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CONSTRUCTION OF TRIPLE COVERINGS OF A CERTAIN TYPE OF ALGEBRAIC SURFACES

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Introduction. In algebraic geometry, double coverings are very useful, and many results on them are well-known. On the other hand, there are not so many papers about triple coverings ([1], [5], [7]). In this paper, we construct a triple covering of a certain type of an algebraic surface by using a different method from Miranda's in [5], and consider its application. For general references on triple coverings of algebraic surfaces, see Miranda [5] or Tokunaga [7].

Let Σ be a nonsingular algebraic surface and let A_0, A_∞, B_0, B_∞ be smooth divisors on Σ satisfying the following conditions:

(i) $A_0 \sim A_\infty$ and $B_0 \sim B_\infty$.

(ii) The divisor $A_0 + A_\infty + B_0 + B_\infty$ has only simple normal crossings as its singularities.

By a_0, a_∞, b_0 and b_∞, we denote the defining equations for A_0, A_∞, B_0 and B_∞, respectively.

(iii) For suitable $\alpha, \beta \in \boldsymbol{C}$, the divisor defined by the equation

$$\alpha^3 a_0^3 b_\infty^2 + \beta^2 a_\infty^3 b_0^2 = 0$$

is reduced and the divisor defined by the equation

$$a_\infty(\alpha^3 a_0^3 b_\infty^2 + \beta^2 a_\infty^3 b_0^2) = 0$$

has singularities at most at $A_0 \cap A_\infty$, $B_0 \cap B_\infty$, $A_0 \cap B_0$ and $A_\infty \cap B_\infty$.

Under the above conditions, we consider a cubic extension of $\boldsymbol{C}(\Sigma)$ defined by the equation

$$X^3 + 3\alpha\left(\frac{a_0}{a_\infty}\right)X + 2\beta\left(\frac{b_0}{b_\infty}\right) = 0 .$$

Let θ be a solution of the above equation, and let S' be a $\boldsymbol{C}(\Sigma)(\theta)$-normalization of Σ. Then S' is a normal finite triple covering of Σ. By $p: S' \to \Sigma$ we denote its covering map. We now state our results:

THEOREM (A). *Singularities of S' are rational triple points of the following form:*

(1) *The points lying over $A_0 \cap A_\infty$ and $B_0 \cap B_\infty$. The singular points whose minimal*

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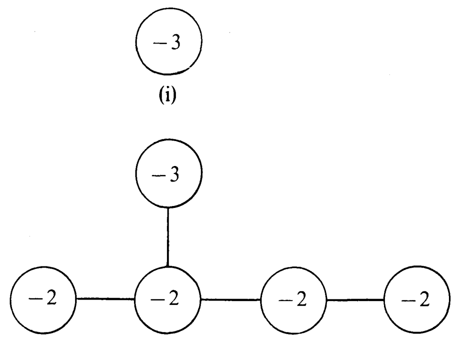

(i)

(ii)

FIGURE 1

resolutions have the following configuration of the exceptional set as in Figure 1, (i).

(2) *The points lying over* $A_\infty \cap B_\infty$. *The singular points whose minimal resolutions have the following configuration of the exceptional set as in Figure* 1, (ii).

THEOREM (B). *The branch divisor of* p *is a divisor on* Σ *defined by the equations*

$$a_\infty(\alpha^3 a_0^3 b_\infty^2 + \beta^2 a_\infty^3 b_0^2) = 0\,, \quad \textit{and} \quad b_\infty = 0\,,$$

Moreover, $p^{-1}(x)$ *for* $x \in A_\infty \cup (\alpha^3 a_0^3 b_\infty^2 + \beta^2 a_\infty^3 b_0^2)$ *consists of two points, while* $p^{-1}(y)$ *for* $y \in B_\infty$ *consists of one point.*

By using the above Theorems (A) and (B), we can compute c_1^2 and c_2 for a smooth model of S', and we obtain the following result as an application of Theorems (A) and (B).

THEOREM 5.2. *There exists a minimal surface of general type* S *with invariants*

$$c_1^2(S) = 4n - 8\,, \qquad c_2(S) = 20n - 4\,, \qquad p_g(S) = 2n - 2\,,$$

which has the structure of a non-Galois trigonal fiber space over $\boldsymbol{P}^1$.

As for the definition of a trigonal fiber space, see Defintion 5.1.

Note that all surfaces which have the numerical invariants as above satisfy Noether's equality $c_1^2 = 2p_g - 4$.

Section 1 starts with a summary on triple coverings of algebraic surfaces without proof. In Section 2, we consider the ramification in codimension one for p, and prove Theorem (B). In Section 3, we examine the singularities of S' and its resolutions, and prove Theorem (A). In Section 4, we give easy examples and compute their c_1^2 and c_2. In Section 5, we define trigonal fiber spaces and prove Theorem 5.2.

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NOTATION AND CONVENTIONS. In this paper, the ground field is always the complex number field $\boldsymbol{C}$.

$g(C):=$ the genus of a curve C.

$q(X):=\dim_{\mathbf{C}} H^1(X, \mathcal{O}_X)$.

$p_g(X):=\dim_{\mathbf{C}} H^n(X, \mathcal{O}_X)$ with $n=\dim X$.

$\mathbf{C}(X)$: the rational function field of X.

$\operatorname{Sing}(X)$: the singular locus of X.

$c_i(X)$: the i-th Chern class of X.

Let $f\colon X \to Y$ be a morphism from a normal variety to another normal variety Y. f is said to be ramified at $x \in X$ if f is not étale at x. f is said to be branched over $y \in Y$ if f is not étale over y. Hence the ramification divisor is a divisor on X, while the branch divisor is a divisor on Y.

For a divisor D on Y, $f^{-1}(D)$ denotes the set-theoretic inverse image of D, while $f^*(D)$ denotes the ordinary pullback.

Let D_1, D_2 be divisors.

$D_1 \sim D_2$: linear equivalence of divisors.

$D_1 \sim D_2$: numerical equivalence of divisors.

A rational curve with self-intersection number $-n$ $(n>0)$ is called a $(-n)$-curve and is represented by a circle with $-n$ inside. A possibly irrational curve with self-intersection number $-n$ is represented by a line with $-n$ beside it.

1. A triple covering of an algebraic surface. Let Σ be an algebraic surface and $\mathbf{C}(\Sigma)$ be its rational function field. Let K be an algebraic extension of $\mathbf{C}(\Sigma)$ determined by an equation

$$X^3+3aX+2b=0$$

with $a, b \in \mathbf{C}(\Sigma)$. Let S be a K-normalization of Σ so that $\mathbf{C}(S)=K$. (For the definition of K-normalization and its property, see Iitaka [3].) Assume that K is not a cyclic extension. Then as in [7] there exists a double covering $\beta_1\colon D(S/\Sigma) \to \Sigma$ of Σ associated with the triple covering $p\colon S \to \Sigma$. We call $D(S/\Sigma)$ the *discriminant surface of* $p\colon S \to \Sigma$. Moreover, there exists a cyclic triple covering $\beta_2\colon \hat{S} \to D(S/\Sigma)$ of $D(S/\Sigma)$ associated with the triple covering $p\colon S \to \Sigma$. We call $\hat{S}$ the *minimal splitting surface of* $p\colon S \to \Sigma$. For details, see [7]. We obtain a diagram

$$\begin{tikzcd}
 & \hat{S} \arrow[dl, "\alpha"'] \arrow[dd, "p_1"] \arrow[dr, "\beta_2"] & \\
S \arrow[dr, "p"'] & & D(S/\Sigma) \arrow[dl, "\beta_1"] \\
 & \Sigma &
\end{tikzcd}$$

where $p_1\colon \hat{S} \to \Sigma$ is the Galois covering with Galois group of $\mathfrak{S}_3$.

REMARK. If $p\colon S \to \Sigma$ is a cyclic triple covering, then $D(S/\Sigma)=\Sigma$ and $\hat{S}=S$.

In the above notation, the following hold:

PROPOSITION 1.1 (cf. [7]). *Let $p: S \to \Sigma$ and $p_1: \hat{S} \to \Sigma$ be as above. By construction, $\boldsymbol{C}(\hat{S})$ is a Galois extension of $\boldsymbol{C}(\Sigma)$ with the Galois group* $\mathrm{Gal}(\boldsymbol{C}(\hat{S})/\boldsymbol{C}(\Sigma)) \simeq \mathfrak{S}_3$ *or $\boldsymbol{Z}/3\boldsymbol{Z}$. The birational maps of $\hat{S}$ into itself over Σ induced by the elements of* $\mathrm{Gal}(\boldsymbol{C}(\hat{S})/\boldsymbol{C}(\Sigma))$ *are automorphisms of $\hat{S}$.*

LEMMA 1.2 (cf. [7]). *Let $p: S \to \Sigma$, and $p_1: \hat{S} \to \Sigma$ be as above. Assume that Σ is smooth. Then by the purity of branch locus (see Zariski [6]), the branch loci of p and p_1 are divisors on Y. We denote their support by $\Delta(S/\Sigma)$ and $\Delta(\hat{S}/\Sigma)$, respectively. Then*

$$\Delta(S/\Sigma) = \Delta(\hat{S}/\Sigma)\,.$$

REMARK. There is another approach to triple coverings due to Miranda [5]. He studied triple coverings by means of rank two vector bundles called Tschirnhausen modules.

2. The codimension one ramification of a triple covering.

In this section, we assume that Σ is always smooth. By the purity of branch locus, the branch locus $\Delta(S'/\Sigma)$ of the triple covering $p: S' \to \Sigma$ is a divisor on Σ.

LEMMA 2.1. *Let $p: S' \to \Sigma$ be a normal finite triple covering over a smooth surface Σ. Assume that $\boldsymbol{C}(S') = \boldsymbol{C}(\Sigma)(\theta)$, where θ satisfies an equation*

$$X^3 + 3aX + 2b = 0\,, \qquad \textit{with} \quad a = \frac{a_0}{a_\infty}\,, \quad b = \frac{b_0}{b_\infty} \in \boldsymbol{C}(\Sigma)\,.$$

Then

$$\Delta(S'/\Sigma) \subset (a_\infty = 0) \cup (b_\infty = 0) \cup (a_0^3 b_\infty^2 + a_\infty^3 b_0^2 = 0)\,.$$

PROOF. Consider the surface

$$S'' := \{(x, t) \in \Sigma \times \boldsymbol{P}^1 \mid t^3 + 3a(x)t + 2b(x) = 0\}\,,$$

where t is an inhomogeneous coordinate of $\boldsymbol{P}^1$. Let S''' be a normalization of S'''. Then we get a commutative diagram

$$\begin{tikzcd}
 & S''' \arrow[dl, "s"'] \arrow[dd, "q_1 \circ \eta"] \\
S' \arrow[dr, "p"'] & \\
 & \Sigma
\end{tikzcd}$$

where s is the Stein factorization, n is the normalization $S''' \to S''$ and q_1 is the projection: $\Sigma \times \boldsymbol{P}^1 \to \Sigma$.

Let x be a point which is not contained in the closed subset

$$(a_\infty=0)\cup(b_\infty=0)\cup(a_0^3b_\infty^2+a_\infty^3b_0^2=0)\,.$$

Then $q_{1|S''}: S''\to\Sigma$ is étale over x. Therefore by the above diagram, p is étale over x.
q.e.d.

By Lemmas 1.2 and 2.1, to study the ramification of $p: S'\to\Sigma$, it is enough to investigate the ramification of $p_1: \hat{S}'\to\Sigma$. Hence we consider ramifications of two cyclic coverings

$$\beta_1: D(S'/\Sigma)\to\Sigma \quad\text{and}\quad \beta_2: \hat{S}'\to D(S'/\Sigma)\,.$$

Assume that a finite triple covering $p: S'\to\Sigma$ is obtained by an algebraic extension associate with

$$X^3+3\alpha\frac{a_0}{a_\infty}X+2\beta\frac{b_0}{b_\infty}=0$$

which satisfies the conditions (i), (ii) and (iii) in the introduction.

Put

$$R=\frac{\alpha^3a_0^3b_\infty^2+\beta^2a_\infty^3b_0^2}{a_\infty^3b_\infty^2}\,.$$

Then $C(D(S'/\Sigma))=C(\Sigma)(\sqrt{R})$. Therefore, the ramification locus of $\beta_1: D(S'/\Sigma)\to\Sigma$ is the divisor

$$B=A_\infty\cup(\alpha^3a_0^3b_\infty^2+\beta^2a_\infty^3b_0^2=0)\,.$$

We next consider the branch locus of β_2 along divisors. By Lemma 2.1, and the above argument, if β_2 is ramified over some divisors, then it must be $\beta_1^*(B_\infty)$. (Note that β_2 cannot be ramified over the ramification divisor of β_1.)

CLAIM 2.2. *β_2 is ramified over $\beta_1^*(B_\infty)$.*

PROOF. Since the problem is local, we restrict ourselves to an affine neighborhood where $\beta_1^*(B_\infty)$ is smooth. Moreover, it is enough to consider our problem over an affine open subset U in Σ which satisfies the following:

(i) $U=\mathrm{Spec}(C[x,y])$.

(ii) The defining equation of B_∞ in U is $x=0$.

(iii) The equation $X^3+(3\alpha a_0/a_\infty)X+(2\beta b_0/b_\infty)=0$ is represented in U as

$$X^3+3X+2/x=0\,.$$

Under the above assumption, we obtain $R=(x^2+1)/x^2$. Hence the double covering $\beta_1^{-1}(U)$ is of the form

$$\beta_1^{-1}(U)=\mathrm{Spec}(C[x,y,\zeta]/(\zeta^2-x^2-1))\,.$$

We investigate the ramification of β_2 over the double covering $\beta_1^{-1}(U)$. It is well-known that

$$C(\hat{S}')=C(\Sigma)\left(\sqrt[3]{-\beta_1^*\left(\frac{b_0}{b_\infty}\right)+\sqrt{R}}\right) \qquad \text{(Cardano's formula)}\,.$$

Therefore, the cyclic triple covering $\beta_2^{-1}(\beta_1^{-1}(U))$ is obtained as

$$\beta_2^{-1}(\beta_1^{-1}(U))=\mathrm{Spec}(A[\eta]/(\eta^3-f))\,,$$

where

$$A=C[x, y, \zeta]/(\zeta^2-x^2-1)\,.$$

Note that

$$C(\beta_2^{-1}(\beta_1^{-1}(U)))=C(\beta_1^{-1}(U))\left(\sqrt[3]{\frac{\bar{\zeta}-1}{x}}\right)=C(\beta_1^{-1}(U))\left(\frac{\sqrt[3]{(\bar{\zeta}+1)(\bar{\zeta}-1)^2}}{x}\right),$$

where the bar means the equivalence class in A. Then f can be written as

$$f=(\bar{\zeta}+1)(\bar{\zeta}-1)^2\,.$$

Therefore, β_2 is ramified over the divisor defined by $\bar{\zeta}=\pm 1$, that is, $x=0$. q.e.d

By the above argument, it is easy to show that the ramification index of $p^{-1}(B)$ is equal to 2, while the ramification index of $p^{-1}(B_\infty)$ is equal to 3.

We summarize what we obtained in this section.

THEOREM (B). *Let $p: S'\to\Sigma$ be a finite normal triple covering. Assume that Σ is smooth and the rational function field $C(S')$ is an algebraic extension of $C(\Sigma)$ satisfying the conditions in the introduction. Then the branch divisor of p is a divisor on Σ defined by the local equations*

$$a_\infty(\alpha^3 a_0^3 b_\infty^2+\beta^2 a_\infty^3 b_0^2)=0 \quad \textit{and} \quad b_\infty=0\,.$$

Moreover, $p^{-1}(x)$ consists of two points for a general $x\in A_\infty\cup(\alpha^3 a_0^3 b_\infty^2+\beta^2 a_\infty^3 b_0^2=0)$, while $p^{-1}(y)$ consists of one point for a general $y\in B_\infty$.

3. Singular points of S' and their resolutions. In this section, we investigate the singularities of S' and their resolutions. To this aim, we examine the singularities of $D(S'/\Sigma)$ and $\hat{S}'$ and their resolutions.

(I) Singularities of $D(S'/\Sigma)$. First of all, we investigate the singularities of $D(S/\Sigma)$. Since $D(S'/\Sigma)$ is a normal finite double covering, the singularities of $D(S'/\Sigma)$ lie over those of the branch locus. In Section 2, we have seen that the branch locus of β_1 to be $A_\infty\cup(\alpha^3 a_0^3 b_\infty^2+\beta^2 a_\infty^3 b_0^2=0)$. Hence, by our assumption, its singularities are

$$A_0 \cap A_\infty : x(y^3 + x^3) = 0$$

$$A_0 \cap B_0 : x^3 + y^2 = 0$$

$$A_\infty \cap B_\infty : x(y^2 + x^3) = 0$$

$$B_0 \cap B_\infty : x^2 + y^2 = 0 .$$

Note that we always take a suitable local coordinate system. Therefore the singularities of $D(S'/\Sigma)$ are

simple elliptic singularities over $A_0 \cap A_\infty$,

A_2-singularities over $A_0 \cap B_0$,

D_5-singularities over $A_\infty \cap B_\infty$,

A_1-singularities over $B_0 \cap B_\infty$.

We now investigate the singularities of $\hat{S}'$. For this purpose, we consider a smooth model $\hat{S}$ of $\hat{S}'$. From now on, we will use the following notation:

$$\begin{tikzcd} \hat{S}' \arrow[d, "\beta_2"'] & & \hat{S} \arrow[ll] \arrow[d, "\tilde{\beta}_2"] \\ D(S'/\Sigma) \arrow[d, "\beta_1"'] & & D(S'/\Sigma)^{\sim} \arrow[ll, "\mu"'] \arrow[d, "\tilde{\beta}_1"] \\ \Sigma & & \tilde{\Sigma} \arrow[ll, "\pi"'] \end{tikzcd}$$

where π is a succession of blowing-ups, μ: $D(S/\Sigma)^{\sim} \rightarrow D(S/\Sigma)$ is the minimal resolution of $D(S'/\Sigma)$, $\tilde{\beta}_2$ is the not necessarily finite morphism induced by β_2, $\tilde{\beta}_1$ is the induced double covering, and $\hat{S}$ is a smooth model of $\hat{S}'$.

(II) Analysis of the morphism $\tilde{\beta}_2$, and a resolution of the singularities of $\hat{S}'$. In a neighborhood of a smooth point of $D(S'/\Sigma)$, μ is an isomorphism. Hence $\tilde{\beta}_2$ is the same as β_2. Therefore, it is sufficient to examine $\tilde{\beta}_2$ in a neighborhood of each exceptional set. We study $\tilde{\beta}_2$ for each type of singularities of $D(S'/\Sigma)$.

Case (i) A simple elliptic singularity.

Let $\mathfrak{p}_1$ be a point of $A_0 \cap A_\infty$. It is enough to consider our problem over an affine open subset U_1 in Σ such that

$$U_1 = \mathrm{Spec}(\boldsymbol{C}[x, y])$$

and that $x=0$, $y=0$ are the defining equations for A_∞ and A_0, respectively. Moreover, the equation $X^3 + (3a_0/a_\infty)X + (2b_0/b_\infty) = 0$ is represented in U as $X^3 + (3y/x)X + 2 = 0$.

Let $\pi_{\mathfrak{p}_1}$: $\tilde{U}_1 \rightarrow U_1$ be the blowing-up at $\mathfrak{p}_1$, and choose an affine open cover of $\tilde{U}_1$ defined by

$$\tilde{U}_1 = V_1 \cup V_2 \quad \text{with } V_1 = \mathrm{Spec}(\boldsymbol{C}[x, s]) , \ V_2 = \mathrm{Spec}(\boldsymbol{C}[y, t]) , \ y = xs , \ x = yt .$$

Under the above assumption, on U_1, V_1 and V_2, we obtain

$$R=\frac{x^3+y^3}{x^3}=\frac{x(x^3+y^3)}{x^4} \quad \text{on} \quad U_1$$

$$=\begin{cases} \dfrac{x^4(s^3+1)}{x^4} & \text{on} \quad V_1 \\[2ex] \dfrac{y^4(t+t^4)}{y^4t^4} & \text{on} \quad V_2 \,. \end{cases}$$

Therefore, the double coverings $\tilde{\beta}_1^{-1}(V_1)$ and $\tilde{\beta}_1^{-1}(V_2)$ are

$$\begin{cases} \tilde{\beta}_1^{-1}(V_1)=\mathrm{Spec}(\boldsymbol{C}[x, s, \zeta_1]/(\zeta_1^2-s^3-1)) \\ \tilde{\beta}_1^{-1}(V_2)=\mathrm{Spec}(\boldsymbol{C}[y, t, \zeta_2]/(\zeta_2^2-t^4-t)) \,. \end{cases}$$

Moreover,

$$-\tilde{\beta}_1^*(1)+\sqrt{\frac{x(x^3+y^3)}{x^4}}=\begin{cases} -1+\tilde{\zeta}_1 \\ -1+\dfrac{\tilde{\zeta}_2}{t^2} \end{cases} \qquad \cdots\cdots(*_1)$$

Let E denote the exceptional elliptic curve of the above simple elliptic singularity. By the results in Section 2 and the fact that $\tilde{\beta}_2$ is not ramified over the ramification divisor of $\tilde{\beta}_1$, we see that if $\tilde{\beta}_2$ is not ramified along some divisor over $\tilde{\beta}_1^{-1}(V_1)$ and $\tilde{\beta}_1^{-1}(V_2)$, then the divisor is the exceptional divisor E. But, from $(*_1)$, it is easy to see that $\tilde{\beta}_2$ is not ramified over E. Therefore, $\tilde{\beta}_2$ is étale over E. Hence, as the inverse image of E, there are two possibilities:

(i) $\tilde{\beta}_2^{-1}(E)$ is irreducible,

(ii) $\tilde{\beta}_2^{-1}(E)$ has three irreducible components which are isomorphic to each other.

CLAIM 3.1. *$\tilde{\beta}_2^{-1}(E)$ is irreducible.*

PROOF OF CLAIM 3.1. Since our concern is a cyclic triple covering over E, it is enough to consider the restricted morphisms $\tilde{\beta}_2|_{\tilde{\beta}_2^{-1}(E)}$ and $\tilde{\beta}_1|_E$. By our construction, the rational function field of the elliptic curve E is $\boldsymbol{C}(E)=\boldsymbol{C}(\boldsymbol{P}^1)(x, y)$, where $y^2=x^3+1$. By the theory of elliptic functions, we may assume that $x=\wp$, $y=\wp'$, where $\wp$ is the Weierstrass $\wp$-function of E, while $\wp'$ is the differential of $\wp$. Moreover, $\boldsymbol{C}(\tilde{\beta}_2^{-1}(E))$ is equal to

$$\boldsymbol{C}(E)(\sqrt[3]{\wp'+1}) \,.$$

From a general theory of cyclic coverings (see [3]), if $\tilde{\beta}_2^{-1}(E)$ is reducible, then $\wp'+1$ must have the form

$$\wp'+1=f^3 \,, \qquad \text{for} \quad f\in\boldsymbol{C}(E) \,,$$

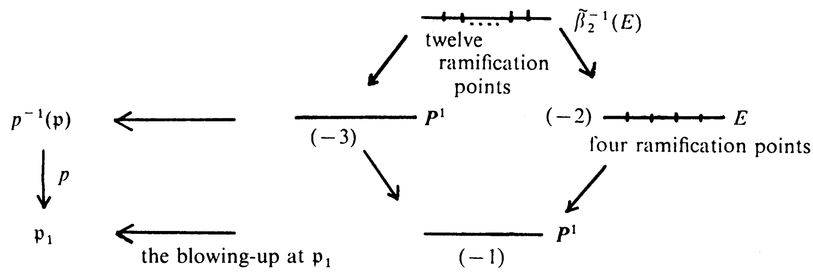


FIGURE 2

which is impossible. Therefore, $\tilde{\beta}_2^{-1}(E)$ is irreducible. q.e.d.

A smooth model S of S' is obtained as a quotient surface $\hat{S}/\iota$ where ι is an appropriate involution induced by an element of the Galois group. It is clear that ι has a fixed point on $\tilde{\beta}_2^{-1}(E)$. There are four fixed points. Hence a resolution of the singular point $p^{-1}(\mathfrak{p}_1)\in p^{-1}(A_0 \cap A_\infty)$ is

$$\tilde{\beta}_2^{-1}(\tilde{\beta}_1^{-1}(V_1)) \cup \tilde{\beta}_2^{-1}(\tilde{\beta}_1^{-1}(V_2))/\langle \iota \rangle$$

and its exceptional set is $\tilde{\beta}(E)/\langle \iota \rangle$, that is, a rational curve whose self-intersection number is -3. Figure 2 explains the above argument.

Case (ii) A_2-singularity.

Let $\mathfrak{p}_2$ be a point of $A_0 \cap B_0$. In the same way as in Case (i), we consider our problem over an affine open set

$$U_2 = \mathrm{Spec}(\boldsymbol{C}[x, y])$$

such that $x=0$, $y=0$ are the defining equations for A_0 and A_∞, respectively. Moreover, the equation $X^3+(3a_0/a_\infty)X+(2b_0/b_\infty)=0$ is represented in U as $X^3+3xX+2y=0$.

Let $\pi_{\mathfrak{p}_2}: \tilde{U}_2 \to U_2$ be the blowing-up at $\mathfrak{p}_2$, and choose an affine open cover of $\tilde{U}_2$ defined by

$$\tilde{U}_2 = V_1 \cup V_2 \quad \text{with} \quad V_1=\mathrm{Spec}(\boldsymbol{C}[x, s])\,,\ V_2=\mathrm{Spec}(\boldsymbol{C}[y, t])\,,\ y=xs\,,\ x=yt\,.$$

In the above notation, we obtain

$$\begin{aligned} R &= y^2+x^3 && \text{on} \quad U_2 \\ &= \begin{cases} x^2(x+s^2) \\ y^2(yt^3+1) \end{cases} && \begin{matrix} \text{on} \quad V_1 \\ \text{on} \quad V_2\,. \end{matrix} \end{aligned}$$

Therefore, the double coverings $\tilde{\beta}_1^{-1}(V_1)$ and $\tilde{\beta}_1^{-1}(V_2)$ are

$$\begin{cases} \tilde{\beta}_1^{-1}(V_1) = \mathrm{Spec}(\boldsymbol{C}[x, s, \zeta_1]/(\zeta_1^2 - x - s^2)) \\ \tilde{\beta}_1^{-1}(V_2) = \mathrm{Spec}(\boldsymbol{C}[y, t, \zeta_2]/(\zeta_2^2 - yt^3 - 1))\,. \end{cases}$$

Moreover,

$$-\tilde{\beta}_1^*(y)+\sqrt{y^2+x^3}=\begin{cases} -(\bar{s}+\bar{\zeta}_1)(\bar{s}-\bar{\zeta}_1)^2 & \text{on } \tilde{\beta}_1^{-1}(V_1) \\ \dfrac{1}{\bar{t}^3}(\bar{\zeta}_2+1)(\bar{\zeta}_2-1)^2 & \text{on } \tilde{\beta}_1^{-1}(V_2) \end{cases} \qquad \cdots(*_2)$$

Clearly,

$$\begin{cases} \bar{s}+\bar{\zeta}_1=\dfrac{\bar{\zeta}_2+1}{\bar{t}} \\ -\bar{s}+\bar{\zeta}_1=\dfrac{\bar{\zeta}_2-1}{\bar{t}}. \end{cases}$$

Therefore, $\tilde{\beta}_2$ is ramified over the divisors defined by the equations

$$\bar{\zeta}_1+\bar{s}=0 \quad \text{and} \quad \bar{\zeta}_1-\bar{s}=0\,.$$

Note that these divisors are the inverse images of the exceptional curve of the blowing-up at $\mathfrak{p}_2$.

By the above argument, in a neighborhood $p^{-1}(\mathfrak{p}_2)$, the surface $\hat{S}$ can be regarded as a resolution of the singularity defined by an equation

$$z^3-uv^2=0\,,$$

which is a rational triple point. The configurations of exceptional sets on $\pi^{-1}(U)$, $\tilde{\beta}_1^{-1}(V_1 \cup V_2)$ and $\tilde{\beta}_1^{-1}(V_1 \cup V_2)$, respectively, are as in Figure 3. Note that the above resolution is not minimal. By contracting (-1)-curves, we obtain the minimal resolution. S' turns out to be smooth over $A_0 \cap B_0$, and the structure of the triple covering is the same as that in [7, §2, Example 3].

Case (iii) D_5-singularity.

Let $\mathfrak{p}_3$ be a point of $A_\infty \cap B_\infty$. In the same way as in the preceding two cases, it is enough to consider our problem over an affine open set

$$U_3=\operatorname{Spec}(\boldsymbol{C}[x, y])$$

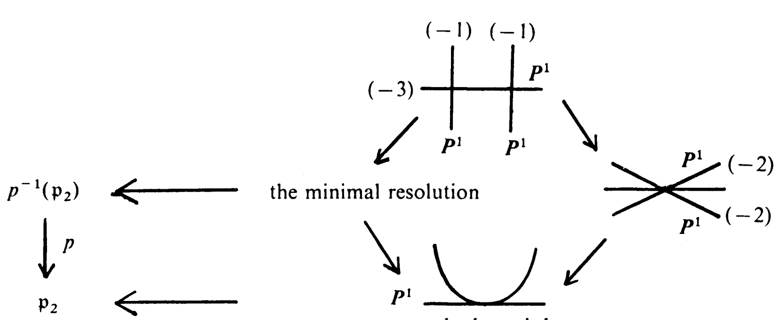


FIGURE 3

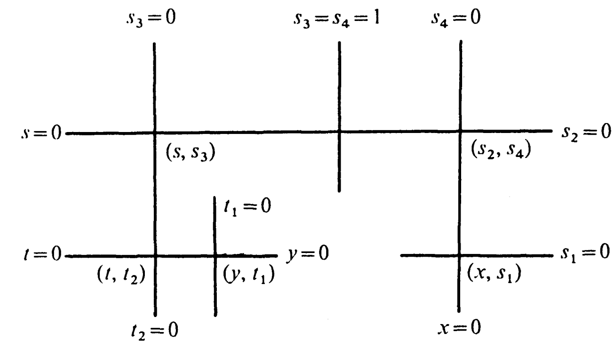


FIGURE 4

such that $x=0$, $y=0$ are the defining equations for A_∞ and B_∞, respectively. Moreover, the equation $X^3+(3a_0/a_\infty)X+(2b_0/b_\infty)=0$ is represented in U as $X^3+(3/x)X+(2/y)=0$.

Let $\pi_{p_3}\colon \tilde{U}_3 \to U_3$ be a succession of blowing-ups such that the branch locus of $\tilde{\beta}_1$ is a smooth divisor on U_3. We introduce an affine open cover

$$\tilde{U}_3 = V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5$$

with $V_1=\mathrm{Spec}(\boldsymbol{C}[x, s_1])$, $V_2=\mathrm{Spec}(\boldsymbol{C}[s_2, s_4])$, $V_3=\mathrm{Spec}(\boldsymbol{C}[s, s_3])$, $V_4=\mathrm{Spec}(\boldsymbol{C}[t, t_2])$, $V_5=\mathrm{Spec}(\boldsymbol{C}[y, t_1])$, where

$$\begin{cases} x = ss_2 = s^2 s_3 = s_2^2 s_4 \\ y = xs = x^2 s = s^2 s_2 = s^3 s_3 = s_2^3 s_4^2 , \end{cases}$$

$$\begin{cases} x = yt = y^2 t_1 = t^2 t_2 \\ y = tt_2 . \end{cases}$$

Figure 4 describes the configuration of the exceptional curves, coordinates and the branch locus on $\tilde{U}_3$. In this notation, we get the following forms of R on each affine open set V_i $(i=1, 2, 3, 4, 5)$:

$$R=\frac{x(y^2+x^3)}{x^4y^2}=\frac{xs_1^2+1}{x^4s_1^2}=\frac{s_4+1}{s_2^6s_4^4}=\frac{s_3(s_3+1)}{s^6s_3^4}=\frac{t_2(1+t^4t_2)}{t_2^4t^6}=\frac{t_1(1+y^4t_1^3)}{t_1^4y^6} .$$

Therefore, the double coverings $\tilde{\beta}_1^{-1}(V_i)$ $(i=1, 2, 3, 4, 5)$ are

$$\tilde{\beta}_1^{-1}(V_1)=\mathrm{Spec}(\boldsymbol{C}[x, s_1, \zeta_1]/(\zeta_1^2 - xs_1^2 - 1))$$

$$\tilde{\beta}_1^{-1}(V_2)=\mathrm{Spec}(\boldsymbol{C}[s_2, s_4, \zeta_2]/(\zeta_2^2 - s_4 - 1))$$

$$\tilde{\beta}_1^{-1}(V_3)=\mathrm{Spec}(\boldsymbol{C}[s, s_3, \zeta_3]/(\zeta_3^2 - s_3(s_3+1)))$$

$$\tilde{\beta}_1^{-1}(V_4)=\mathrm{Spec}(\boldsymbol{C}[t, t_2, \zeta_4]/(\zeta_4^2 - t_2(1+t^4t_2)))$$

$$\tilde{\beta}_1^{-1}(V_5)=\mathrm{Spec}(\boldsymbol{C}[y, t_1, \zeta_5]/(\zeta_5^2 - t_1(1+y^4t_1^3))) .$$

We obtain the following on each open set $\tilde{\beta}_1^{-1}(V_i)$, $(i=1, 2, 3, 4, 5)$:

$$-\tilde{\beta}_1^*\left(\frac{1}{y}\right)+\frac{\sqrt{x(y^2+x^3)}}{x^2y}=\frac{\bar{\zeta}_1-1}{\bar{x}^2\bar{s}_1}=\frac{\bar{\zeta}_2-1}{\bar{s}_2^3\bar{s}_4^2}=\frac{\bar{\zeta}_3-\bar{s}_3}{\bar{s}^3\bar{s}_3^2}=\frac{\bar{\zeta}_4-\bar{t}^2\bar{t}_2}{\bar{t}^3\bar{t}_2^2}=\frac{\bar{\zeta}_5-\bar{t}_1^2\bar{y}^2}{\bar{t}_1^2\bar{y}^3}.$$

Let us analyze $\tilde{\beta}_2$ on each affine open set $\tilde{\beta}_1^{-1}(V_i)$ $(i=1, 2, 3, 4, 5)$.

On $\tilde{\beta}_1^{-1}(V_1)$, the action of the Galois group is

$$\bar{\zeta}_1 \mapsto -\bar{\zeta}_1 .$$

Moreover, by the relation $\bar{\zeta}_1^2-1=\bar{x}^2\bar{s}_1$ we see that the branch locus of $\tilde{\beta}_2$ contains the divisors defined by the equations $\bar{x}=0$, or $\bar{s}_1=0$. Hence $\hat{S}$ is obtained as a resolution of the two singularities over $\bar{x}=\bar{s}_1=0$.

On $\tilde{\beta}_1^{-1}(V_2)$, the action of the Galois group is

$$\bar{\zeta}_2 \mapsto -\bar{\zeta}_2 .$$

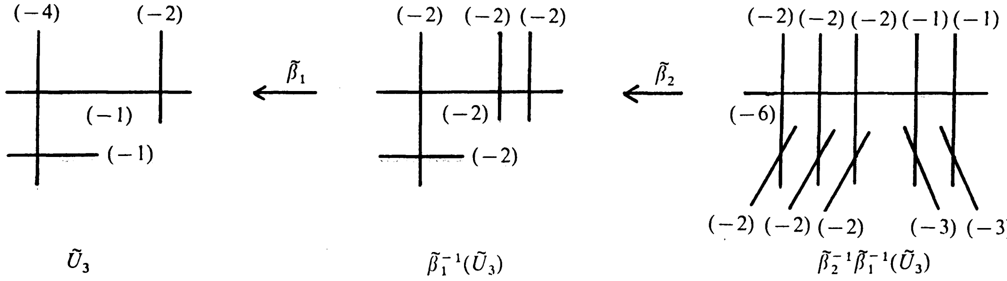
Moreover, by the relation $\bar{\zeta}_2^2-1=\bar{s}_4$ we see that the branch locus of $\tilde{\beta}_2$ is a divisor defined by the equation $\bar{s}_4=0$.

On $\tilde{\beta}_1^{-1}(V_2)$, the action of the Galois group is

$$\bar{\zeta}_3 \mapsto -\bar{\zeta}_3 .$$

Moreover, by the relation $\bar{\zeta}_3^2-\bar{s}_3^2=\bar{s}_3^3$ if $\tilde{\beta}_2$ is ramified over some divisors on $\tilde{\beta}_1^{-1}(V_3)$, then it is the divisor defined by the equation $\bar{s}_3=0$. But this is a part of the ramification divisor of $\tilde{\beta}_1$. Since $\hat{S}$ cannot have a ramification divisor whose ramification index is 6, we see that $\tilde{\beta}_2$ is étale over $\tilde{\beta}_1^{-1}(V_3)$.

By the same argument as above, we can conclude that $\tilde{\beta}_2$ is étale over the double coverings $\tilde{\beta}_1^{-1}(V_4)$, $\tilde{\beta}_1^{-1}(V_5)$. The configuration of the exceptional curves is as in Figure 5. To obtain a minimal resolution of the singularity $p^{-1}(\mathfrak{p}_3)$, we need to investigate the action of the Galois group with respect to exceptional curves. To this aim, it is enough to look at the structure of the Galois covering



All curves are isomorphic to P^1.

FIGURE 5

$$\tilde{\beta}_2^{-1}(\tilde{\beta}_1^{-1}(s=0)) \cup \tilde{\beta}_2^{-1}(\tilde{\beta}_1^{-1}(s_2=0)) \rightarrow \boldsymbol{P}^1 .$$

By our construction, it is easy to show that $\tilde{\beta}_2^{-1}(\tilde{\beta}_1^{-1}(s=0))$ is the Galois covering associated to a triple covering of $\boldsymbol{A}^1(=\operatorname{Spec}(\boldsymbol{C}[u]))$ defined by the equation

$$X^3+(3/u)X+(2/u)=0 ,$$

where u is a coordinate of $\boldsymbol{A}^1$, while $\tilde{\beta}_2^{-1}(\tilde{\beta}_1^{-1}(s_2=0))$ is the Galois covering associated to a triple covering of $\boldsymbol{A}^1(=\operatorname{Spec}(\boldsymbol{C}[u]))$ defined by the equation

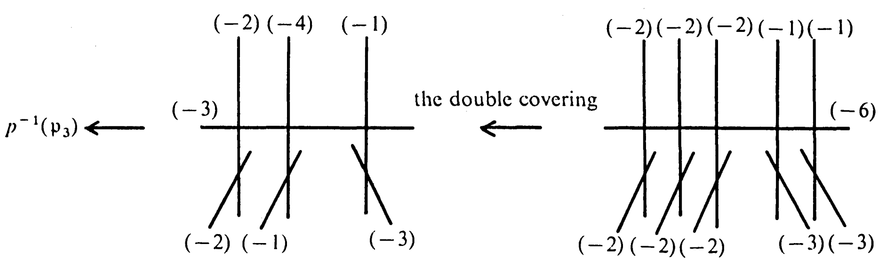
$$X^3+(3/v)X+(2/v^2)=0 ,$$

where v is a coordinate of $\boldsymbol{A}^1$.

By the above fact, and a calculation similar to that in [7, §2, Example 1], we see that the structure of the Galois covering

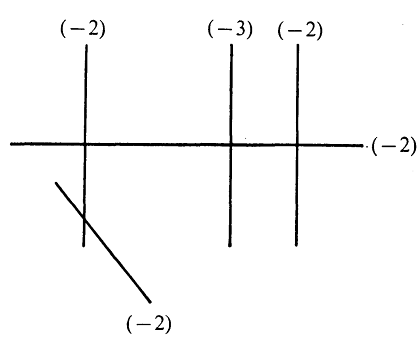
$$\tilde{\beta}_2^{-1}(\tilde{\beta}_1^{-1}(s=0)) \cup \tilde{\beta}_2^{-1}(\tilde{\beta}_1^{-1}(s_2=0)) \rightarrow \boldsymbol{P}^1$$

is the same as that in [5, §2, Example 1] (i.e., the action of the Galois group is the same). Therefore, it is easy to show that the resolution of $p^{-1}(\mathfrak{p}_3)$ has the configuration



All curves are isomorphic to $\boldsymbol{P}^1$.

FIGURE 6



All curves are isomorphic to $\boldsymbol{P}^1$.

FIGURE 7

of the excepional set as in Figure 6. Note that the above resolution is not minimal. By contracting exceptional curves of the first kind, we obtain the minimal resolution of the singularity $p^{-1}(\mathfrak{p}_3)$. The resulting configuration of exceptional curves is as in Figure 7.

Case (iv) A_1-singularity.

Let $\mathfrak{p}_4$ be a point of $B_0 \cap B_\infty$. In the same way as in the preceding cases, we consider our problem over an affine open set

$$U_4 = \mathrm{Spec}(\boldsymbol{C}[x, y])$$

such that $x=0, y=0$ denote the defining equations for B_0 and B_∞, respectively. Moreover, in U_4, the equation $X^3+(3a_0/a_\infty)X+(2b_0/b_\infty)=0$ has the form $X^3+3X+(y/x)=0$.

Let $\pi_{\mathfrak{p}_4}\colon \tilde{U}_4 \to U_4$ be the blowing-up at $\mathfrak{p}_4$. We take an affine open covers

$$\tilde{U}_4 = V_1 \cup V_2 \quad \text{with } V_1 = \mathrm{Spec}(\boldsymbol{C}[x, s]),\ V_2 = \mathrm{Spec}(\boldsymbol{C}[y, t]),\ y = xs,\ x = yt.$$

In this notation, R has the following form on each open set:

$$R = \frac{x^2+y^2}{x^2} = \begin{cases} 1+s^2 & \text{on} \quad (x, s), \\ \dfrac{t^2+1}{t^2} & \text{on} \quad (y, t). \end{cases}$$

Therefore, the double coverings $\tilde{\beta}_1^{-1}(V_i)$ $(i=1, 2)$ are

$$\begin{cases} \tilde{\beta}_1^{-1}(V_1) = \mathrm{Spec}(\boldsymbol{C}[x, s, \zeta_1]/(\zeta_1^2 - s^2 - 1)) \\ \tilde{\beta}_1^{-1}(V_2) = \mathrm{Spec}(\boldsymbol{C}[y, t, \zeta_2]/(\zeta_2^2 - t^2 - 1)). \end{cases}$$

Moreover,

$$-\tilde{\beta}_1^*\left(\frac{y}{x}\right) + \sqrt{\frac{x^2+y^2}{x^2}} = \begin{cases} -\bar{s}+\bar{\zeta}_1 & \text{on} \quad \tilde{\beta}_1^{-1}(V_1) \\ \dfrac{1+\bar{\zeta}_2}{\bar{t}} & \text{on} \quad \tilde{\beta}_1^{-1}(V_2). \end{cases}$$

Let us analyze $\tilde{\beta}_2$ on each affine open set $\tilde{\beta}_1^{-1}(V_i)$ $(i=1, 2)$.

On $\tilde{\beta}_1^{-1}(V_1)$, the action of the Galois group is

$$\bar{\zeta}_1 \mapsto -\bar{\zeta}_1.$$

Moreover, by the relation $\bar{\zeta}_1^2 - \bar{s}^2 = 1$, we see that $\tilde{\beta}_2$ is étale over $\tilde{\beta}_1^{-1}(V_1)$.

On $\tilde{\beta}_1^{-1}(V_2)$, the action of the Galois group is

$$\bar{\zeta}_2 \mapsto -\bar{\zeta}_2.$$

Moreover, by the relation $\bar{\zeta}_2^2 - 1 = \bar{t}^2$, we see that $\tilde{\beta}_2$ is ramified along the divisor defined by the equation $\bar{t}=0$.

To obtain the minimal resolution of $p^{-1}(\mathfrak{p}_4)$, we use the same argument as in Case (iii). Namely, we look at the structure of the Galois covering

$$\tilde{\beta}_2^{-1}(\tilde{\beta}_1^{-1}(x=0)) \cup \tilde{\beta}_2^{-1}(\tilde{\beta}_1^{-1}(y=0)) \to \boldsymbol{P}^1 .$$

By our construction, it is easy to show that $\tilde{\beta}_2^{-1}(\tilde{\beta}_1^{-1}(x=0))$ is the Galois covering associated to a triple covering of $\boldsymbol{A}^1 = \mathrm{Spec}(\boldsymbol{C}[u])$ defined by the equation,

$$X^3 + 3X + 2u = 0 ,$$

where u is a coordinate of $\boldsymbol{A}^1$. Therefore the Galois covering

$$\tilde{\beta}_2^{-1}(\tilde{\beta}_1^{-1}(x=0)) \cup \tilde{\beta}_2^{-1}(\tilde{\beta}_1^{-1}(y=0)) \to \boldsymbol{P}^1$$

is the same as that in [7, §2, Example 1]. Therefore, the resolution of $p^{-1}(\mathfrak{p}_4)$ has a rational curve with self-intersection number -3 as its exceptional set.

In summing up, we have obtained the following result in this section:

THEOREM (A). *Let S' be a normal finite triple covering of a smooth surface Σ which satisfies the conditions in the introduction. Then the singularities of S' are rational triple points of the following form*:

(i) *The points lying over $A_0 \cap A_\infty$ and $B_0 \cap B_\infty$. The singular points whose minimal resolutions have the configuration of excepional sets as in Figure* 1, (i).

(ii) *The points lying over $A_0 \cap A_\infty$. The singular points whose minimal resolutions have the configuration of exceptional sets as in Figure* 1, (ii).

4. Elementary examples.

EXAMPLE 4.1. Let $\Sigma = \boldsymbol{P}^2$ and consider the equation

$$X^3 + (3l_0/l_\infty)X + 2 = 0 ,$$

where l_0, l_∞ are linear forms. Assume that l_0, l_∞ satisfy the three conditions in the introduction. Let S' be the corresponding normal finite triple covering of $\boldsymbol{P}^2$, and S be a smooth model. Then S is a minimal rational ruled surface of degree 3, and we obtain S' by contracting the negative section of S. Moreover, both $D(S'/\Sigma)$ and $\hat{S}$ are ruled surfaces whose base curves are elliptic curves. We see these properties by the blowing-up at $l_0 \cap l_\infty$.

REMARK 4.2. In the above example, S' is isomorphic to a triple covering of $\boldsymbol{P}^2$ in [7, §2, Example 2], while both $\hat{S}$ and $D(S'/\Sigma)$ are different from those of [7, §2, Example 2].

EXAMPLE 4.3. Let $\Sigma = \boldsymbol{P}^2$ and consider an equation

$$X^3 + 3X + 2G_0/G_\infty = 0 ,$$

where both G_0 and G_∞ are homogeneous polynomials of degree n. Assume that the divisors $G_0 = 0$, $G_\infty = 0$ satisfy three conditions in the introduction for some $\alpha, \beta \in \boldsymbol{C}$. For brevity, let us assume $\alpha = \beta = 1$. Let S' be the corresponding normal finite triple covering of $\boldsymbol{P}^2$, and S be a smooth model. Then

$$\begin{cases} c_1^2(S) = (n-3)(5n-9) \\ c_2(S) = 7n^2 - 12n + 9 \,. \end{cases}$$

Indeed, let $\pi: \hat{\boldsymbol{P}}^2 \to \boldsymbol{P}^2$ be a succession of blowing-ups at the n^2 intersection points of $G_0 = 0$ and $G_\infty = 0$. Since

$$R = \frac{G_0^2 + G_\infty^2}{G_\infty^2} = \frac{(G_0 + \sqrt{-1}G_\infty)(G_0 - \sqrt{-1}G_\infty)}{G_\infty^2},$$

the branch locus of $\tilde{\beta}_1: D(S'/\boldsymbol{P}^2)^\sim \to \hat{\boldsymbol{P}}^2$ is reducible of the form

$$\bar{B} = \bar{B}_1 + \bar{B}_2 \,, \qquad \bar{B}_1 \bar{B}_2 = 0 \,.$$

Let B be the divisor $G_0^2 + G_\infty^2 = 0$, and B_∞ the strict transformation of the divisor $G_\infty = 0$. As is well-known, we have

$$K_{D(S'/\boldsymbol{P}^2)^\sim} \approx \tilde{\beta}_1^* \pi^* \left(K_{\boldsymbol{P}^2} + \frac{1}{2} B \right).$$

By what we saw in Section 3, we have

$$K_{\hat{S}} \approx \tilde{\beta}_2^* \left(K_{D(S'/\boldsymbol{P}^2)^\sim} + \frac{2}{3} \tilde{\beta}_1^*(B_\infty) \right).$$

We need to consider the ramification locus of $\alpha: \hat{S} \to S$. It is easy to show that

$$\begin{cases} \tilde{\beta}_2^*(\tilde{\beta}_1^* B_1) = 2(R_1' + R_2' + R_3') \\ \tilde{\beta}_2^*(\tilde{\beta}_1^* B_2) = 2(R_1'' + R_2'' + R_3'') \,, \end{cases}$$

and that the ramification divisor of α is of the form $R_i' + R_i''$, for some i. We may assume $i = 1$. Then, we obtain

$$\alpha^* K_S + (R_1' + R_1'') \sim K_{\hat{S}} \approx \tilde{\beta}_2^* \left(K_{D(S'/\boldsymbol{P}^2)^\sim} + \frac{2}{3} \tilde{\beta}_1^* B_\infty \right).$$

Since $R_1'^2 = R_1''^2 = 0$, and since both R_1' and R_1'' are smooth divisors isomorphic to B_1 and B_2, respectively, we obtain

$$K_{\hat{S}}(R_1' + R_1'') = 2n(n-3) \,, \quad \text{and} \quad K_{\hat{S}}^2 = 2(n-3)(7n-9) \,.$$

Therefore, we get

$$c_1^2(S) = (n-3)(5n-9) \,.$$

Let us now compute the second Chern class $c_2(S)$ of S by a Hurwitz type argument. We need the following important claim, which easily follows from Claim 2.2.

Claim. *The divisor $\tilde{\beta}_1^*(B_\infty)$ on $D(S'/\boldsymbol{P}^2)^\sim$ consists of two components, both of which* *are isomorphic to* B_∞.

By the above claim and a Hurwitz type argument, we see that

$$c_2(\hat{P}^2)=c_2(P^2)+n^2\,,$$
$$c_2(D(S'/P^2)^{\sim})=2c_2(\hat{P}^2)+2(2g(B_1)-2)\,,$$
$$c_2(\hat{S})=3c_2(D(S'/P^2)^{\sim})+2\times 2(2g(B_\infty)-2)=2c_2(S)+2(2g(R_1')-2)\,,$$

where $g(C)$ denotes the genus of a curve C. By the above equalities, we obtain

$$c_2(S)=7n^2-12n+9\,.$$

REMARK 4.4. Set $n=1$ in the above example. Then, S is the same as that in [7, §2, Example 2]. We can easily check that the above formulas hold in this case.

REMARK 4.5. Note that S is not necessarily minimal.

EXAMPLE 4.6. Let $\Sigma=P^1\times P^1$ with homogeneous coordinates $([s_0:s_\infty],[t_0:t_\infty])$. Let S' be the $C(\Sigma)(\theta)$-normalization of Σ where θ satisfies the cubic equation

$$X^3+(3s_0/s_\infty)X+(2t_0/t_\infty)=0\,.$$

It is clear that the conditions in the introduction are satisfied, and we have

$$R=\frac{s_\infty(s_0^3t_\infty^2+s_\infty^3t_0^2)}{s_\infty^4t_\infty^2}\,.$$

Hence the branch locus of $p\colon S'\to P^1\times P^1$ is the divisor defined by the equation $s_\infty(s_0^3t_\infty^2+s_\infty^3t_0^2)t_\infty=0$, and p is totally ramified along $t_\infty=0$ (see, Figure 8). There is a unique singularity of S lying over $(s_\infty=0)\cap(t_\infty=0)$. In the following, we study the

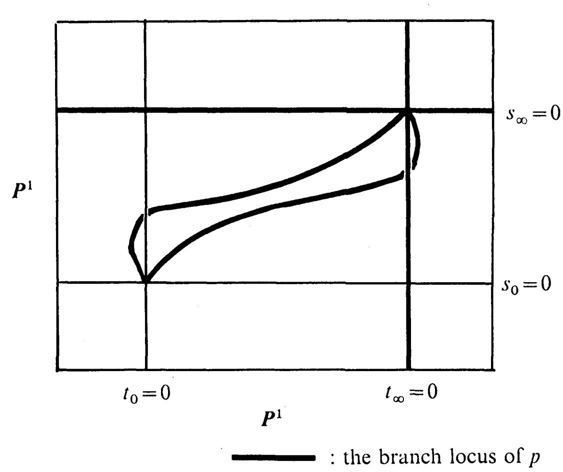


FIGURE 8

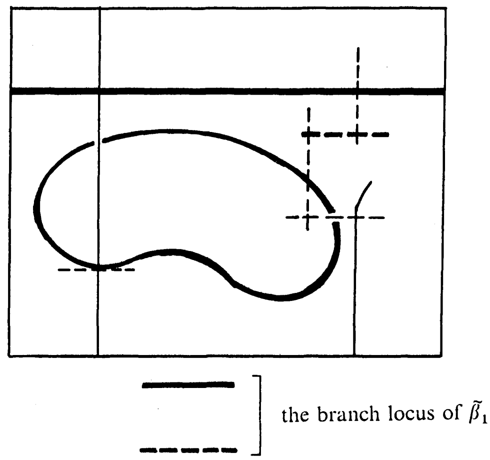


FIGURE 9

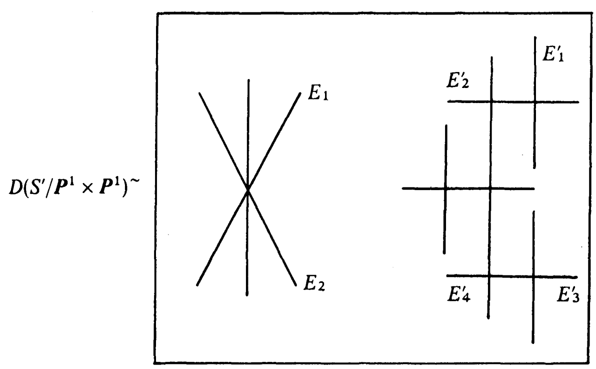


FIGURE 10

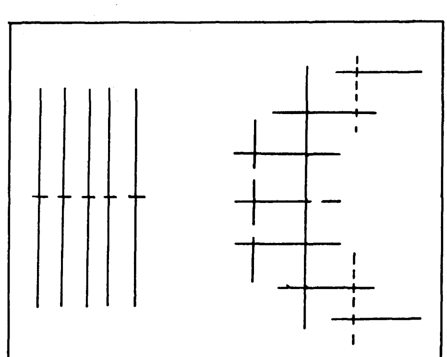


FIGURE 11

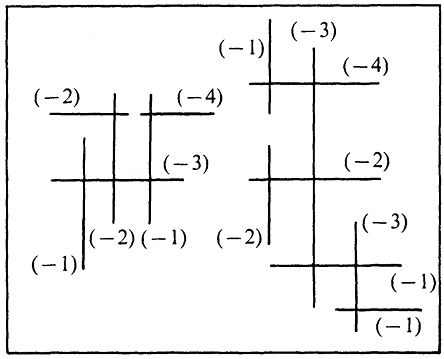


FIGURE 12

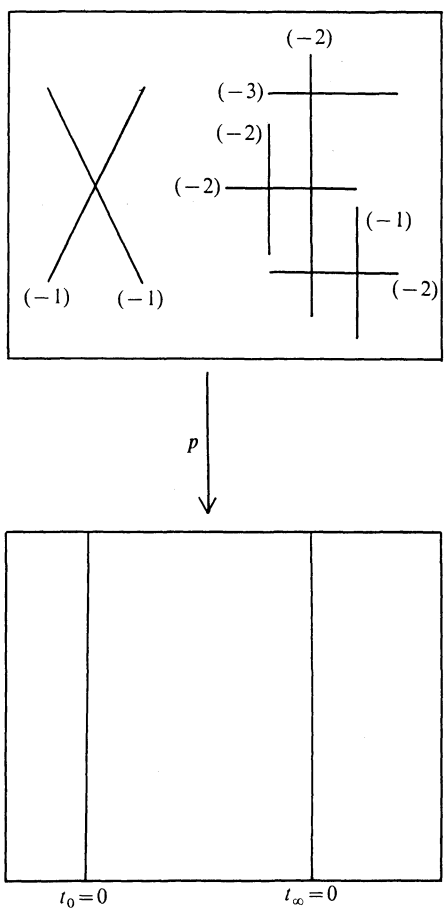


FIGURE 13

smooth models of S', $\hat{S}'$, and $D(S'/\boldsymbol{P}^1 \times \boldsymbol{P}^1)$. Consider a succession of blowing-ups at $(s_0=0) \cap (t_0=0)$ and $(s_\infty=0) \cap (t_\infty=0)$ as in Figure 9. Note that by blowing down suitable rational curves, we obtain a rational ruled surface Σ_2, and the corresponding branch locus is an effective divisor which is linearly equivalent to $\Delta_0+3\Delta_\infty$, where we denote the negative section and the positive section on Σ_2 by Δ_0 and Δ_∞, respectively. Hence, it is easy to check that the minimal resolution of $D(S'/\boldsymbol{P}^1 \times \boldsymbol{P}^1)$, which we denote by $D(S'/\boldsymbol{P}^1 \times \boldsymbol{P}^1)^\sim$, is a rational elliptic surface with sections and its two singular fibers are of types IV and IV* in the notation due to Kodaira [4] (See Figure 10). From results in Section 3, the branch locus of $\tilde{\beta}_2$ consists of E_i $(i=1, 2)$ and E'_j $(j=1, 2, 3, 4)$, and we have a smooth model of $\hat{S}'$ as in Figure 11. Hence, by blowing down suitable rational curves, the above smooth model becomes a rational elliptic surface of the same type as $D(S'/\boldsymbol{P}^1 \times \boldsymbol{P}^1)^\sim$. We thus have a smooth model of S' as in Figure 12. Finally, we have the minimal resolution S of S' as in Figure 13. Note that the (-1)-curves map to the divisors $t_0=0$ and $t_\infty=0$, respectively, and that S is also a rational ruled surface.

5. Trigonal fiber spaces. In this section, we apply the preceding results to construction of surfaces of general type.

DEFINITION 5.1. Let S be a surface and f: $S \to C$ a morphism from S to a curve C with a connected fiber. We call f: $S \to C$ a *trigonal fiber space* if there exists a dominant rational map Φ of degree 3 from S to a ruled surface Σ over C such that the following diagram commutes.

$$\begin{tikzcd} S \arrow[rr, dotted, "\Phi"] \arrow[dr, "f"'] & & \Sigma \arrow[dl] \\ & C & \end{tikzcd}$$

We call f: $S \to C$ a non-Galois trigonal fiber space if a general fiber of f is equipped with the structure of a non-Galois triple covering of $\boldsymbol{P}^1$ through the rational map Φ.

We devote this section to proving the following theorem:

THEOREM 5.2. *There exists a minimal surface S of general type with invariants*

$$c_1^2(S)=4n-8\,, \qquad c_2(S)=20n-4\,, \qquad p_g(S)=2n-2 \quad (n\geq 3)\,,$$

which has structure of a non-Galois trigonal fiber space over $\boldsymbol{P}^1$.

REMARK 5.3. Note that the surface whose numerical invariants are the same as above satisfies Noether's equality, $c_1^2=2p_g-4$. It is known (cf. Horikawa [2]) that such a surface is always a double covering over a suitable rational surfaces. It is an interesting problem to express S in Theorem 5.2 in this manner.

From now on, we use the following notation:

Σ_n: a rational ruled surface of degree n $(n \geq 2)$.
s_0 : the negative section of Σ_n.
s_∞: the positive section of Σ_n.
$\mathfrak{f}$: a fiber of the ruling for Σ_n.
Theorem 5.2 is an easy consequence of the following:

PROPOSITION 5.4. *Let $\boldsymbol{C}(\Sigma_n)(\theta)$ be an algebraic extension of $\boldsymbol{C}(\Sigma_n)$ defined by an equation*

$$X^3+3X+2b=0\,, \qquad b\in \boldsymbol{C}(\Sigma_n)\,.$$

Assume that b satisfies the condition in the introduction for $\alpha=\beta=1$, and denote $B_0=(b)_0$, $B_\infty=(b)_\infty \in |2s_\infty|$. Then the smooth model S of the $\boldsymbol{C}(\Sigma_n)(\theta)$-normalization S' of Σ_n (cf. Iitaka [2, §2.14]) *as in the preceding section is a minimal surface with numerical invariants*

$$c_1^2(S)=4n-8\,, \qquad c_2(S)=20n-4\,, \qquad p_g(S)=2n-2\,.$$

PROOF. We first compute $c_1^2(S)$. Let $\pi : \hat{\Sigma}_n \to \Sigma_n$ be a succession of blowing-ups at the $4n$ intersection points of B_0 and B_∞. In the same way as in Example 4.3, the branch locus of $\tilde{\beta}_1$ is reducible of the form

$$D=D_1+D_2\,, \qquad D_1D_2=0\,.$$

As is well-known, we have

$$K_{D(S'/\Sigma_n)^\sim} \sim \tilde{\beta}_1^*\pi^*(K_{\Sigma_n}+2s_\infty) \sim \tilde{\beta}_1^*\pi^*((n-2)\mathfrak{f})\,,$$

where $\tilde{\beta}_1^* : D(S'/\Sigma_n)^\sim \to \hat{\Sigma}_n$. Note that $|K_{D(S'/\Sigma_n)^\sim}|$ is base-point-free since $n \geq 2$.

Since $\tilde{\beta}_2 : \hat{S} \to D(S'/\Sigma_n)^\sim$ is a cyclic triple covering branched along $\tilde{\beta}_1^*\bar{B}_\infty$, where $\bar{B}_\infty$ is the strict transformation of B_∞, we obtain

$$3K_{\hat{S}} \sim \tilde{\beta}_2^*(3K_{D(S'/\Sigma_n)^\sim}+2\tilde{\beta}_1^*\bar{B}_\infty) \sim \tilde{\beta}_2^*\tilde{\beta}_1^*(3\pi^*((n-2)\mathfrak{f})+2\bar{B}_\infty)\,.$$

To compute $c_1^2(S)$, we now represent $K_{\hat{S}}$ in terms of $K_{\hat{S}}$ and the ramification locus of $\alpha : \hat{S} \to S$, which has the following form:

Set

$$\tilde{\beta}_2^*\tilde{\beta}_1^*(D_1)=2(R_1'+R_2'+R_3')\,, \qquad \tilde{\beta}_2^*\tilde{\beta}_1^*(D_2)=2(R_1''+R_2''+R_3'')\,.$$

Then we may assume $R_1=R_1'+R_1''$ to be the ramification locus of α. We get $\alpha^*K_S+R_1=K_{\hat{S}}$. Hence,

$$\alpha^*K_S \sim K_{\hat{S}}-R_1 \sim \tilde{\beta}_2^*\tilde{\beta}_1^*(3\pi^*((n-2)\mathfrak{f})+2\bar{B}_\infty)-R_1\,.$$

By our construction, we can easily show

$$R_1'^2=R_2'^2=R_3'^2=R_1''^2=R_2''^2=R_3''^2=0\,,$$

$$R_i'R_j'=R_i''R_j''=0 \qquad \text{for} \quad i>j\,,$$

$$K_{\hat{S}}R'_1 = K_{\hat{S}}R''_1 = 2n-4 \, .$$

Hence we get $(\alpha^* K_S)^2 = 8n-16$, and $K_S^2 = 4n-8$.

Next, we compute $c_2(S)$ by a Hurwitz type argument. Note that the divisor $\tilde{\beta}_1^* \bar{B}_\infty$ on $D(S'/\Sigma_n)$ consists of two components, both of which are isomorphic to $\bar{B}_\infty$ by the Calim in Example 4.3. Hence it is easy to show that

$$c_2(\hat{\Sigma}_n) = c_2(\Sigma_n) + 4n$$

$$c_2(D(S'/\Sigma_n)^{\sim} = 2c_2(\hat{\Sigma}_n) + (2g(D_1)-2) + (2g(D_2)-2)$$

$$c_2(\hat{S}) = 3c_2(D(S'/\Sigma_n)^{\sim} + 2 \times 2(2g(B_\infty)-2) = 2c_2(S) + (2g(R'_1)-2) + (2g(R''_1)-2) \, .$$

Since,

$$2g(R'_1)-2 = 2g(R''_1)-2 = 2g(B_\infty)-2 = 2g(D_1)-2 = 2g(D_2)-2 = 2n-4 \, ,$$

we obtain $c_2(S) = 2n-4$. As for the equality $p_g(S) = 2n-2$, we use Noether's formula

$$\chi(\mathcal{O}_S) = \frac{1}{12}(c_1^2 + c_2) \, ,$$

by which we have $p_g = 2n-2+q$. Hence it suffices to show $q=0$.

LEMMA 5.5. *Let f: $S \to C$ be a surjective morphism from a surface to a curve of genus g with connected fibers. Assume that there exists a singular fiber of f whose irreducible components are all rational curves. Then $q(S) = g$.*

PROOF OF LEMMA 5.5. It is clear that $q(S) \geq g$. Assume that $q(S) > g$. Consider the Albanese mapping α: $S \to \mathrm{Alb}(S)$. Since $q(S) > g$, the image of a general fiber of f is a curve in $\mathrm{Alb}(S)$. Let L be an ample line bundle over $\mathrm{Alb}(S)$. Then

$$L \cdot (\alpha \text{ (a general fiber)}) > 0 \, .$$

On the other hand, let F be the singular fiber as above. Then $\alpha(F)$ is a point. Therefore, $L \cdot (\alpha(F)) = 0$. This contradicts the fact that F is numerically equivalent to a general fiber.

q.e.d.

We now continue the proof of Theorem 5.2. Consider the fibration induced by

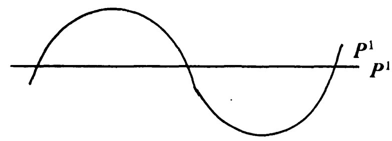
$$f \colon S \to \hat{\Sigma}_n \to \Sigma_n \to \boldsymbol{P}^1 \, .$$

By Lemma 5.4, it is enough to show that the fibration f has a singular fiber whose irreducible components are all rational curves. To see this, consider a fiber $\mathfrak{f}$ of $\Sigma_n \to \boldsymbol{P}^1$ such that $\mathfrak{f} \cap (B_0 \cap B_\infty) \neq \emptyset$. There are two possibilities:

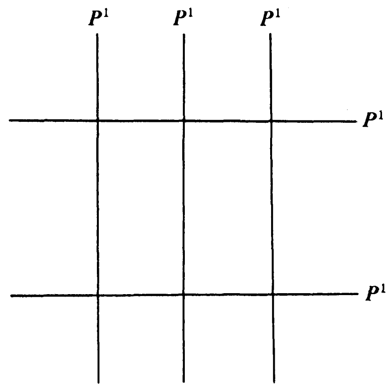
(i) $\mathfrak{f} \cap (B_0 \cap B_\infty)$ is a point.

(ii) $\mathfrak{f} \cap (B_0 \cap B_\infty)$ consists of two points.

In Case (i), we can show that the singular fiber of f over $\mathfrak{f}$ consists of two rational



(i)



(ii)

FIGURE 14

curves with three intersection points with the configuration as in Figure 14, (i).

In Case (ii), we can show that the singular fiber of f over $\mathfrak{f}$ consists of five rational curves with the configuration as in Figure 14, (ii).

We thus conclude that f: $S \to \boldsymbol{P}^1$ has a singular fiber whose irreducible components are all rational curves. Therefore, we obtain $q=0$.

It remains to show that S is a minimal surface. It sufficies to show that $\alpha^* K_S$ is numerically effective. Since

$$3K_{\hat{S}} \sim \tilde{\beta}_2^* \tilde{\beta}_1^* (3\pi^*((n-2)\mathfrak{f}) + 2B_\infty) \quad \text{and} \quad K_{\hat{S}} \sim \alpha^* K_S + R_1 ,$$

we get

$$3\alpha^* K_S \sim \tilde{\beta}_2^* \tilde{\beta}_1^* (3\pi^*((n-2)\mathfrak{f}) + 2\tilde{\beta}_2^* \tilde{\beta}_1^* B_\infty - 3R_1 .$$

By our construction, $\tilde{\beta}_2^* \tilde{\beta}_1^* (B_\infty) \sim \tilde{\beta}_2^* \tilde{\beta}_1^* (D_i)$ $(i=1,\ 2)$ and R_i', R_i'' $(i=1,\ 2,\ 3)$ are all numerically equivalent to one another. Hence

$$3\alpha^* K_S \approx \tilde{\beta}_2^* \tilde{\beta}_1^* (3\pi^*((n-2)\mathfrak{f})) + 6R_1' .$$

Moreover, both $\tilde{\beta}_2^* \tilde{\beta}_1^* (\pi^*(\mathfrak{f}))$ and R_1' are numerically effective divisors. Therefore, $\alpha^* K_S$ is numerically effective, and S is a minimal surface. This completes the proof of Proposition 5.4.

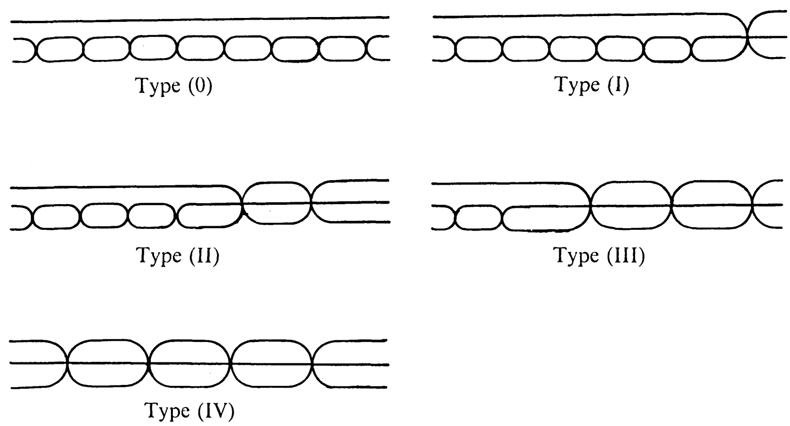


FIGURE 15

REMARK 5.6. Note that a general fiber of f: $S \to \boldsymbol{P}^1$ is a curve of genus 2 as we show in the Appendix below.

APPENDIX. A curve of genus 2 as a trigonal curve.

It is well-known that a curve of genus 2 is hyperelliptic. On the other hand, for any divisor $\mathfrak{d}$ of degree 3 on a curve C of genus 2, we have

$$\dim_{\boldsymbol{C}} H^0(C, \mathcal{O}_C(\mathfrak{d}))=2$$

by the Riemann-Roch theorem. Therefore, C can be regarded as a trigonal curve. As for their structure of ramification as triple coverings over $\boldsymbol{P}^1$, there are five types (0) ~ (IV) as in Figure 15.

For each type, there exists a cubic equation corresponding to the covering. The proof of the following is easy.

THEOREM. *Let t be an inhomogeneous coordinate of $\boldsymbol{P}^1$. Then the following holds: The covering corresponding to the equation*

$$X^3+3\frac{(t-a)(t-b)}{t}X+2=0 \qquad (a \neq b,\ a, b \neq 0, \infty)$$

is of Type (0).

The covering corresponding to the equation

$$X^3+3\frac{(t-a)}{(t-b)}X+2t=0 \qquad (a \neq b,\ a, b=0, \infty)$$

is of Type (I).

The covering corresponding to the equation

$$X^3+3X+2\frac{g_0(t)}{g_\infty(t)}=0\,,$$

where $g_0(t)$, $g_\infty(t)$ *are polynomials with* $\deg g_0=\deg g_\infty=2$, *and no common zeros, is of Type* (II).

The covering corresponding to the equation

$$X^3+3X+2\frac{t^2(t-a)+(t-1)^2(t-b)}{\sqrt{-1}\{(t-b)(t-1)^2-t^2(t-a)\}}=0 \qquad \begin{pmatrix} a\neq b,\ a, b\neq 0,\ 1, \\ 4ab-8b+1\neq 0 \end{pmatrix}$$

is of Type (III).

The covering corresponding to the equation

$$X^3+\frac{g_0(t)}{g_\infty(t)}=0\,,$$

where g_0 *and* g_∞ *satisfy the same condition as in the third case, is of Type* (IV).

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