphs of the form $E_{p_1,\cdots,p_r}$ and solutions in integers of the equation*

$$w_0 = \sum_{i=1}^{r} \frac{A_i}{B_i} + \frac{1}{\prod_{i=1}^{r} B_i}$$

with $w_0 \geq 2$, $B_i \geq 2$, $0 < A_i < B_i$, and $(A_i, B_i) = 1$.

First we need a lemma.

(4.2) LEMMA. *Given relatively prime positive integers $A < B$, there is a unique weighted chain G (cf. Figure 9) such that $|G| = B$, $|G'| = A$, and $w_i \geq 2$ for all i.*

PROOF (induction on B). If $B = 2$, then $A = 1$, and the unique solution is that G consists of one vertex of weight 2 and G' is empty. Now let $B > 2$ and assume that the result is true for all smaller numbers. Given $0 < A < B$ with $(A, B) = 1$, if $A = 1$, then again one solution is for G to be a vertex of weight B and G' an empty graph. This solution is unique since no non-empty chain G' with all weights ≥ 2 can have determinant 1.

If $A > 1$, let w_0 be the unique integer ≥ 2 for which $B < w_0 A < A + B$. Put $A' = w_0 A - B$. Then $0 < A' < A < B$, and $(A', A) = 1$, so by the induction hypothesis there is a unique weighted chain G' (cf. Figure 10) with $|G'| = A$ and $|G''| = A'$. But then for this choice of weights the graph G in Figure 9 has $|G'| = A$ and $|G| = w_0 |G'| - |G''| = B$ as required. G is unique since if $\bar{G}$, $\bar{G}'$ is another solution, with weights $\bar{w}_0, \cdots, \bar{w}_{\bar{n}}$, then $B = |\bar{G}| = \bar{w}_0 |\bar{G}'| - |\bar{G}''| = \bar{w}_0 A - |\bar{G}''|$. But $\bar{w}_2, \cdots, \bar{w}_{\bar{n}} \geq 2$ implies that $|\bar{G}''| < A$. (To see this, note that the chain $A_{\bar{n}}(\bar{w}_1, \cdots, \bar{w}_{\bar{n}})$ is positive definite if $\bar{w}_i \geq 2$ for all i, so by Lemma (3.3), $|\bar{G}''|/A = |\bar{G}''|/|\bar{G}'| < |A_{\bar{n}-1}(2, \cdots, 2)|/|A_{\bar{n}}(2, \cdots, 2)| = \bar{n}/(\bar{n}+1) < 1$.) Thus $|\bar{G}''| = A'$ and $\bar{w}_0 = w_0$, so uniqueness of the solution for A, A' implies that $G = \bar{G}$.

REMARK. The proof shows that the unique solution $\{w_1, \cdots, w_n\}$ is just the set of integers that appear in the continued fraction expansion

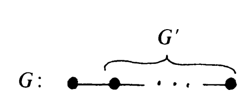


FIGURE 9

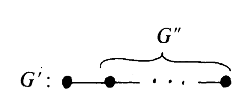


FIGURE 10

$$[[w_0, w_1, \cdots, w_n]] = w_0 - \cfrac{1}{w_1 - \cfrac{1}{w_2 - \cfrac{1}{\ddots - \cfrac{1}{w_{n-1} - \cfrac{1}{w_n}}}}}$$

for B/A (cf. Brieskorn [6, Section 2.4] for example).

We now prove Proposition (4.1). Let G be a perfect weighted graph of type $E_{p_1,\cdots,p_r}$ whose central vertex v_0 is joined to terminal vertices $v_1, \cdots, v_r$ of graphs $G_1, \cdots, G_r$ of types $A_{p_1}, \cdots, A_{p_r}$ respectively. Then by (2.14) the weight of v_0 satisfies

$$w_0 = \sum_{i=1}^{r} \frac{A_i}{B_i} + \frac{1}{\prod_{r=1}^{r} B_i},$$

where $B_i = |G_i|$ and $A_i = |G_i - \{v_i\}|$. Clearly $(A_i, B_i) = 1$, as is seen by clearing denominators. Also $A_i/B_i < 1$ since if all weights on G_i are 2 then $A_i/B_i = p_i/(p_i+1) < 1$, and A_i/B_i is a decreasing function of its weights by Lemma (3.3).

Conversely, let $w_0 = \sum_{i=1}^{r}(A_i/B_i) + 1/(\prod_{i=1}^{r} B_i)$ be a solution to this Diophantine equation, with $w_0 \geq 2$, $B_i \geq 2$, and $A_i < B_i$ for all i. By Lemma (4.2), for each i there exists a unique weighted chain G_i as in Figure 9 with $|G_i| = B_i$ and $|G_i'| = A_i$. Then the graph G of type $E_{p_1,\cdots,p_r}$ whose central vertex v_0 has weight w_0 and whose arms are $G_1, \cdots, G_r$, with v_0 joined to G_i at the vertex of $G_i - G_i'$, is the required perfect graph. This completes the proof.

In a similar fashion, given any special type of graph we can identify the Diophantine equation that must be solved to produce the perfect weights. In particular, we will do this for the graphs of the type shown in Figure 1 (with no restrictions on n) for which we do not know all minimal solutions.

Given a rational number A/B, and *Egyptian fraction expansion* for A/B is a decomposition of the form

$$\frac{A}{B} = \sum_{i=1}^{N} \frac{1}{w_i}$$

with the w_i's distinct positive integers. It is well known that every positive rational number can be so expressed, and in many different ways. Indeed, papers such as [10], [11], [2], and [3] either prove this fact in an especially nice way or give algorithms for producing such expansions with particular features, such as a minimal number of summands.

(4.3) LEMMA. *Let G be the weighted graph in Figure* 11 *with all weights* ≥ 2. *Then* $|G|=1$ *if and only if the following equation is satisfied*:

$$\frac{A}{B}=\sum_{i=1}^{k}\frac{1}{x_i}+\frac{1}{B\prod_{i=1}^{k}x_i} \tag{4.4}$$

where A/B has the "continued fraction" expansion

$$\left[\left[w_0, w_1, \cdots, w_n, w_{n+1}-\sum_{j=1}^{l}\frac{1}{y_j}\right]\right].$$

PROOF. Expand $|G|$ about the vertex w_0 and apply (2.14) to obtain

$$w_0=\sum_{i=1}^{k}\frac{1}{x_i}+\frac{|G_2|}{|G_1|}+\frac{|G|}{|G_1|\prod_{i=1}^{k}x_i},$$

where G_1 and G_2 are the graphs in Figure 12. As in the remark following Lemma (4.2), it is easy to check by induction that

$$\frac{|G_2|}{|G_1|}=\left[\left[w_1, \cdots, w_n, w_{n+1}-\sum_{j=1}^{l}\frac{1}{y_j}\right]\right].$$

The assertion now follows by putting $B=|G_1|$ and $A=w_0B-|G_2|$.

For $n\geq 3$ a complete set of minimal solutions to this equation is represented by graphs (20), (22), (23), (24), and (25) of Table I. For instance, the example (22) corresponds to the solution

$$[[2, 2, 2, 2, 2, 3]]=\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{179}+\frac{1}{24323}+\frac{1}{11\cdot 2\cdot 3\cdot 5\cdot 7\cdot 179\cdot 24323}.$$

To complete our analysis of perfectable graphs, then, we must find all minimal solutions for $n=0$, 1, and 2. Some solutions that may be minimal are represented by the seven

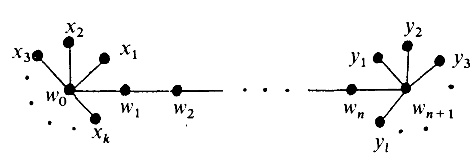


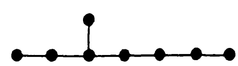
FIGURE 11

FIGURE 12

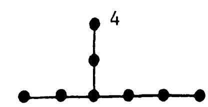
TABLE I. Minimal Perfectable Graphs.

Each graph is pictured with a set of perfect weights. (Unlabelled vertices have weight 2.) The choice of perfect weights is not unique in general.

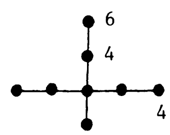
(1)



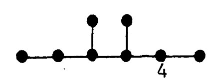
(2)



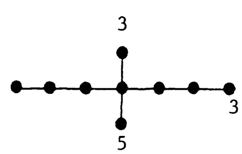
(3)



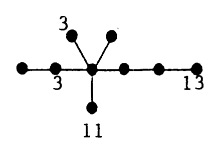
(4)



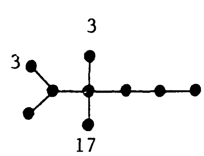
(5)



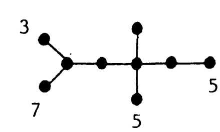
(6)



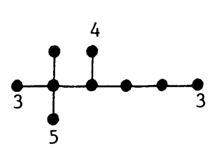
(7)



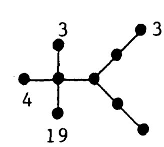
(8)



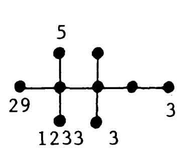
(9)



(10)



(11)



(12)

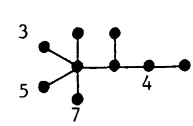
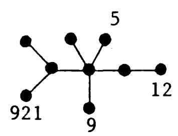
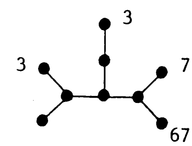
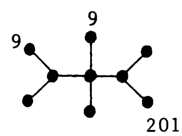
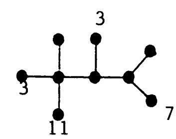


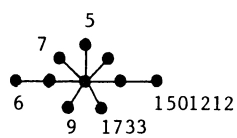
TABLE I. (Continued).

(13)


(14)


(15)


(16)


(17)


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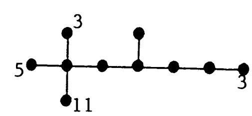
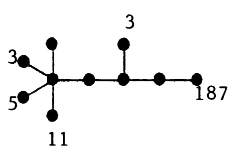
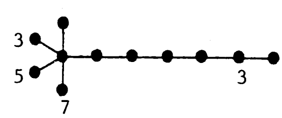
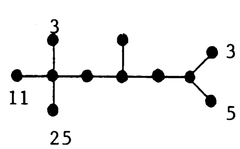
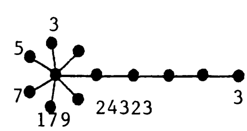
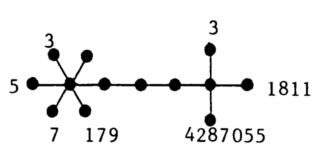


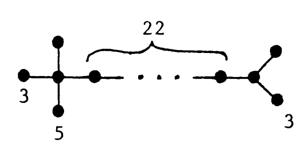
(19)


(20)


(21)


(22)


(23)


(24)


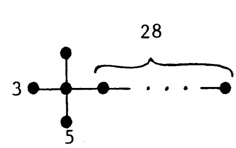
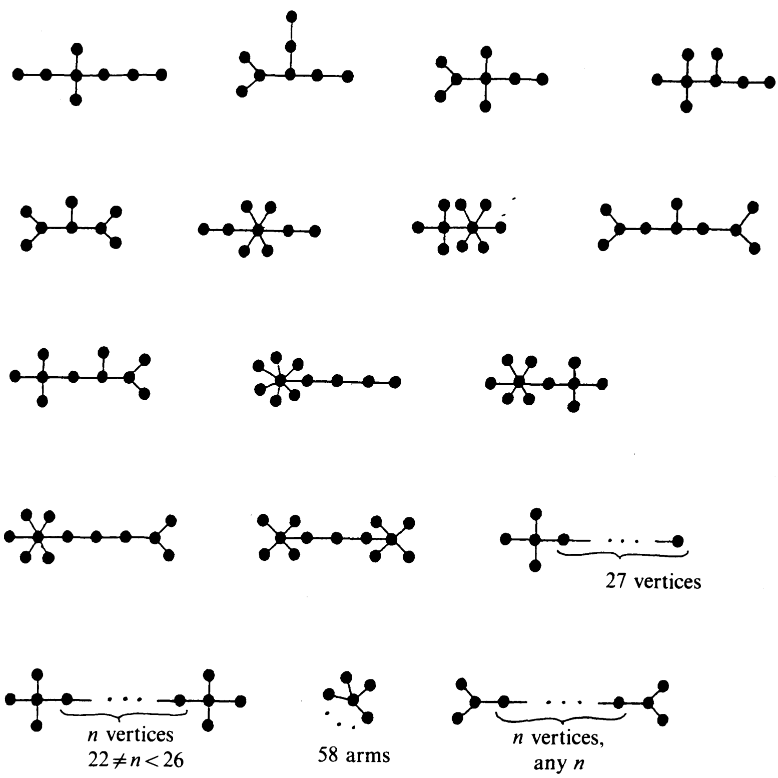
(25)


TABLE II. Examples of Non-perfectable Graphs.



weighted graphs of Table III.

To illustrate the role of Egyptian fractions in problems of this kind we will show how we determined the weights for the fourth example of Table III. (Examples 1, 2, 3, and 7 are similar.) Suppose that we wish to find perfect weights for the graph in Figure 13. By (4.4) we have

$$\frac{A}{B} = \sum_{i=1}^{N} \frac{1}{x_i} + \frac{1}{B\prod_{i=1}^{N} x_i}$$

with

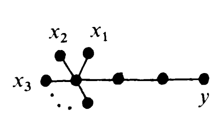
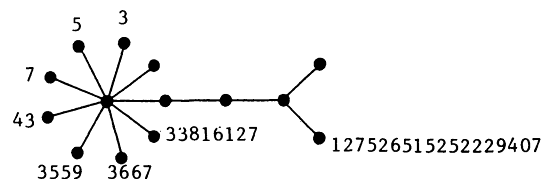


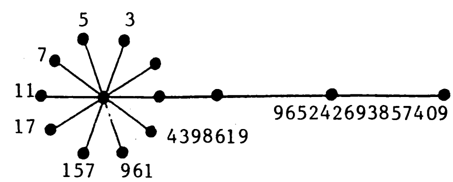
FIGURE 13

TABLE III. Some Special Perfect Graphs.
(Unlabelled vertices have weight 2).

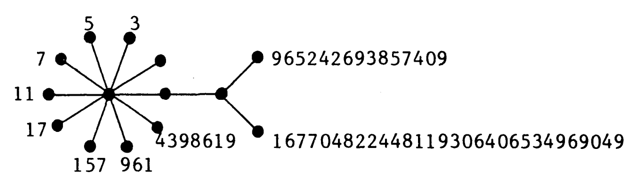
(1)



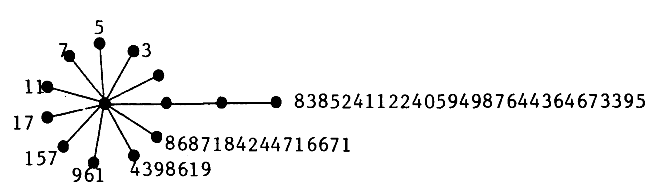
(2)



(3)



(4)



(5)

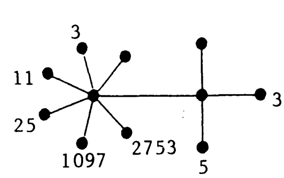
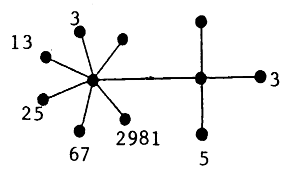
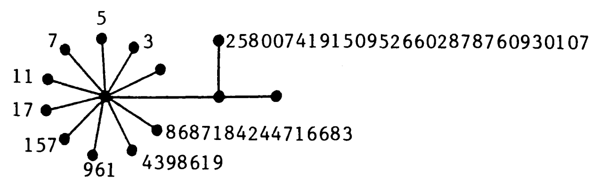


TABLE III. (Continued).

(6)



(7)



$$\frac{A}{B}=[[2,2,2,y]]=\frac{4y-3}{3y-2}=1+\frac{1}{3}-\frac{1}{3(3y-2)}.$$

Thus if one of the x_i, say x_N, is equal to 3, we have

$$1=\sum_{i=1}^{N-1}\frac{1}{x_i}+\frac{1}{3(3y-2)}+\frac{1}{3(3y-2)\prod_{i=1}^{N-1}x_i}.$$

That is, we seek a solution to the unit fraction equation

$$1=\sum_{i=1}^{N+1}\frac{1}{n_i} \tag{4.5}$$

with the additional constraints that $n_N=3(3y-2)\equiv 3 \bmod 9$ and that $n_{N+1}=\prod_{i=1}^{N}n_i$. For then the perfect weights are $x_i=n_i$ for $i=1,\cdots,N-1$, $x_N=3$, and $y=(n_N+6)/9$.

Now the problem (4.5) of expressing 1 as the sum of unit fractions has been much studied and has a substantial literature. (See for example the bibliographies in [9] and [12].) In particular, in [4] we considered the condition $n_{N+1}=\prod_{i=1}^{N}n_i$ in some detail, and, by computer search techniques, obtained a list of solutions for small N. $N=9$ is the smallest value of N for which we have a solution $(n_1,\cdots,n_{N+1})$ that satisfies the extra condition $n_N\equiv 3 \bmod 9$ (but $n_N>3$ so that $y=(n_N+6)/9\geq 2$). The solution is

(4.6) 2, 5, 7, 11, 17, 157, 961, 4398619, 8687184244716671,
754671701016535488879928206O5571,
5695293763151911320400374304363730155668749225304912374335630470.

These numbers give the perfect weights for the graph of Example 4.

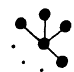


FIGURE 14

It is worth noting that from a purely number-theoretic viewpoint the most interesting of our unsolved cases is the star-shaped graph in Figure 14. For this graph, Equation (4.4) is simply

$$w_0 = \sum_{i=1}^{N} \frac{1}{x_i} + \frac{1}{\prod_{i=1}^{N} x_i} \tag{4.7}$$

(cf. (4.5)), which we wish to solve in integers w_0, x_i, all ≥ 2. In [4] we show that there is no solution for $N \leq 58$. No solution to (4.7) is known for any N.

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