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EXISTENCE AND CONTINUOUS DEPENDENCE OF MILD SOLUTIONS TO SEMILINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS IN BANACH SPACES

Dedicated to Professor Junji Kato on his sixtieth birthday

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Abstract. This paper is concerned with a general existence and continuous dependence of mild solutions to semilinear functional differential equations with infinite delay in Banach spaces. In particular, our results are applicable to the equations whose C_0-semigroups and nonlinear operators, defined on an open set, are noncompact.

Introduction. Let E be a Banach space over the real field R with norm $|\cdot|$ and $\mathcal{B}$ a phase space satisfying the fundamental axioms given in [3], [4], [15]. If $x : (-\infty, \sigma + a) \to E$, $0 < a \le +\infty$, then for any $t \in (-\infty, \sigma + a)$ define a mapping $x_t : (-\infty, 0] \to E$ by $x_t(\theta) = x(t + \theta)$, $-\infty < \theta \le 0$. Denote by $C([a, b], E)$ the space of all continuous functions from $[a, b]$ into E with the supremum norm. Let A be the infinitesimal generator of a C_0-semigroup $T(t)$ on E.

In this paper we deal with the initial-value problem for the semilinear functional differential equation with infinite delay in E (for brevity, IP(σ, φ)):

$$\frac{d}{dt}u(t) = Au(t) + F(t, u_t), \quad \sigma < t \le \sigma + a,$$

with $u_\sigma = \varphi \in \mathcal{B}$, where $(\sigma, \varphi) \in R \times \mathcal{B}$ is given initial data and F is a (strongly) continuous function mapping an open subset D in $R \times \mathcal{B}$ into E. If $u : (-\infty, \sigma + a] \to E$ is a continuous function satisfying the integral equation

$$u(t) = \begin{cases} T(t-\sigma)\varphi(0) + \displaystyle\int_\sigma^t T(t-s)F(s, u_s)ds & \text{for } t \in [\sigma, \sigma + a] \\ \varphi(t - \sigma) & \text{for } t \in (-\infty, \sigma], \end{cases}$$

then u is called a mild solution of IP(σ, φ).

Roughly speaking, the study of the existence of mild solutions to IP(σ, φ) has been developed in two different directions. One direction is to find conditions to guarantee the existence and uniqueness of mild solutions for IP(σ, φ); for instance, refer to Iwamiya [8], Martin [11], Schumacher [16], Shin [21], [22] and Travis and Webb [24], etc. The other is to find conditions to ensure only the existence of mild solutions to IP(σ, φ), which is mainly

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described in terms of the measure of noncompactness (α-measure for short) introduced by Kuratowskii; for instance, refer to [1], [7], [9], [10], [13], [14], [17], [19], [20].

In the present paper we will investigate the existence and the continuous dependence of mild solutions to IP(σ, φ) in the latter direction.

First, we will establish a general existence theorem on mild solutions for IP(σ, φ). The fundamental results on the existence of mild solutions for the case of non-delay were established by Krasnoselskii, Krein and Sobolevskii [9] and Pazy [14] in which it is assumed that $T(t)$ is a C_0-compact semigroup on E or F is a compact operator. Recently, in the work of Henriquez [7] the above result was extended to IP(σ, φ). Thus, in the case that both $T(t)$ and F are noncompact operators, we will develop an existence theorem of mild solutions to IP(σ, φ) in the present paper. In such a direction Bothe [1] showed a result on the existence of mild solutions to the multivalued semilinear differential equation on a closed set, which is a partial extension of the one due to Mönch and Harten [13] for ordinary differential equations. However, even Bothe's result cannot directly extend to IP(σ, φ), because, contrary to the case of non-delay, it is difficult to obtain the compactness of a sequence $\{z^n\}_{n\in N} \subset C([a, b], E)$ of approximate solutions for IP(σ, φ).

To overcome this difficulty, we establish the following inequality (Theorem 1) on the α-measure: For a bounded subset $\mathcal{U}$ in $C([a, b], E)$

$$\alpha\left(\left\{\int_a^{\cdot} T(\cdot - s)f(s)ds|[a, b] \;\middle|\; f \in \mathcal{U}\right\}\right) \le \gamma_T \sup_{a\le t\le b} \alpha\left(\left\{\int_a^t T(t - s)f(s)ds \;\middle|\; f \in \mathcal{U}\right\}\right),$$

where

$$\int_a^{\cdot} T(\cdot - s)f(s)ds|[a, b] \in C([a, b], E) \quad \text{and} \quad \gamma_T = \limsup_{\delta\to 0+} \|T(\delta)\|.$$

Using this result and the integral inequality [6] (refer to [2], [13]) on the α-measure, we can prove our existence theorem (Theorem 2) for IP(σ, φ). Of course, our result extends Mönch's and Harten's one [13] and contains Bothe's one [1] for the single valued case on an open set as well as Henriquez's one [7] (see Remark 2.2). See [23] for an application of the above inequality.

Secondly, based on our existence theorem, general results (Theorem 3 and Proposition 4.7) on the continuous dependence of mild solutions are formulated in semilinear functional differential equations. Our theorem is an extension of Kamke's theorem given in functional and ordinary differential equations in finite dimensional spaces or in infinite dimensional spaces (refer to [4], [15], [16], [19], [25]). In the case $A = 0$, a similar result to Theorem 3 can be found in [19], but its proof is based on the assumption that F is uniformly continuous. We note that the result given in [16, Theorem 3.1] is related to the local Lipschitz condition on F.

1. Phase space $\mathcal{B}$, C_0-semigroup and α-measure. Let $R^- = (-\infty, 0]$, $R^+ = [0, \infty)$ and $R = (-\infty, \infty)$. Let $\mathcal{B} = \mathcal{B}(R^-, E)$ be a linear space, with semi-norm $|\cdot|_{\mathcal{B}}$, consisting of functions mapping R^- into E. Throughout this paper we assume that the following axioms on the phase space $\mathcal{B}$ are satisfied:

(B_1) If $x : (-\infty, \sigma + a) \to E$ is continuous on $[\sigma, \sigma + a)$ and $x_\sigma \in \mathcal{B}$, then $x_t \in \mathcal{B}$ for all $t \in [\sigma, \sigma + a)$ and x_t is continuous in $t \in [\sigma, \sigma + a)$.

(B_2) There exist a continuous function $K(t) > 0$ and a locally bounded function $M(t) \geq 0$ such that

$$|x_t|_{\mathcal{B}} \leq K(t - \sigma) \sup\left\{|x(s)| \,\middle|\, \sigma \leq s \leq t\right\} + M(t - \sigma)|x_\sigma|_{\mathcal{B}}$$

for $t \in [\sigma, \sigma + a)$ and x having the properties in (B_1).

(B_3) There exists a constant $L > 0$ such that $|\varphi(0)| \leq L|\varphi|_{\mathcal{B}}$ for all $\varphi \in \mathcal{B}$.

(B_4) The quotient space $\hat{\mathcal{B}} = \mathcal{B}/|\cdot|_{\mathcal{B}}$ is a Banach space.

For examples of the phase space $\mathcal{B}$ refer to [3], [4], [15]. Frequently, we will use the following notations in this paper: $K_a = \sup\left\{K(t) \,\middle|\, 0 \leq t \leq a\right\}$ and $\mathcal{B}(\varphi, r) = \left\{\psi \in \mathcal{B} \,\middle|\, |\varphi - \psi|_{\mathcal{B}} \leq r\right\}$.

Let Y be a linear space with a semi-norm $|\cdot|_Y$ and the quotient space $\hat{Y} = Y/|\cdot|_Y$ be a Banach space. For a bounded subset Ω in Y, the α-measure of Ω is defined as follows:

$$\alpha(\Omega) = \inf\left\{d > 0 \,\middle|\, \Omega \text{ has a finite cover of diameter } < d\right\}.$$

Hereafter, we will use the same notation α for Kuratowski's measure of noncompactness in any linear and semi-normed space whose quotient space is a Banach space. Refer to [10], [17], [19] for elementry properties of the α-measure.

Denote by $C[a, b]$ for short the space $C([a, b], E)$ and by $|x|_{[a,b]}$ the supremum norm of x in $C[a, b]$. Let $\mathcal{X}$ be a set of functions x from $(-\infty, \sigma + a)$, $0 < a \leq \infty$, to E such that $x_\sigma \in \mathcal{B}$ and x is continuous on $[\sigma, \sigma + a)$. Then we will use the following notations:

$$\mathcal{X}(t) = \left\{x(t) \in E \,\middle|\, x \in \mathcal{X}\right\}, \quad \mathcal{X}_t = \left\{x_t \in \mathcal{B} \,\middle|\, x \in \mathcal{X}\right\} \quad \text{for } t \in [\sigma, \sigma + a),$$

$$\mathcal{X}|[c, d] = \left\{x|[c, d] \in C[c, d] \,\middle|\, x \in \mathcal{X}\right\} \quad \text{and} \quad \dot{\mathcal{X}}(t) = \left\{\dot{x}(t) \in E \,\middle|\, x \in \mathcal{X}\right\},$$

where $\sigma \leq c \leq d < \sigma + a$, $x|[c, d]$ stands for the restriction of x to $[c, d]$ and $\dot{x}(t)$ denotes the differential of x at t. If $\mathcal{X}|[\sigma, t]$, $t \in [\sigma, \sigma + a)$, and $\mathcal{X}_\sigma$ are bounded, then the relation

$$\frac{1}{L}\alpha(\mathcal{X}(t)) \leq \alpha(\mathcal{X}_t) \leq K(t - \sigma)\alpha(\mathcal{X}|[\sigma, t]) + M(t - \sigma)\alpha(\mathcal{X}_\sigma) \tag{1.1}$$

holds (see [17, Theorem 2.1] and [19, Lemma 1.5]). We denote by $\mathcal{L}^1[a, b]$ the space of all integrable functions from $[a, b]$ to R with the norm $|f|_{\mathcal{L}^1} = \int_a^b |f(t)|dt$.

The following result is found in [6, Theorem 2.1] (refer to [2], [13]).

LEMMA 1.1 *Let $\mathcal{W}$ be a countable set of strongly measurable functions from $[a, b]$ to E. Assume that there exists a $\mu \in \mathcal{L}^1[a, b]$ such that $|x(t)| \leq \mu(t)$ for all $x \in \mathcal{W}$ and for a.a. $t \in [a, b]$. Then $\alpha(\mathcal{W}(t))$ is integrable on $[a, b]$ and*

$$\alpha\left(\left\{\int_a^b x(t)dt \,\middle|\, x \in \mathcal{W}\right\}\right) \leq 2\int_a^b \alpha(\mathcal{W}(t))dt.$$

If $T : Y \to Y$ is continuous and takes bounded sets into bounded sets and if there is a $\kappa \geq 0$ such that $\alpha(TB) \leq \kappa\alpha(B)$ for all bounded sets $B \subset Y$, we define

$$\alpha(T) = \inf\left\{\kappa \in R^+ \,\middle|\, \alpha(TB) \leq \kappa\alpha(B) \text{ for all bounded sets } B \subset Y\right\}.$$

Then we have $\alpha(TB) \leq \alpha(T)\alpha(B)$ for every bounded set $B \subset Y$.

Similarly,

$$\hat{\alpha}(T) = \inf\left\{\kappa \in R^+ \,\middle|\, \alpha(TB) \leq \kappa\alpha(B) \text{ for all bounded countable sets } B \subset Y\right\}.$$

REMARK 1.2. Let $T : E \to E$ be a bounded linear operator. Then T is a compact operator if and only if $\hat{\alpha}(T) = 0$.

It is well-known that if $T(t)$ is a C_0-semigroup on E, then $\|T(t)\| \leq M_\omega e^{\omega t}$ for all $t \in R^+$, where $M_\omega \geq 1$ and $\omega \in (-\infty, \infty)$ (see [14]). Hence it follows that

$$\hat{\alpha}(T(t)) \leq \alpha(T(t)) \leq \|T(t)\| \leq M_\omega e^{\omega t} \quad \text{for all } t \in R^+. \tag{1.2}$$

If $\mathcal{H}$ is a bounded subset in $C[a, b]$, then for $\delta > 0$ and $t \in [a, b]$ we set $\alpha(t, \delta; \mathcal{H}) = \alpha(\mathcal{H} \,|\, [t - \delta, t + \delta])$ and define

$$\alpha(t; \mathcal{H}) = \inf\left\{\alpha(t, \delta; \mathcal{H}) \,\middle|\, \delta > 0\right\} = \lim_{\delta \to 0+} \alpha(t, \delta; \mathcal{H}).$$

The following result is found in [12, Lemma 1].

LEMMA 1.3. *Let $\mathcal{H} \subset C[a, b]$ and $\alpha(t; \mathcal{H})$ be as above. Then $\alpha(\mathcal{H}) = \sup_{a \leq t \leq b} \alpha(t; \mathcal{H})$.*

For a bounded set $\mathcal{H} \subset C[a, b]$ and for $t \in [a, b]$ we use the following notations:

$$\omega(\delta; t, \mathcal{H}) = \sup\left\{|g(\tau) - g(s)| \,\middle|\, \tau, s \in [t - \delta, t + \delta],\, g \in \mathcal{H}\right\},$$

$$\omega(t, \mathcal{H}) = \inf\left\{\omega(\delta; t, \mathcal{H}) \,\middle|\, \delta > 0\right\} = \lim_{\delta \to 0+} \omega(\delta; t, \mathcal{H})$$

and

$$\omega(\mathcal{H}) = \sup_{a \leq t \leq b} \omega(t, \mathcal{H}).$$

Clearly, $\mathcal{H}$ is uniformly equicontinuous on $[a, b]$ if and only if $\omega(\mathcal{H}) = 0$.

For a continuous function $u : [a, b] \to E$ we put

$$T * u(t) = \int_a^t T(t - s)u(s)ds \quad \text{for } t \in [a, b],$$

and for a subset $\mathcal{U} \subset C[a, b]$ we put $T * \mathcal{U} = \left\{T * u \,\middle|\, u \in \mathcal{U}\right\}$.

We are now in a position to prove the main theorem in this section.

THEOREM 1. *Let $\mathcal{U}$ be a bounded set in $C[a, b]$ and $T(t)$ a C_0-semigroup on E. Then*

$$\frac{1}{2\gamma_T}\omega(t, T * \mathcal{U}) \leq \sup_{a \leq \tau \leq t} \alpha(T * \mathcal{U}(\tau)) \leq \alpha(T * \mathcal{U} \,|\, [a, t]) \leq \gamma_T \sup_{a \leq \tau \leq t} \alpha(T * \mathcal{U}(\tau)) \tag{1.3}$$

for all $t \in [a, b]$, where $\gamma_T = \limsup_{\delta \to 0+} \|T(\delta)\|$.

In particular, if $T(t)$ is a C_0-contraction semigroup on E, then the above relation (1.3) is reduced to

$$\frac{1}{2}\omega(t, T * \mathcal{U}) \le \alpha(T * \mathcal{U} \mid [a, t]) = \sup_{a \le \tau \le t} \alpha(T * \mathcal{U}(\tau)). \tag{1.4}$$

PROOF. Since $\mathcal{U}$ is a bounded subset in $C[a, b]$, there is an $L > 0$ such that $|f|_{[a,b]} \le L$ for all $f \in \mathcal{U}$. Set $\mathcal{K} = T * \mathcal{U}$. For any $t \in (a, b]$ and for any ε, $0 < \varepsilon < t - a$, there exist $\mathcal{K}_i(t-\varepsilon) \subset \mathcal{K}(t-\varepsilon)$, $i = 1, 2, \dots, m$, such that

$$\operatorname{dia} \mathcal{K}_i(t-\varepsilon) \le \alpha(\mathcal{K}(t-\varepsilon)) + \frac{\varepsilon}{6} \ (i = 1, 2, \dots, m), \quad \mathcal{K}(t-\varepsilon) = \bigcup_{i=1}^{m} \mathcal{K}_i(t-\varepsilon). \tag{1.5}$$

Set

$$\mathcal{K}_i \mid [t-\varepsilon, t+\varepsilon] = \left\{ T * h \mid [t-\varepsilon, t+\varepsilon] \in \mathcal{K} \mid [t-\varepsilon, t+\varepsilon] \,\middle|\, T * h(t-\varepsilon) \in \mathcal{K}_i(t-\varepsilon) \right\}.$$

Then we have

$$\mathcal{K} \mid [t-\varepsilon, t+\varepsilon] = \bigcup_{i=1}^{m} \mathcal{K}_i \mid [t-\varepsilon, t+\varepsilon].$$

Now we will prove the first inequality in (1.3). For any $T * f \in \mathcal{K}$ there is a $j \in \{1, 2, \dots, m\}$ such that $T * f \mid [t-\varepsilon, t+\varepsilon] \in \mathcal{K}_j \mid [t-\varepsilon, t+\varepsilon]$. Select then a $T * g \in \mathcal{K}$ such that $T * g \mid [t-\varepsilon, t+\varepsilon] \in \mathcal{K}_j \mid [t-\varepsilon, t+\varepsilon]$. Since $T * g$ is uniformly continuous on $[a, b]$, there is a δ ($\varepsilon > \delta > 0$) such that $|T * g(\tau) - T * g(s)| < \varepsilon/3$ if $|\tau - s| < \delta$. Thus we have, for $\tau, s \in [t-\delta, t+\delta]$,

$$|T * f(\tau) - T * f(s)| \le |T * g(\tau) - T * f(\tau)| + |T * g(s) - T * f(s)| + \frac{\varepsilon}{3}. \tag{1.6}$$

The first term in the right hand side of (1.6) is estimated as follows. Let $C = \sup_{0 \le s \le b-a} \|T(s)\|$. Then

$$\begin{aligned}
&|T * g(\tau) - T * f(\tau)| \\
&\quad \le \left| \int_a^{t-\varepsilon} T(\tau - s) g(s) ds - \int_a^{t-\varepsilon} T(\tau - s) f(s) ds \right| \\
&\qquad + \left| \int_{t-\varepsilon}^{\tau} T(\tau - s) g(s) ds - \int_{t-\varepsilon}^{\tau} T(\tau - s) f(s) ds \right| \\
&\quad \le \|T(\tau - t + \varepsilon)\| \left| \int_a^{t-\varepsilon} T(t - \varepsilon - s) g(s) ds - \int_a^{t-\varepsilon} T(t - \varepsilon - s) f(s) ds \right| \\
&\qquad + 2CL|\tau - t + \varepsilon| \\
&\quad \le \sup_{0 < \gamma \le \varepsilon_1} \|T(\gamma)\| |T * g(t-\varepsilon) - T * f(t-\varepsilon)| + 2CL\varepsilon_1,
\end{aligned} \tag{1.7}$$

where $\varepsilon_1 = \varepsilon + \delta$. Since the second term in the right hand side in (1.6) is similarly estimated as (1.7), the inequality (1.6) becomes

$$|T * f(\tau) - T * f(s)| \le 2 \sup_{0 < \gamma \le \varepsilon_1} \|T(\gamma)\| |T * g(t-\varepsilon) - T * f(t-\varepsilon)| + 4CL\varepsilon_1 + \frac{\varepsilon}{3}.$$

Using (1.5), we have

$$\begin{aligned}|T * f(\tau) - T * f(s)| &\le 2 \sup_{0<\gamma\le\varepsilon_1} \|T(\gamma)\| \operatorname{dia} \mathcal{K}_j(t-\varepsilon) + 4CL\varepsilon_1 + \frac{\varepsilon}{3} \\ &\le 2 \sup_{0<\gamma\le\varepsilon_1} \|T(\gamma)\| \sup_{a\le\tau\le t} \alpha(\mathcal{K}(\tau)) + (4CL + C)\varepsilon_1 ,\end{aligned}$$

and hence

$$\omega(\delta; t, \mathcal{K}) \le 2 \sup_{0<\gamma\le\varepsilon_1} \|T(\gamma)\| \sup_{a\le\tau\le t} \alpha(\mathcal{K}(\tau)) + (4CL + C)\varepsilon_1 .$$

Therefore, letting $\varepsilon \to 0+$ in both sides of the above inequality, we have $\omega(t, \mathcal{K}) \le 2\gamma_T \sup_{a\le\tau\le t} \alpha(\mathcal{K}(\tau))$ as required.

Next, we will prove the third inequality in (1.3). In view of (1.7) we have that for any $T * f \mid [t-\varepsilon, t+\varepsilon], T * h \mid [t-\varepsilon, t+\varepsilon] \in \mathcal{K}_i \mid [t-\varepsilon, t+\varepsilon]$

$$\begin{aligned}&\sup\left\{|T * f(s) - T * h(s)| \,\middle|\, t-\varepsilon \le s \le t+\varepsilon\right\} \\ &\quad = |T * f(\tau) - T * h(\tau)| \quad \text{for some } \tau \in [t-\varepsilon, t+\varepsilon] \\ &\quad \le \sup_{0<\gamma\le\varepsilon_1} \|T(\gamma)\| |T * f(t-\varepsilon) - T * h(t-\varepsilon)| + 2CL\varepsilon_1\end{aligned}$$

and hence,

$$\begin{aligned}\alpha(t, \varepsilon; \mathcal{K}) &\le \operatorname{dia} \mathcal{K}_i \mid [t-\varepsilon, t+\varepsilon] \\ &\le \sup_{0<t\le\varepsilon_1} \|T(\tau)\| \sup_{a\le s\le t} \alpha(\mathcal{K}(s)) + (2CL + C)\varepsilon_1 ,\end{aligned}$$

from which it follows that $\alpha(t, \mathcal{K}) \le \gamma_T \sup_{a\le s\le t} \alpha(\mathcal{K}(s))$. Using Lemma 1.3 we can obtain

$$\alpha(\mathcal{K} \mid [a, t]) = \sup_{a\le\tau\le t} \alpha(\tau, \mathcal{K}) \le \gamma_T \sup_{a\le\tau\le t} \alpha(\mathcal{K}(\tau))$$

as required.

If $T(t)$ is a C_0-contraction semigroup on E, then $\|T(t)\| \le 1$ on R^+. Hence we have $\gamma_T \le 1$ and so, $\gamma_T = 1$, because of (1.3). q.e.d.

COROLLARY 1.4. *Let $T * \mathcal{U}$ be as in Theorem* 1. *Then $\alpha(T * \mathcal{U}) = 0$ if and only if $\alpha(T * \mathcal{U}(t)) = 0$ for all $t \in [a, b]$.*

REMARK 1.5. In general, if $\mathcal{H} \subset C[a, b]$ is a bounded set, then it follows that

$$\max\left\{\frac{1}{2}\omega(\mathcal{H}), \sup_{a\le t\le b} \alpha(\mathcal{H}(t))\right\} \le \alpha(\mathcal{H}) \le 2\omega(\mathcal{H}) + \sup_{a\le t\le b} \alpha(\mathcal{H}(t))$$

(see [12, Theorem 1]). Also refer to [5]. Theorem 1 refines on the above result for a special case.

Combining Theorem 1 with the relation (1.1) we can easily obtain the following result.

PROPOSITION 1.6. *Let $\mathcal{U}$ be a bounded set in $C[a,b]$ and $T(t)$ a C_0-semigroup on E. Put $\mathcal{K} = \{G_f : (-\infty, b] \to E \mid f \in \mathcal{U}\}$*, where

$$G_f(t) = \begin{cases} \int_a^t T(t-s)f(s)ds & \text{for } t \in [a,b], \\ 0 & \text{for } t \in (-\infty, a]. \end{cases}$$

Then

$$\frac{1}{L}\alpha(\mathcal{K}(t)) \le \alpha(\mathcal{K}_t) \le \gamma_T K(t-a) \sup_{a \le s \le t} \alpha(\mathcal{K}(s)).$$

Define linear operators $S(t) : \mathcal{B} \to \mathcal{B}$, $t \ge 0$, by

$$[S(t)\varphi](\theta) = \begin{cases} \varphi(t+\theta) & \text{if } t+\theta < 0, \\ \varphi(0) & \text{if } t+\theta \ge 0. \end{cases}$$

Furthermore, according to [19], we here assume that $\beta_\mu \in R$, where

$$\beta_\mu = \limsup_{t \to 0+} \frac{1}{t}\{\alpha(S(t)) - 1\}.$$

In general, the following result holds, which refines on Theorem 1.12 in [19] for a special case. Denote by $C^1[a,b]$ the set of all continuously differentiable functions from $[a,b]$ to E and denote by N the set of all positive integers.

PROPOSITION 1.7. *Let $a < b$ and $\mathcal{U} = \{u^n : (-\infty, b] \to E \mid u^n \mid [a,b] \in C^1[a,b]$, $u^n_a \in \mathcal{B}, n \in N\}$. Suppose that $\mathcal{U} \mid [a,b]$ is bounded and equicontinuous and that there exists a $\mu \in \mathcal{L}^1[a,b]$ such that $|\dot{u}^n(t)| \le \mu(t)$ for all $n \in N$ and for a.a. $t \in [a,b]$. Then*

$$\bar{D}_+\alpha(\mathcal{U}_t) \le 2K(0)\alpha(\dot{\mathcal{U}}(t)) + \beta_\mu \alpha(\mathcal{U}_t), \tag{1.8}$$

$$\frac{d}{dt}\alpha(\mathcal{U} \mid [a,t]) \le 2\alpha(\dot{\mathcal{U}}(t)) \quad \text{and} \quad \frac{d}{dt}\alpha(\mathcal{U}(t)) \le 2\alpha(\dot{\mathcal{U}}(t)) \tag{1.9}$$

for a.a. $t \in [a,b]$, where $\bar{D}_+$ denotes the right-hand upper derivative.

PROOF. First, we shall show that the inequality (1.8) holds. Combining the relation (1.1) and Lemma 1.1, we have, for $t \in [a,b)$ and for $h > 0$,

$$\begin{aligned} \alpha(\{u^n_{t+h} - S(h)u^n_t \mid n \in N\}) &\le K(h) \sup_{t \le s \le t+h} \alpha(\{u^n(s) - u^n(t) \mid n \in N\}) \\ &\le K(h) \sup_{t \le s \le t+h} \alpha\left(\left\{\int_t^s \dot{u}^n(\tau)d\tau \,\Big|\, n \in N\right\}\right) \\ &\le 2K(h) \sup_{t \le s \le t+h} \int_t^s \alpha(\dot{\mathcal{U}}(\tau))d\tau \\ &\le 2K(h) \int_t^{t+h} \alpha(\dot{\mathcal{U}}(\tau))d\tau. \end{aligned}$$

Hence we get

$$\alpha(\mathcal{U}_{t+h}) - \alpha(\mathcal{U}_t) \tag{1.10}$$
$$\leq \alpha\left(\left\{u^n_{t+h} - S(h)u^n_t \mid n \in N\right\}\right) + \alpha\left(\left\{S(h)u^n_t \mid n \in N\right\}\right) - \alpha(\mathcal{U}_t)$$
$$\leq 2K(h)\int_t^{t+h} \alpha(\dot{\mathcal{U}}(\tau))d\tau + \{\alpha(S(h)) - 1\}\alpha(\mathcal{U}_t)\,.$$

Moreover, we have that for *a.a.* $t \in [a, b)$,

$$\frac{1}{h}K(h)\int_t^{t+h} \alpha(\dot{\mathcal{U}}(\tau))d\tau \to K(0)\alpha(\dot{\mathcal{U}}(t)) \quad \text{as } h \to 0+\ .$$

Thus, dividing the both sides by h and letting $h \to 0+$ in the above inequality (1.10), we can easily obtain the inequality (1.8).

Next, we shall prove that the inequality (1.9) holds. For $t \in [a, b)$ and for $h > 0$ we have

$$\begin{aligned}\alpha(\mathcal{U} \mid [a, t+h]) - \alpha(\mathcal{U} \mid [a, t]) &= \sup_{a \leq s \leq t+h} \alpha(\mathcal{U}(s)) - \sup_{a \leq s \leq t} \alpha(\mathcal{U}(s)) \\ &\leq \alpha(\mathcal{U}(t+\tau)) - \alpha(\mathcal{U}(t)) \quad \text{for some } \tau \in [0, h] \\ &\leq \alpha\left(\left\{u^n(t+\tau) - u^n(t) \mid n \in N\right\}\right) \\ &\leq 2\int_t^{t+\tau} \alpha(\dot{\mathcal{U}}(s))ds\,,\end{aligned}$$

from which we can easily obtain the relation (1.9). q.e.d.

2. Existence of mild solutions for IP(σ, φ). In this section we will prove existence theorems for IP(σ, φ). For a compact set Γ in $\mathcal{B}$, we set

$$\rho_S(t, \Gamma) = \sup_{0 \leq \tau \leq t} \sup\left\{|(S(\tau) - S(0))\psi|_{\mathcal{B}} \mid \psi \in \Gamma\right\}$$

and

$$r_T(t, \Gamma(0)) = \sup_{0 \leq \tau \leq t} \sup\left\{|(T(\tau) - T(0))\psi(0)| \mid \psi \in \Gamma\right\}.$$

In particular, if $\Gamma = \{\varphi\}$, then we denote $\rho_S(t, \Gamma)$ and $r_T(t, \Gamma(0))$ by $\rho_S(t, \varphi)$ and $r_T(t, \varphi(0))$, respectively. It is obvious that

$$\rho_S(t, \Gamma) \to 0 \quad \text{and} \quad r_T(t, \Gamma(0)) \to 0 \quad \text{as } t \to 0+\ . \tag{2.1}$$

We make the following hypotheses for IP(σ, φ):

(H1) $F : [\sigma, \sigma + a] \times \mathcal{B}(\varphi, r) \to E$, $0 < a < \infty$, is a continuous function such that $|F(t, \psi)| \leq H$ over there.

(H2) A is the infinitesimal generator of a C_0-semigroup $T(t)$ on E.

Throughout this paper we put $C_a = \sup\left\{\|T(t)\| \mid 0 \leq t \leq a\right\}$ in (H2).

LEMMA 2.1. *Suppose that the hypotheses* (H1) *and* (H2) *are satisfied for* IP(σ, φ). *Let*

$$\gamma = \sup\left\{t \in [0, a] \mid \rho_S(t, \varphi) + K_a r_T(t, \varphi(0)) + K_a H C_a t \leq r\right\}. \tag{2.2}$$

Then for each $\varepsilon_n \in (0, \min\{r, \gamma\})$, $\varepsilon_n \to 0$ *monotone as* $n \to \infty$, *there exists a function* $z^n : (-\infty, \sigma + \gamma] \to E$ *such that* $z^n_\sigma = \varphi$, $z^n \mid [\sigma, \sigma + \gamma] \in C[\sigma, \sigma + \gamma]$ *and*

$$\left| z^n(t) - T(t-\sigma)\varphi(0) - \int_\sigma^t T(t-s)F(s, z^n_s)ds \right| \leq 3C_a\varepsilon_n(t-\sigma) \tag{2.3}$$

for all $t \in [\sigma, \sigma + \gamma]$.

For the proof refer to [22, Lemma 5.3].

A function $\omega : (a, b) \times [0, c) \to R$ is said to be a Kamke-type function if the following conditions hold:

(ω_1) $\omega := \omega(t, s)$ is Lebesgue measurable in t for each $s \in [0, c)$ and is continuous in s for a.a. $t \in (a, b)$.

(ω_2) For each $r \in (0, c)$ there exists a function m_r, defined on (a, b) and locally integrable on (a, b), such that $|\omega(t, s)| \leq m_r(t)$ for a.a. $t \in (a, b)$ and all $s \in [0, r]$.

Sometimes, the following condition is needed for a Kamke-type function ω.

(ω_3) $\omega(t, s)$ is nondecreasing in s for a.a. $t \in (a, b)$.

We are now in a position to state the main result in this paper.

THEOREM 2. *Suppose that the hypotheses* (H1) *and* (H2) *are satisfied for* IP(σ, φ). *Then* IP(σ, φ) *has a mild solution existing on* $[\sigma, \sigma + \gamma]$, *where* γ *is as in Lemma* 2.1, *under the assumption that either* $T(t)$ *is a* C_0*-compact semigroup on E or the following conditions are satisfied: There exists a Kamke-type function* $\omega : (\sigma, \sigma + a] \times [0, 2r] \to R^+$ *with* (ω_3) *such that*

(1) $\omega(t, K(t-\sigma)u(t)) \to 0$ *as* $t \to \sigma+$, *where* $K(t)$ *is as in* (B_2) *and* $u : [\sigma, \sigma + a] \to R^+$ *is any continuous function satisfying the condition*

$$\lim_{t \to \sigma+} \frac{u(t)}{t - \sigma} = u(\sigma) = 0; \tag{2.4}$$

(2) *the inequality* $\alpha(F(t, B)) \leq \omega(t, \alpha(B))$ *holds for each bounded set* $B \subset \mathcal{B}(\varphi, r)$ *and for a.a.* $t \in (\sigma, \sigma + a)$; *and*

(3) $u(t) \equiv 0$ *is the unique absolutely continuous function satisfying the equation*

$$\frac{d}{dt}u(t) = 2\gamma_T \sup_{0<\tau\leq a} \hat{\alpha}(T(\tau))\omega(t, K(t-\sigma)u(t)) \quad \textit{for a.a. } t \in (\sigma, \sigma + a) \tag{2.5}$$

with the condition (2.4), *where* γ_T *is as in Theorem* 1.

PROOF. Set $\mathcal{Z}^k = \{z^n : (-\infty, \sigma + \gamma] \to E \mid n \geq k\}$, $k \in N$, and $\mathcal{X}^k = \{x^n : (-\infty, \sigma + \gamma] \to E \mid n \geq k\}$, where z^n is as in Lemma 2.1 and

$$x^n(t) = \begin{cases} T(t-\sigma)\varphi(0) + \displaystyle\int_\sigma^t T(t-s)F(s, z^n_s)ds & \text{for } t \in [\sigma, \sigma + \gamma], \\ \varphi(t - \sigma) & \text{for } t \in (-\infty, \sigma]. \end{cases}$$

Also put $\mathcal{Z} = \mathcal{Z}^1$ and $\mathcal{X} = \mathcal{X}^1$. Using (2.3) we have that for $t \in [\sigma, \sigma + \gamma]$ and for $k \in N$,

$$
\begin{aligned}
|\alpha(\mathcal{Z} \mid [\sigma, t]) - \alpha(\mathcal{X} \mid [\sigma, t])| &\le \alpha\big(\{(z^n - x^n) \mid [\sigma, t] \,\big|\, n \ge k\}\big) \\
&\le 2 \sup_{n \ge k} |z^n - x^n|_{[\sigma,t]} \le 6C_a \varepsilon_k (t - \sigma) .
\end{aligned} \tag{2.6}
$$

Letting $k \to \infty$ in both sides of the above inequality, we get

$$
\alpha(\mathcal{Z} \mid [\sigma, t]) = \alpha(\mathcal{X} \mid [\sigma, t]) . \tag{2.7}
$$

On the other hand, by Theorem 1 and Lemma 1.1 we have

$$
\begin{aligned}
\alpha(\mathcal{X} \mid [\sigma, t]) &\le \alpha\left(\left\{\int_\sigma^\cdot T(\cdot - s)F(s, z_s^n)ds \mid [\sigma, t] \,\Big|\, n \in N\right\}\right) \\
&\le \gamma_T \sup_{\sigma \le \tau \le t} \alpha\left(\left\{\int_\sigma^\tau T(\tau - s)F(s, z_s^n)ds \,\Big|\, n \in N\right\}\right) \\
&\le 2\gamma_T \sup_{\sigma \le \tau \le t} \int_\sigma^\tau \alpha\big(\{T(\tau - s)F(s, z_s^n) \,\big|\, n \in N\}\big)ds \\
&\le 2\gamma_T \sup_{0 < \tau \le a} \hat{\alpha}(T(\tau)) \int_\sigma^t \alpha\big(\{F(s, z_s^n) \,\big|\, n \in N\}\big)ds .
\end{aligned} \tag{2.8}
$$

If $\{T(t)\}_{t \ge 0}$ is a C_0-compact semigroup on E, then it follows from Remark 1.2 that $\hat{\alpha}(T(t)) = 0$ for all $t \in (0, a]$. Hence, from (2.7) and (2.8) it follows that the set $\mathcal{Z} \mid [\sigma, \sigma + \gamma]$ is relatively compact in $C[\sigma, \sigma + \gamma]$.

Let us consider the case where $\sup_{0 < \tau \le a} \hat{\alpha}(T(\tau)) > 0$. If we put

$$
v(t) = 2\gamma_T \sup_{0 < \tau \le a} \hat{\alpha}(T(\tau)) \int_\sigma^t \alpha\big(\{F(s, z_s^n) \,\big|\, n \in N\}\big)ds ,
$$

then $v(t)$ is continuous on $[\sigma, \sigma + \gamma]$, $v(\sigma) = 0$ and $\alpha(\mathcal{X} \mid [\sigma, t]) \le v(t)$. We claim that $\lim_{t \to \sigma+} v(t)/(t - \sigma) = 0$. Using Lemma 2.1 we have

$$
\begin{aligned}
|z_t^n - \varphi|_{\mathcal{B}} &\le |z_t^n - S(t - \sigma)\varphi|_{\mathcal{B}} + |S(t - \sigma)\varphi - \varphi|_{\mathcal{B}} \\
&\le K(t - \sigma) \sup_{\sigma \le s \le t} |z^n(s) - \varphi(0)| + |S(t - \sigma)\varphi - \varphi|_{\mathcal{B}} \\
&\le K_a C_a (H + 3)(t - \sigma) + K_a r_T(t - \sigma, \varphi(0)) + \rho_S(t - \sigma, \varphi) \\
&\to 0 \quad \text{as } t \to \sigma+ .
\end{aligned}
$$

Hence, from the continuity of F we have that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $|F(t, z_t^n) - F(\sigma, \varphi)| < \varepsilon/2$ for all $n \in N$ if $|t - \sigma| < \delta$. From this we have, for $t \in (\sigma, \sigma + \delta)$,

$$
\begin{aligned}
\frac{1}{t - \sigma} \int_\sigma^t \alpha\big(\{F(s, z_s^n) \,\big|\, n \in N\}\big)ds &= \frac{1}{t - \sigma} \int_\sigma^t \alpha\big(\{F(s, z_s^n) - F(\sigma, \varphi) \,\big|\, n \in N\}\big)ds \\
&\le \frac{1}{t - \sigma} \int_\sigma^t \varepsilon \, ds = \varepsilon ,
\end{aligned}
$$

which implies that $\lim_{t \to \sigma+} v(t)/(t - \sigma) = 0$.

Furthermore, using (1.1), (2.7) and (2.8) we see that $\alpha(\mathcal{Z}_s) \leq K(s-\sigma)\alpha(\mathcal{Z} \mid [\sigma, s]) \leq K(s-\sigma)v(s)$ for $s \in (\sigma, \sigma+\gamma]$. Therefore we get, together with the assumption (2),

$$\int_{\sigma}^{t} \alpha\big(\{F(s, z_s^n) \,|\, n \in N\}\big)ds \leq \int_{\sigma+}^{t} \omega(s, K(s-\sigma)v(s))ds < \infty .$$

If we set

$$u(t) = 2\gamma_T \sup_{0<\tau\leq a} \hat{\alpha}(T(\tau)) \int_{\sigma+}^{t} \omega(s, K(s-\sigma)v(s))ds ,$$

then $v(t) \leq u(t)$ and $u(t)$ is absolutely continuous. Hence we find

$$\frac{d}{dt}u(t) \leq 2\gamma_T \sup_{0<\tau\leq a} \hat{\alpha}(T(\tau))\omega(t, K(t-\sigma)u(t)) \quad \text{a.a. } t \in (\sigma, \sigma+\gamma) .$$

Using the assumption (1), we can easily see that $u(t)$ satisfies the condition (2.4). Combining a comparison theorem ([18, Lemma 4.1]) with the assumption (3) we have $u(t) \equiv 0$ and hence, $\alpha(\mathcal{Z} \mid [\sigma, t]) = 0$ for all $t \in [\sigma, \sigma+\gamma]$. Therefore, $\mathcal{Z} \mid [\sigma, \sigma+\gamma]$ is relatively compact in $C[\sigma, \sigma+\gamma]$. By Ascoli-Arzela's theorem, we see that there are a sequence $\{n(i)\} \subset N$ and a function $z : (-\infty, \sigma+\gamma] \to E$ such that $z_\sigma = \varphi$ and $|z^{n(i)} - z|_{[\sigma,\sigma+\gamma]} \to 0$ as $i \to \infty$. Hence it follows from the axiom (B_2) that $z_t^{n(i)} \to z_t$ uniformly on $[\sigma, \sigma+\gamma]$ as $i \to \infty$. By Lebesgue's dominated convergence theorem, we see that the function z is a mild solution of IP(σ, φ). q.e.d.

REMARK 2.2. Recently, Henriquez [7] showed the existence of mild solutions to IP(σ, φ) under the condition that $\alpha(T(t)F([\sigma, \sigma+a] \times \mathcal{B}(\varphi, r))) = 0$ for each $t \in (0, a]$. This condition is satisfied whenever $T(t)$ is a C_0-compact semigroup on E or $F : [\sigma, \sigma+a] \times \mathcal{B}(\varphi, r) \to E$ is a compact operator. Our condition states a sufficient condition on the existence of mild solutions to IP(σ, φ) for the case where both $T(t)$ and F are noncompact operators.

COROLLARY 2.3. *If $T(t)$ is a C_0-contraction semigroup on E in Theorem* 2, *then the equation* (2.5) *is reduced to the equation*

$$\frac{d}{dt}u(t) = 2\omega(t, K(t-\sigma)u(t)) \quad \textit{for a.a. } t \in (\sigma, \sigma+a] .$$

We note that if $\sigma = 0$ and $T(t)$ is a C_0-contraction semigroup on E, then the function $\omega(t, s) = (1+\varepsilon(t))s/2K(t)t$ satisfies the assumption (1) and (3) in Theorem 2, where $\varepsilon : (0, a] \to R^+$ is continuous and $\int_{0+} \varepsilon(t)/t\, dt < \infty$.

COROLLARY 2.4. *If $F(t, \psi) = F(t, \psi(0))$ in Theorem* 2, *then the equation* (2.5) *is reduced to the equation*

$$\frac{d}{dt}u(t) = 2 \sup_{0<\tau\leq a} \hat{\alpha}(T(\tau))\omega(t, u(t)) \quad \textit{for a.a. } t \in [\sigma, \sigma+a] .$$

PROOF. Using the same argument as in the proof of Theorem 2, $\alpha(\mathcal{X}(t))$ is easily estimated as follows:

$$\alpha(\mathcal{X}(t)) \le 2 \sup_{0<\tau\le a} \hat{\alpha}(T(\tau)) \int_\sigma^t \omega(s, \alpha(\mathcal{X}(s)))ds .$$

Hence, we can easily prove the corollary. q.e.d.

REMARK 2.5. Corollary 2.4 generalizes Bothe's result in the single valued case with non-delay on an open set, in which it is assumed that

$$\lim_{h\to 0+} \alpha(F(J_{t,h} \times B)) \le k(t)\alpha(B)$$

on $[\sigma, \sigma + a]$ and for all bounded set $B \subset E$, where $J_{t,h} = [t - h, t] \cap [\sigma, \sigma + a]$ and $k \in \mathcal{L}^1[\sigma, \sigma + a]$.

PROPOSITION 2.6. *Suppose that* $A = 0$ *in* IP(σ, φ) *and Hypothesis* (H1) *is satisfied. Then* IP(σ, φ) *has a solution existing on* $[\sigma, \sigma + \gamma]$ *for some* $\gamma > 0$, *under the following assumptions*:

(1) *There exists a Kamke-type function* $\omega : (\sigma, \sigma + a] \times [0, 2r] \to R^+$ *such that the inequality* $\alpha(F(t, B)) \le \omega(t, \alpha(B))$ *holds for a bounded set* $B \subset \mathcal{B}(\varphi, r)$ *and for a.a.* $t \in [\sigma, \sigma + a]$.

(2) $u(t) \equiv 0$ *is the unique absolutely continuous function with* (2.4), *which satisfies the equation*

$$\frac{d}{dt}u(t) = 2\omega(t, K(t-\sigma)u(t)) \quad \textit{for a.a. } t \in [\sigma, \sigma + a], \tag{2.9}$$

provided that ω *satisfies the condition* (ω_3), *or*

$$\frac{d}{dt}u(t) = 2K(0)\omega(t, u(t)) + \beta_\mu u(t) \quad \textit{for a.a. } t \in [\sigma, \sigma + a]. \tag{2.10}$$

PROOF. Since $A = 0$, we have $T(t) = I$ (the identity operator). Let $\mathcal{Z}$ and $\mathcal{X}$ be as in the proof of Theorem 2 with $T(t) = I$ and $\mathcal{W} = \{w^n : (-\infty, \sigma + \gamma] \to E \mid n \in N\}$, where

$$w^n(t) = \begin{cases} \int_\sigma^t F(s, z_s^n)ds & \text{for } t \in [\sigma, \sigma + \gamma] \\ 0 & \text{for } t \in (-\infty, \sigma]. \end{cases}$$

Then we have that $x_t^n = S(t-\sigma)\varphi + w_t^n$ for $t \in [\sigma, \sigma + \gamma]$. We can easily obtain the following properties:

(i) $|w_t^n - w_s^n|_{\mathcal{B}} \le K_a H|t - s|$ for $t, s \in [\sigma, \sigma + \gamma]$ and for all $n \in N$;

(ii) $|\alpha(\mathcal{W}_t) - \alpha(\mathcal{W}_s)| \le 2K_a H|t - s|$ for all $t, s \in [\sigma, \sigma + \gamma]$; and

(iii) $\alpha(\mathcal{W}_t) = \alpha(\mathcal{X}_t)$ for $t \in [\sigma, \sigma + \gamma]$

(see [19, Lemma 2.1]).

Now, we will show that $\alpha(\mathcal{Z}_t) = \alpha(\mathcal{X}_t)$ for $t \in [\sigma, \sigma+\gamma]$. Using (2.6) and the relation (1.1), we have

$$\begin{aligned}|\alpha(\mathcal{Z}_t) - \alpha(\mathcal{X}_t)| &\le \alpha\big(\{z_t^n - x_t^n \,|\, n \ge k\}\big)\\ &\le K_a\alpha\big(\{(z^n - x^n) \mid [\sigma, t] \,|\, n \ge k\}\big) \le 6K_a\varepsilon_k(t-\sigma),\end{aligned}$$

which implies the assertion.

To complete the proof, it is sufficient to show that $\alpha(\mathcal{Z}_t) = 0$ for all $t \in [\sigma, \sigma+\gamma]$. Using Proposition 1.7, we get

$$\begin{aligned}\bar{D}_+\alpha(\mathcal{W}_t) &\le 2K(0)\alpha(\dot{\mathcal{W}}(t)) + \beta_\mu\alpha(\mathcal{W}_t)\\ &\le 2K(0)\alpha\big(\{F(t, z_t^n) \,|\, n \in N\}\big) + \beta_\mu\alpha(\mathcal{W}_t)\end{aligned} \tag{2.11}$$

for a.a. $t \in [\sigma, \sigma+\gamma]$. Since $\alpha(\mathcal{Z}_t) = \alpha(\mathcal{X}_t) = \alpha(\mathcal{W}_t)$ for $t \in [\sigma, \sigma+\gamma]$, we have, together with the assumption (1),

$$\frac{d}{dt}\alpha(\mathcal{X}_t) \le 2K(0)\omega(t, \alpha(\mathcal{X}_t)) + \beta_\mu\alpha(\mathcal{X}_t) \quad \text{for a.a. } t \in [\sigma, \sigma+\gamma],$$

because of the property (ii). We note that $\lim_{t\to\sigma+}\alpha(\mathcal{X}_t)/(t-\sigma) = 0$ by using (2.11). Put $v(t) = \alpha(\mathcal{X}_t)$ for $t \in [\sigma, \sigma+\gamma]$. Then $v(t)$ is absolutely continuous on $[\sigma, \sigma+\gamma]$ and satisfies the differential inequality

$$\frac{d}{dt}v(t) \le 2K(0)\omega(t, v(t)) + \beta_\mu v(t) \quad \text{for a.a. } t \in [\sigma, \sigma+\gamma].$$

Hence, from a comparison theorem ([18, Lemma 4.1]) and the assumption (2) it follows that $v(t) = 0$ for all $t \in [\sigma, \sigma+\gamma]$. The rest of the proof is easily proved by using Proposition 1.7. q.e.d.

REMARK 2.7. (1) If F is uniformly continuous, then the equations (2.9) and (2.10) in Proposition 2.6 can be replaced by the equations

$$\frac{d}{dt}u(t) = \omega(t, K(t-\sigma)u(t)) \quad \text{for a.a. } t \in [\sigma, \sigma+\gamma]$$

and

$$\frac{d}{dt}u(t) = K(0)\omega(t, u(t)) + \beta_\mu u(t) \quad \text{for a.a. } t \in [\sigma, \sigma+\gamma],$$

respectively (refer to [17], [19], [20]).

(2) Proposition 2.6 is an extension of the result due to Mönch and Harten [13].

EXAMPLE. Let us consider the initial value problem of the integro-partial differential equation:

$$\begin{aligned}\frac{\partial u(t,x)}{\partial t} =& \frac{1}{4}\frac{\partial^2 u(t,x)}{\partial x^2} + \frac{1}{2}\int_{-\infty}^{t} A(s-t, x) f(t, u(s,x))ds\\ &+ \frac{1}{2}B(t,x)u(t-r, x) + \int_{-\infty}^{\infty} G(t,x,y)g(t, u(t-r, y))dy,\end{aligned} \tag{2.12}$$

for $(t, x) \in [0, \infty) \times R$, with the initial condition

$$u(t, x) = \varphi(t, x) \quad \text{for } (t, x) \in (-\infty, 0] \times R\,, \tag{2.13}$$
$$\lim_{t \to 0+} u(t, x) = \varphi(0, x) \quad \text{and} \quad \varphi \in \mathcal{L}\,.$$

Here we take

$$C = \Big\{f : R \to R \,\Big|\, f \text{ is a continuous function such that } \lim_{x \to +\infty} f(x) \text{ and } \lim_{x \to -\infty} f(x) \text{ exist}\Big\}$$

and

$$\mathcal{L} = \Big\{\varphi : (-\infty, 0] \to C \,\Big|\, \varphi \text{ is measurable on } (-\infty, -r],\ r > 0, \text{ continuous on } [-r, 0] \text{ and } |\varphi|_{\mathcal{L}} < \infty\Big\}\,,$$

where

$$|\varphi|_{\mathcal{L}} = \sup_{-r \le \theta \le 0} |\varphi(\theta)|_C + \int_{-\infty}^{0} e^{\theta} |\varphi(\theta)|_C d\theta\,.$$

Then C is a Banach space with the supremum norm $|\cdot|_C$ and the phase space $\mathcal{L}$ satisfies the axioms (B_1)–(B_4) with $K(t) = 2 - e^{-t}$.

Further, we define a C_0-semigroup $T(t)$ on C with $\|T(t)\| = 1$, as

$$[T(t)u](x) = \frac{1}{(\pi t)^{1/2}} \int_{-\infty}^{\infty} e^{-(x-y)^2/t} u(y)dy \quad \text{for } t > 0 \quad \text{and} \quad u \in C\,,$$

$T(0) = I$. Then the infinitesimal generator $\mathcal{A}$ of the C_0-semigroup $T(t)$ is given by

$$(\mathcal{A}u)(x) = \frac{1}{4}\frac{d^2}{dx^2} u(x) \quad \text{for } u \in \mathcal{D}(\mathcal{A})\,,$$

$$\mathcal{D}(\mathcal{A}) = \left\{ u \in C \,\middle|\, \frac{d}{dx}u(x) \in C,\ \frac{d^2}{dx^2}u(x) \in C \right\}.$$

Assume that

(1) $A : (-\infty, 0] \times R \to R$, $B : [0, \infty) \times R \to R$ are continuous functions such that $|A(\theta, x)| \le e^{\theta}$, $|B(t, x)| \le 1$, $A(\theta, \cdot) \in C$ and $B(t, \cdot) \in C$;

(2) $G : [0, \infty) \times R \times R \to R$ is a continuous function with compact support with respect to $(x, y) \in R \times R$ for each $t \in [0, \infty)$ and $g : [0, \infty) \times R \to R$ is a continuous function; and

(3) $f : [0, \infty) \times R \to R$ is a bounded continuous function and satisfies the inequality

$$|f(t, x) - f(t, y)| \le \frac{|x - y|}{t} \quad \text{for } t > 0\,.$$

As such a function, we can take the function $f(t, x) = t \sin(x/t^2)$.

Now, we define functions as follows: For $(t, \psi) \in [0, \infty) \times \mathcal{L}$,

$$F_1(t, \psi)(x) = \frac{1}{2}\int_{-\infty}^{0} A(\theta, x) f(t, \psi(\theta, x))d\theta + \frac{1}{2}B(t, x)\psi(-r, x)\,,$$

$$F_2(t, \psi)(x) = \int_{-\infty}^{\infty} G(t, x, y) g(t, \psi(-r, y)) dy$$

and

$$\mathcal{F}(t, \psi)(x) = F_1(t, \psi)(x) + F_2(t, \psi)(x) .$$

Then the existence of mild solutions to the initial value problem (2.12)–(2.13) is reduced to the existence of mild solutions to the abstract initial value problem given as

$$\frac{dw}{dt} = \mathcal{A}w + \mathcal{F}(t, w_t), \quad t > 0, \quad \text{and} \quad w_0 = \varphi \in \mathcal{L}. \tag{2.14}$$

Applying Theorem 2 to the above problem, we shall show the local existence of mild solutions to the initial value problem (2.14). Hence, we may assume that $\mathcal{D}(\mathcal{F}) = [0, 1] \times \mathcal{L}(\varphi, 1)$, where $\mathcal{L}(\varphi, 1) = \{\eta \in \mathcal{L} \mid |\eta - \varphi|_{\mathcal{L}} \le 1\}$. Then it is easy to see that $\mathcal{F}$ is bounded and continuous.

First, we consider the function F_1. For $(t, \varphi_1), (t, \varphi_2) \in \mathcal{D}(\mathcal{F})$, $t \neq 0$, we have, by the assumptions (1) and (3),

$$\begin{aligned} &|F_1(t, \varphi_1)(x) - F_1(t, \varphi_2)(x)| \\ &\quad \le \frac{1}{2} \left\{ \int_{-\infty}^{0} e^{\theta} |f(t, \varphi_1(\theta, x)) - f(t, \varphi_2(\theta, x))| d\theta + |\varphi_1(-r, x) - \varphi_2(-r, x)| \right\} \\ &\quad \le \frac{1}{2t} \left\{ \int_{-\infty}^{0} e^{\theta} |\varphi_1(\theta, x) - \varphi_2(\theta, x)| d\theta + |\varphi_1(-r, x) - \varphi_2(-r, x)| \right\}, \end{aligned}$$

from which we get

$$|F_1(t, \varphi_1) - F_1(t, \varphi_2)|_C \le \frac{1}{2t} |\varphi_1 - \varphi_2|_{\mathcal{L}} .$$

Next, we show that $\alpha(F_2(t, \mathcal{L}(\varphi, 1))) = 0$ for every $t \in [0, 1]$. Take any $t \in [0, 1]$ and any sequence $\{h_n(t)\} \subset F_2(t, \mathcal{L}(\varphi, 1))$. Then there exists a $\varphi_n \in \mathcal{L}(\varphi, 1)$ such that $h_n(t) = F_2(t, \varphi_n)$. From the assumption (2) we have that there exists a positive number M such that $G(t, x, y) = 0$ for $(x, y) \in R \times R \setminus \{[-M, M] \times [-M, M]\}$ and that $G(t, x, y)$ is uniformly continuous in $(x, y) \in R \times R$. Put $G(t) = \max \{|G(t, x, y)| \mid (x, y) \in R \times R\}$ and $g(t) = \max \{|g(t, u)| \mid |u| \le |\varphi|_{\mathcal{L}} + 1\}$. Since $|\varphi_n(-r, y)| \le |\varphi_n(-r)|_C \le |\varphi_n|_{\mathcal{L}} \le |\varphi|_{\mathcal{L}} + 1$, $n \in N$, we have

$$\begin{aligned} |h_n(t)(x)| &\le \int_{-M}^{M} |G(t, x, y)| \, |g(t, \varphi_n(-r, y))| dy \\ &\le 2MG(t) g(t) , \end{aligned}$$

which implies that $\{h_n(t)(x)\}$ is uniformly bounded.

Furthermore, we have, for $u, v \in R$,

$$\begin{aligned} |h_n(t)(u) - h_n(t)(v)| &\le \int_{-M}^{M} |G(t, u, y) - G(t, v, y)| \, |g(t, \varphi_n(-r, y))| dy \\ &\le g(t) \int_{-M}^{M} |G(t, u, y) - G(t, v, y)| dy . \end{aligned}$$

From this and the uniform continuity of $G(t, x, y)$ it follows that $\{h_n(t)(x)\}$ is uniformly equicontinuous. Therefore, using Ascoli-Arzela's theorem, we see that $\alpha(F_2(t, \mathcal{L}(\varphi, 1))) = 0$ for every $t \in [0, 1]$.

From the above results, we have the following: For $B \subset \mathcal{L}(\varphi, 1)$,

$$\begin{aligned}\alpha(\mathcal{F}(t, B)) &\leq \alpha(F_1(t, B)) + \alpha(F_2(t, B)) \\ &= \alpha(F_1(t, B)) \\ &\leq \frac{1}{2t}\alpha(B).\end{aligned}$$

Hence we can take $\omega(t, s) = s/2t$ as a Kamke-type function in Theorem 2. So, the equation (2.5) becomes

$$\frac{dz(t)}{dt} = \frac{1}{t}(2 - e^{-t})z(t),$$

because $\|T(t)\| = 1$ for all $t \geq 0$. Therefore all conditions in Theorem 2 are satisfied (cf. [17, Corollary 3.1]) and hence, there exists a mild solution to the initial value problem (2.14).

We note that, as shown in the above, Theorem 2 is applicable to the above initial value problem, but in general, Henriquez's result is not.

3. Hypotheses and some lemmas. In this section we shall give some lemmas to show the continuous dependence of mild solutions for IP(σ, φ). If u is a mild solution of IP(σ, φ), then we say that u is a solution of IP(T, F, σ, φ).

Hereafter, in IP(T, F, σ, φ) we will use the following hypothesis instead of (H1):

(H1-0) F is continuous on D, whee D is an open set of $R \times \mathcal{B}$.

Put $\mathrm{pr}\, D = \{t \in R \mid (t, \psi) \in D$ for some $\psi \in \mathcal{B}\}$ for a $D \subset R \times \mathcal{B}$ and $\mathcal{L}_{loc}(\mathrm{pr}\, D) = \{g : \mathrm{pr}\, D \to E \mid g$ is locally integrable$\}$.

First, we list the following hypotheses to discuss a continuous dependence of mild solutions for IP(σ, φ) in Section 4.

(C1) $(\sigma_n, \varphi_n) \to (\sigma, \varphi) \in D$ as $n \to \infty$.

(C2) $T_n(t)$, $n \in N$, is a C_0-semigroup on E and for each $x \in E$, $T_n(t)x \to T(t)x$ as $n \to \infty$ uniformly on every compact interval $[0, a]$ in R^+.

(C3) $F_n : D \to E$, $n \in N$, is continuous and $\{F_n\}_{n \in N}$ is uniformly bounded on every closed bounded subset of D.

(C4) $F_n(t, \psi) \to F(t, \psi)$ as $n \to \infty$ uniformly on every compact subset of D.

(C5) $\int_J |g_n(t)|dt \to 0$ as $n \to \infty$ for every compact interval $J \subset \mathrm{pr}\, D$, where $g_n \in \mathcal{L}_{loc}(\mathrm{pr}\, D)$, $n \in N$.

(C6) For every $(\tau, \psi) \in D$ and $n \in N$, IP$(T_n, F_n + g_n, \tau, \psi)$ has a local solution.

LEMMA 3.1. *Let $a > 0$ and suppose that Hypotheses* (H2) *and* (C2) *are satisfied. Then*

$$\sup\{\|T_n(t)\| \mid n \in N, t \in [0, a]\} < \infty.$$

PROOF. It follows from (C2) that for any $x \in E$ there exists an $N_x \in \boldsymbol{N}$ such that $\sup_{t\in[0,a]} |T_n(t)x - T(t)x| \le 1$ for all $n \ge N_x$. Thus we have

$$\begin{aligned}\sup_{t\in[0,a]} |T_n(t)x| &\le \sup_{t\in[0,a]} |T(t)x| + \sup_{t\in[0,a]} |T_n(t)x - T(t)x| \\ &\le \sup_{t\in[0,a]} \|T(t)\||x| + 1 \quad \text{for all } n \ge N_x ,\end{aligned}$$

and hence $\sup_{t\in[0,a]} |T_n(t)x| \le C_x$ for all $n \in \boldsymbol{N}$ and for some $C_x > 0$. This implies that $\sup_{n\in\boldsymbol{N}} \sup_{t\in[0,a]} |T_n(t)x| < \infty$ for all $x \in E$. Hence the assertion of the lemma follows from the uniform boundedness theorem. q.e.d.

LEMMA 3.2. *Let $a > 0$ and Ω be a compact subset in E. Suppose that Hypotheses* (H2) *and* (C2) *are satisfied. Then for any $\varepsilon > 0$ there exist a positive integer N_0 and a positive number δ such that*

$$|T_n(t)x - T(s)y| < \varepsilon \quad \text{for all } n \ge N_0 , \tag{3.1}$$

if $|t - s| < \delta$, $t, s \in [0, a]$, and $|x - y| < \delta$, $x, y \in \Omega$.

PROOF. Assume that the conclusion is not true. Then we may assume that there exist some $\varepsilon_0 > 0$, $\{n(k)\} \subset \boldsymbol{N}$, $n(k) > k$, $\{t_k\}$, $\{s_k\}$, $\{x_k\}$ and $\{y_k\}$ such that $t_k \to \tau_0$, $s_k \to \tau_0$, $x_k \to z_0$ and $y_k \to z_0$ as $k \to \infty$, where $\tau_0 \in [0, a]$ and $z_0 \in \Omega$, and that $\varepsilon_0 \le |T_{n(k)}(t_k)x_k - T(s_k)y_k|$ for all $k \in \boldsymbol{N}$. From (C2) we have

$$\begin{aligned} &|T_{n(k)}(t_k)z_0 - T(\tau_0)z_0| \\ &\quad\le |T_{n(k)}(t_k)z_0 - T(t_k)z_0| + |T(t_k)z_0 - T(\tau_0)z_0| \\ &\quad\le \sup_{0\le t\le a} |T_{n(k)}(t)z_0 - T(t)z_0| + |T(t_k)z_0 - T(\tau_0)z_0| \\ &\quad\to 0 \quad \text{as } k \to \infty . \end{aligned} \tag{3.2}$$

Put $C = \sup\left\{\|T_n(t)\| \,\middle|\, n \in \boldsymbol{N}, t \in [0, a]\right\}$. Then $1 \le C < \infty$ by Lemma 3.1. Hence we have, by (3.2),

$$\begin{aligned} \varepsilon_0 &\le |T_{n(k)}(t_k)x_k - T(s_k)y_k| \\ &\le |T_{n(k)}(t_k)x_k - T_{n(k)}(t_k)z_0| + |T_{n(k)}(t_k)z_0 - T(\tau_0)z_0| \\ &\quad + |T(\tau_0)z_0 - T(s_k)z_0| + |T(s_k)z_0 - T(s_k)y_k| \\ &\le \|T_{n(k)}(t_k)\|\,|x_k - z_0| + |T_{n(k)}(t_k)z_0 - T(\tau_0)z_0| \\ &\quad + |T(\tau_0)z_0 - T(s_k)z_0| + \|T(s_k)\|\,|y_k - z_0| \\ &\le C|x_k - z_0| + C_a|y_k - z_0| + |T_{n(k)}(t_k)z_0 - T(\tau_0)z_0| \\ &\quad + |T(\tau_0)z_0 - T(s_k)z_0| \\ &\to 0 \quad \text{as } k \to \infty , \end{aligned}$$

which yields a contradiction. q.e.d.

The following results are directly obtained from Lemma 3.2.

COROLLARY 3.3. *Suppose that Hypotheses* (H2) *and* (C2) *are satisfied. Then*

$$|T_n(t_n)x^n - T(t_0)x^0| \to 0 \quad \text{as } n \to \infty\,,$$

if $t_n \to t_0 \in R^+$ *and* $x^n \to x^0 \in E$ *as* $n \to \infty$.

COROLLARY 3.4. *Let the same assumptions as in Lemma* 3.2 *be satisfied. Then for any* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that* $|T_n(t)x - T_n(s)y| < \varepsilon$ *for all* $n \in N$, *if* $|t - s| < \delta$, $s, t \in [0, a]$, *and* $|x - y| < \delta$, $x, y \in \Omega$.

Denote by $\mathrm{NS}(T, F, \sigma, \varphi)$ the set of all noncontinuable solutions for $\mathrm{IP}(T, F, \sigma, \varphi)$ and by τ_x the final time of the existence-interval of x in $\mathrm{NS}(T, F, \sigma, \varphi)$.

LEMMA 3.5. *Let* $(\sigma, \varphi) \in D$ *and* $g \in \mathcal{L}_{loc}(\mathrm{pr}\, D)$. *Suppose that Hypotheses* (H1-0) *and* (H2) *are satisfied and that for each* $(\tau, \psi) \in D$, $\mathrm{IP}(T, F + g, \tau, \psi)$ *has a local solution. Then there exists a positive number* γ *such that*

$$[\sigma, \sigma + \gamma] \subset [\sigma, \tau_x) \quad \text{for all } x \in \mathrm{NS}(T, F + g, \sigma, \varphi)\,.$$

PROOF. Since $(\sigma, \varphi) \in D$, it follows from (H1-0) that there exists a positive numbers a, r and H such that $\Omega := [\sigma - a, \sigma + a] \times \mathcal{B}(\varphi, r) \subset D$ and $|F| \le H$ over there. Set

$$\delta(t) = K_a r_T(t, \varphi(0)) + \rho_S(t, \varphi) + K_a H C_a t + K_a C_a \int_\sigma^{\sigma+t} |g(s)|ds \quad \text{for } t \in [0, a]$$

and $\gamma_0 = \sup\{t \in [0, a] \mid \delta(t) \le r\} > 0$. Clearly, $\delta(t)$ is continuous in $t \in [0, a]$, $\delta(t) > \delta(s)$ if $t > s$, and $\delta(\gamma_0) \le r$.

Now we shall show that the number γ_0 is a required one. Suppose it is not true. Then there exists a solution $z \in \mathrm{NS}(T, F, \sigma, \varphi)$ such that $\tau_0 := \tau_z - \sigma \le \gamma_0$. Set

$$\tau = \sup\{t > \sigma \mid (s, z_s) \in \Omega \text{ for all } s \in [\sigma, t]\} - \sigma\,.$$

Then it is obvious that $0 < \tau \le \tau_0$. Assume that $\tau = \tau_0$. Then we have $(t, z_t) \in \Omega$ for $t \in [\sigma, \sigma + \tau)$. Using the same argument as in the proof of [15, Theorem 2.2], we see that $z(t)$ can be continued beyond $\sigma + \tau$, which yields a contradiction with the maximality of τ_0. Hence $\tau < \tau_0$. Since $\tau_0 \le \gamma_0$, we have $\tau < \gamma_0$. On the other hand, we have

$$\begin{aligned}
|z_{\sigma+\tau} - \varphi|_{\mathcal{B}} &\le |z_{\sigma+\tau} - S(\tau)\varphi|_{\mathcal{B}} + |S(\tau)\varphi - \varphi|_{\mathcal{B}} \\
&\le K_a \sup\{|z(t) - \varphi(0)| \mid \sigma \le t \le \sigma + \tau\} + |S(\tau)\varphi - \varphi|_{\mathcal{B}} \\
&\le K_a \sup\{|T(t - \sigma)\varphi(0) - \varphi(0)| \mid \sigma \le t \le \sigma + \tau\} + |S(\tau)\varphi - \varphi|_{\mathcal{B}} \\
&\quad + K_a \sup_{\sigma \le t \le \sigma+\tau} \left| \int_\sigma^t T(t - s)F(s, z_s)ds \right| + K_a C_a \int_\sigma^{\sigma+\tau} |g(s)|ds \\
&\le \delta(\tau) < \delta(\gamma_0) \le r\,,
\end{aligned}$$

which implies that $(\sigma + \tau, z_{\sigma+\tau})$ is an interior point of Ω. From the assumption we see that $z(t)$ can be continued beyond $\sigma + \tau$. This is a contradiction with the definition of τ. q.e.d.

LEMMA 3.6. *Suppose that Hypotheses* (H1-0), (H2), (C1)–(C3) *and* (C5)–(C6) *are satisfied. Let* $x^n \in \mathrm{NS}(T_n, F_n + g_n, \sigma_n, \varphi_n)$, $n \in N$, *and* $\tau_n = \tau_{x^n} - \sigma_n$. *Then there exists a* *positive number γ such that*

$$[\sigma_n, \sigma_n + \gamma] \subset [\sigma_n, \sigma_n + \tau_n) \quad \text{for all } n \in N.$$

PROOF. Let $\Gamma = \{\varphi_n\} \cup \{\varphi\}$. It is obvious that Γ is compact in $\mathcal{B}$. From assumptions, Lemma 3.1 and Lemma 3.2 it follows that there are positive numbers a, r, C and H such that $[\sigma - a, \sigma + a] \times \mathcal{B}(\varphi, r) \subset D$ and that for all $n \in N$ (without loss of generality)

$$(\sigma_n, \varphi_n) \in \left[\sigma - \frac{a}{2}, \sigma + \frac{a}{2}\right] \times \mathcal{B}\left(\varphi, \frac{r}{2}\right),$$

$$\sup_{0 \le t \le a/2} \sup\left\{|[T_n(t) - T(t)]\psi(0)| \,\middle|\, \psi \in \Gamma\right\} < \frac{r}{12K_a},$$

$$\sup\left\{\|T_n(t)\| \,\middle|\, 0 \le t \le a\right\} \le C, \quad \max\{|F|, |F_n|\} \le H + 1 \quad \text{on } [\sigma - a, \sigma + a] \times \mathcal{B}(\varphi, r),$$

and

$$C \sup_{0 \le t \le a/2} \int_{\sigma_n}^{\sigma_n + t} |g_n(s)| ds < \frac{r}{12K_a}.$$

Set

$$\delta_n(t) = K_a r_{T_n}(t, \varphi_n(0)) + \rho_S(t, \varphi_n) + K_a C(H+1)t + K_a C \int_{\sigma_n}^{\sigma_n + t} |g_n(s)| ds,$$

$$\delta(t) = K_a r_T(t, \Gamma(0)) + \rho_S(t, \Gamma) + K_a C(H+1)t,$$

$$\gamma_n = \sup\left\{t \in [0, a/2] \,\middle|\, \delta_n(t) \le r/2\right\} > 0 \quad \text{and} \quad \gamma = \sup\left\{t \in [0, a/2] \,\middle|\, \delta(t) \le r/3\right\} > 0.$$

By Lemma 3.5, we have $[\sigma_n, \sigma_n + \gamma_n] \subset [\sigma_n, \sigma_n + \tau_n)$ for all $n \in N$.

On the other hand, we have, for all $n \in N$ and for $t \in [0, a/2]$,

$$\begin{aligned} r_{T_n}(t, \varphi_n(0)) &= \sup_{0 \le \tau \le t} |[T_n(\tau) - T_n(0)]\varphi_n(0)| \\ &\le \sup_{0 \le \tau \le t} |[T_n(\tau) - T(\tau)]\varphi_n(0)| + \sup_{0 \le \tau \le t} |[T(\tau) - T(0)]\varphi_n(0)| \\ &\le \frac{r}{12K_a} + r_T(t, \Gamma(0)), \end{aligned}$$

and so,

$$\begin{aligned} \delta_n(t) &\le \delta(t) + \frac{r}{12} + K_a C \int_{\sigma_n}^{\sigma_n + t} |g_n(s)| ds \\ &\le \delta(t) + \frac{r}{6}. \end{aligned}$$

This implies that $\delta_n(\gamma) \le r/2$ for all $n \in N$. From the definition of γ_n we see that $0 < \gamma \le \gamma_n$ for all $n \in N$. q.e.d.

4. Continuous dependence of mild solutions for IP(σ, φ). In this section we will discuss a continuous dependence of mild solutions for IP(σ, φ). For a function $x : [a, b] \to E$, define a function $\hat{x} : (-\infty, b] \to E$ as follows:

$$\hat{x}(t) = \begin{cases} x(t) & \text{for } t \in [a, b] \\ x(a) & \text{for } t \in (-\infty, a]. \end{cases}$$

The following result is a modification of Ascoli-Arzelà's theorem. Since the proof is not difficult, it is omitted.

LEMMA 4.1. *Let* $a_n \to a$, $a < b$, *as* $n \to \infty$. *Take a bounded sequence of continuous functions* $x^n : [a_n, b] \to E$, $n \in N$, *such that* $\{x^n(a_n)\}$ *converges. Assume that*

(1) $\{\hat{x}^n \mid [a,b] \bigm| n \in N\}$ *is uniformly equicontinuous*;

(2) $\alpha\big(\{x^n(t) \bigm| n \in N\}\big) = 0$ *for each* $t \in (a, b]$; *and*

(3) $\sup\big\{|\hat{x}^n(t) - x^n(a_n)| \bigm| \min\{a_n, a\} \le t \le a\big\} \to 0$ *as* $n \to \infty$.

Then there exist a continuous function $x \mid [a,b]$ *and a subsequence* $\{n(i)\} \subset N$ *such that* $|x^{n(i)} - x|_{[s(n(i)),b]} \to 0$ *as* $i \to \infty$, *where* $s(n) = \max\{a_n, a\}$.

To state the main result in the present paper, we will make use of the following hypothesis:

(B) $T(t)$ is a C_0-compact semigroup on E or F satisfies the hypothesis (H1-0) and the following condition: For every point $(\tau, \psi) \in D$ there exist positive numbers a, r and a Kamke-type function $\omega : (\tau, \tau + a] \times [0, 2r] \to R^+$ with (ω_3) satisfying the following properties:

(1) $[\tau - a, \tau + a] \times \mathcal{B}(\psi, r) \subset D$.

(2) $\alpha(F(t, B)) \le \omega(t, \alpha(B))$ holds for each bounded set $B \subset \mathcal{B}(\psi, r)$ and for a.a. $t \in (\tau, \tau + a]$.

(3) $\omega(t, K(t-\tau)u(t)) \to 0$ as $t \to \tau+$, where $K(t)$ is as in (B_2) and $u : [\tau, \tau + a] \to R$ is any continuous function such that

$$\lim_{t \to \tau+} \frac{u(t)}{t - \tau} = u(\tau) = 0. \tag{4.1}$$

(4) $u(t) \equiv 0$ is the unique absolutely continuous function with the condition (4.1), which satisfies the equation

$$\frac{d}{dt}u(t) = 2\gamma_T \sup_{0<s\le a} \hat{\alpha}(T(s))\omega(t, K(t-\tau)u(t)) \quad \text{for a.a. } t \in (\tau, \tau + a). \tag{4.2}$$

We are now in a positive to show the main theorem in the present paper.

THEOREM 3. *Let Hypotheses* (H1-0) *and* (H2) *be satisfied and let* $x^n \in \mathrm{NS}(T_n, F_n + g_n, \sigma_n, \varphi_n)$, $n \in N$, *and* $\beta_n = \tau_{x^n} - \sigma_n$. *Assume that*

(1) *Hypotheses* (C1), (C3)–(C6) *and* (B) *are satisfied*;

(2) $T_n(t)$, $n \in N$, *is a* C_0-*semigroup on* E, *and for each closed bounded subset* $\Lambda \subset E$ *and for each* $a \in (0, \infty)$

$$\sup_{x \in \Lambda} \sup_{t \in [0,a]} |T_n(t)x - T(t)x| \to 0 \quad \text{as } n \to \infty; \quad \text{and}$$

(3) *for each closed bounded set* Ω *of* D

$$\alpha\big(\{F_n(t,\eta) - F(t,\eta) \bigm| (t,\eta) \in \Omega, n \ge k\}\big) \to 0 \quad \text{as } k \to \infty.$$

Then there exist a subsequence $\{x^{n(i)}\}$ *of* $\{x^n\}$ *and an* $x^0 \in \mathrm{NS}(T, F, \sigma, \varphi)$ *such that the following conditions hold*:

(i) $\beta_0 \le \liminf_{i\to\infty} \beta_{n(i)}$, *where* $\beta_0 = \tau_{x^0} - \sigma$; *and*

(ii) $|x^{n(i)} - x^0|_{[s(n(i)),d]} \to 0$ *as* $i \to \infty$ *for every* $d \in (\sigma, \sigma + \beta_0)$, *where* $s(n) = \max\{\sigma, \sigma_n\}$.

PROOF. Let $\{(\tau_n, x^n_{\tau_n})\}_{n\in N}$ be any sequence such that $(\tau_n, x^n_{\tau_n}) \to (\tau, \psi) \in D$ as $n \to \infty$. Put $\psi_n = x^n_{\tau_n}$ and $\Gamma = \{\psi_n\}_{n\in N} \cup \{\psi\}$. Then from (H1-0), (C3) and Lemma 3.1 it follows that there are positive numbers a, r, C and H such that $\Omega := [\tau - a, \tau + a] \times \mathcal{B}(\psi, r) \subset D$, $\sup\{\|T_n(t)\| \mid 0 \le t \le a\} \le C$ for all $n \in N$ and $\max\{|F|, |F_n|\} \le H + 1$ on Ω for all $n \in N$. Moreover, using Lemma 3.6 we see that there is a positive number γ, $\tau < \gamma < \tau + a$, such that $(s, x^n_s) \in \Omega$ for all $s \in [\tau_n, \gamma]$. Put $f_n(s) = F_n(s, x^n_s)$ and $f^0_n(s) = F(s, x^n_s)$ for all $t \in [\tau_n, \gamma]$,

$$y^n(t) = \begin{cases} \displaystyle\int_{\tau_n}^t T_n(t-s) f_n(s)ds & \text{for } t \in [\tau_n, \gamma] \\ 0 & \text{for } t \in (-\infty, \tau_n], \end{cases}$$

$$z^n(t) = \begin{cases} \displaystyle\int_{\tau_n}^t T_n(t-s) g_n(s)ds & \text{for } t \in [\tau_n, \gamma] \\ 0 & \text{for } t \in (-\infty, \tau_n] \end{cases}$$

and

$$w^n(t) = \begin{cases} T_n(t-\tau_n)\psi_n(0) & \text{for } t \in [\tau_n, \gamma] \\ \psi_n(t-\tau_n) & \text{for } t \in (-\infty, \tau_n]. \end{cases}$$

Then $x^n(t) = y^n(t) + z^n(t) + w^n(t)$ for $t \in (-\infty, \gamma]$. For the function $w^n \mid [\tau_n, \gamma]$, $n \in N$, we define a function $\hat{w}^n : (-\infty, \gamma] \to E$, as before; that is, $\hat{w}^n(t) = w^n(t)$ for $t \in [\tau_n, \gamma]$, while $\hat{w}^n(t) = w^n(\tau_n)$ for $t \in (-\infty, \tau_n]$. Set $\hat{x}^n \mid [\tau, \gamma] = (y^n + z^n + \hat{w}^n) \mid [\tau, \gamma]$. Clearly, $\{\hat{x}^n \mid [\tau, \gamma] \mid n \in N\}$ is uniformly bounded.

The proof will be divided into three parts as follows.

Step 1. We will prove that there exist a subsequence $\{n(i)\}_{i\in N}$ of N and a solution x^0 of IP(T, F, τ, ψ) such that $|x^{n(i)} - x^0|_{[\hat{s}(n(i)),\gamma]} \to 0$ as $i \to \infty$, where $\hat{s}(n) = \max\{\tau, \tau_n\}$.

To show Step 1, we will check all conditions in Lemma 4.1.

First of all, we shall show that the condition (3) in Lemma 4.1 is satisfied. It is sufficient to see the case where $\tau_n < \tau$. Then we have

$$\begin{aligned} &\sup_{\tau_n \le t \le \tau} |x^n(t) - \psi_n(0)| \\ &\qquad \le \sup_{\tau_n \le t \le \tau} |T_n(t-\tau_n)\psi_n(0) - \psi_n(0)| + C(H+1)|\tau - \tau_n| + C\int_{\tau_n}^{\tau} |g_n(s)|ds. \end{aligned}$$

Since $(\tau_n, \psi_n) \to (\tau, \psi) \in D$ as $n \to \infty$, in view of Corollary 3.3, we can easily see that the condition (3) in Lemma 4.1 is satisfied. Since $(\tau_n, \psi_n) \to (\tau, \psi) \in D$ as $n \to \infty$, $\{\hat{w}^n \mid [\tau, \gamma] \mid n \in N\}$ is uniformly equicontinuous by using Lemma 3.2. From this, $\{\hat{x}^n \mid [\tau, \gamma] \mid n \in N\}$ is also uniformly equicontinuous, and hence the condition (1) in Lemma 4.1 is satisfied.

Next, we will check the condition (2) in Lemma 4.1. Without loss of generality, we may assume that $\tau \le \tau_n$ for all $n \in N$. We here assume that

(C) $\{y^n \mid [\tau,\gamma] \mid n \in N\}$ is relatively compact in $C[\tau,\gamma]$ (for the proof, see Step 2).

To check the condition (2), we shall show that $\alpha(\{\hat{x}^n \mid [\tau,\gamma] \mid n \in N\}) = 0$; that is, $\alpha(\{z^n \mid [\tau,\gamma] \mid n \in N\}) = 0$ and $\alpha(\{\hat{w}^n \mid [\tau,\gamma] \mid n \in N\}) = 0$. We note that this result is used in Step 2. Since $\{\hat{w}^n \mid [\tau,\gamma] \mid n \in N\}$ is uniformly equicontinuous, we have

$$\begin{aligned}\sup_{\tau\le t\le\tau+\delta} \alpha(\{\hat{w}^n(t) \mid n \in N\}) &\le \sup_{\tau\le t\le\tau+\delta} \alpha(\{\hat{w}^n(t) - \hat{w}^n(\tau) \mid n \in N\}) + \alpha(\Gamma(0)) \\ &\le 2 \sup_{\tau\le t\le\tau+\delta} \{|\hat{w}^n(t) - \hat{w}^n(\tau)| \mid n \in N\} \\ &\to 0 \quad \text{as } \delta \to 0\end{aligned}$$

and

$$\begin{aligned}\sup_{\tau+\delta\le t\le\gamma} \alpha(\{w^n(t) \mid n \in N\}) &\le \sup_{\tau+\delta\le t\le\gamma} \alpha(\{T_n(t-\tau_n)\psi_n(0) - T(t-\tau)\psi_n(0) \mid n \ge k\}) \\ &\quad + \sup_{\tau+\delta\le t\le\gamma} \alpha(T(t-\tau))\alpha(\Gamma(0)) \\ &\le 2 \sup_{n\ge k} |T_n(\cdot-\tau_n)\psi_n(0) - T(\cdot-\tau)\psi_n(0)|_{[\tau+\delta,\gamma]} \\ &\to 0 \quad \text{as } k \to \infty,\end{aligned}$$

from which it follows that

$$\alpha(\{\hat{w}^n \mid [\tau,\gamma] \mid n \in N\}) = \sup_{\tau\le t\le\gamma} \alpha(\{\hat{w}^n(t) \mid n \in N\}) = 0.$$

Furthermore, we have, together with the condition (C5),

$$\begin{aligned}\alpha(\{z^n \mid [\tau,\gamma] \mid n \in N\}) &\le 2 \sup \left\{ \left| \int_{\tau_n}^{\cdot} T_n(\cdot - s) g_n(s) ds \right|_{[\tau_n,\gamma]} \,\middle|\, n \ge k \right\} \\ &\le 2C \sup \left\{ \int_{\tau}^{\gamma} |g_n(s)| ds \,\middle|\, n \ge k \right\} \to 0 \quad \text{as } k \to \infty,\end{aligned}$$

from which we see that $\alpha(\{z^n \mid [\tau,\gamma] \mid n \in N\}) = 0$.

In view of Condition (C), we have $\alpha(\{\hat{x}^n \mid [\tau,\gamma] \mid n \in N\}) = 0$, and so $\alpha(\{x^n(t) \mid n \in N\}) = 0$ for $t \in (\tau,\gamma]$. Therefore, all conditions in Lemma 4.1 are satisfied. Using Lemma 4.1 and taking a subsequence if necessary, we may assume that there is a continuous function $x^0 \mid [\tau,\gamma]$ such that $|x^n - x^0|_{[\hat{s}(n),\gamma]} \to 0$ as $n \to \infty$.

Finally, we prove that the limit function $x^0 \mid [\tau,\gamma]$, $x^0_\tau = \psi$, is a solution of IP(T, F, τ, ψ). Put $f(s) = F(s, x^0_s)$ for all $s \in [\tau,\gamma]$ and let any $\varepsilon > 0$ be fixed. Then it follows from (C4) that $f_n(s) \to f(s)$ as $n \to \infty$ for each $s \in [\tau+\varepsilon, t]$. Hence, using Lebesgue's convergence theorem and the condition (C5), we get

$$\int_{\tau+\varepsilon}^{t} T(t-s) f_n(s) ds \to \int_{\tau+\varepsilon}^{t} T(t-s) f(s) ds \quad \text{as } n \to \infty,$$

so that

$$\lim_{n\to\infty}\int_{\tau_n}^{t} T(t-s)[f_n(s)+g_n(s)]ds = \int_{\tau}^{t} T(t-s)f(s)ds\,.$$

Therefore $x^0 \mid [\tau, \gamma]$, $x^0_\tau = \psi$, is a solution of IP(T, F, τ, ψ).

Step 2. We shall show that Condition (C) is verified. Let any t be fixed in $(\tau, \gamma]$. Then for any $\varepsilon > 0$ there exists a $\delta > 0$ and $k_\varepsilon \in N$ such that if $n \geq k_\varepsilon$, then

(i) $\tau + \delta < t$ and $\tau \leq \tau_n \leq \tau + \delta$; and

(ii) $\sup\{|F(s, x^n_s) - F(\tau, \psi)| \mid \tau_n \leq s \leq \tau + \delta\} < \varepsilon$ and $\sup\{|x^n_s - \psi|_{\mathcal{B}} \mid \tau_n \leq s \leq \tau + \delta\} < \varepsilon/2$.

For any $k > k_\varepsilon$ we have

$$\begin{aligned}
&\alpha(\{y^n \mid [\tau, t] \mid n \in N\}) \\
&\quad = \alpha(\{y^n \mid [\tau, t] \mid n \geq k\}) \\
&\quad = \max\{\alpha(\{y^n \mid [\tau, \tau+\delta] \mid n \geq k\}), \alpha(\{y^n \mid [\tau+\delta, t] \mid n \geq k\})\} \\
&\quad \leq \max\left\{2C(H+1)\delta, \alpha\left(\left\{\int_{\tau_n}^{\tau+\delta} T_n(\cdot - s)f_n(s)ds \mid [\tau+\delta, t] \mid n \geq k\right\}\right)\right. \\
&\qquad \left. + \alpha\left(\left\{\int_{\tau+\delta}^{\cdot} T_n(\cdot - s)f_n(s)ds \mid [\tau+\delta, t] \mid n \geq k\right\}\right)\right\} \\
&\quad \leq 2C(H+1)\delta + \alpha\left(\left\{\int_{\tau+\delta}^{\cdot} T_n(\cdot - s)f_n(s)ds \mid [\tau+\delta, t] \mid n \geq k\right\}\right).
\end{aligned} \tag{4.3}$$

We here note that

$$T_n(\theta - s)f_n(s) = T(\theta - s)f^0_n(s) + T(\theta - s)[f_n(s) - f^0_n(s)] + [T_n(\theta - s) - T(\theta - s)]f_n(s)\,.$$

First, we shall show the following assertions:

(iii) $\alpha\left(\left\{\int_{\tau+\delta}^{\cdot} T(\cdot - s)[f_n(s) - f^0_n(s)]ds \mid [\tau+\delta, t] \mid n \geq k\right\}\right) = 0$; and

(iv) $\alpha\left(\left\{\int_{\tau+\delta}^{\cdot} [T_n(\cdot - s) - T(\cdot - s)]f_n(s)ds \mid [\tau+\delta, t] \mid n \geq k\right\}\right) = 0.$

We have, together with the assumption (3),

$$\begin{aligned}
&\alpha(\{T(\theta - s)[f_n(s) - f^0_n(s)] \mid n \geq k\}) \\
&\qquad \leq \sup_{0\leq\theta\leq a} \hat{\alpha}(T(\theta))\alpha(\{f_n(s) - f^0_n(s) \mid n \geq k\}) \\
&\qquad \leq \sup_{0\leq\theta\leq a} \hat{\alpha}(T(\theta))\alpha(\{F_n(s, \eta) - F(s, \eta) \mid (s, \eta) \in \Omega, n \geq k\}) \\
&\qquad \to 0 \quad \text{as } k \to \infty\,.
\end{aligned}$$

Therefore, using Theorem 1, Lemma 1.1 and Lebesgue's convergence theorem, we obtain

$$\begin{aligned}
&\alpha\left(\left\{\int_{\tau+\delta}^{\cdot} T(\cdot - s)[f_n(s) - f_n^0(s)]ds \mid [\tau+\delta, t] \,\middle|\, n \in N\right\}\right) \\
&\qquad \le \gamma_T \sup_{\tau+\delta \le \theta \le t} \alpha\left(\left\{\int_{\tau+\delta}^{\theta} T(\theta - s)[f_n(s) - f_n^0(s)]ds \,\middle|\, n \ge k\right\}\right) \\
&\qquad \le 2\gamma_T \sup_{\tau+\delta \le \theta \le t} \int_{\tau+\delta}^{\theta} \alpha\big(\big\{T(\theta - s)[f_n(s) - f_n^0(s)] \,\big|\, n \ge k\big\}\big)ds \\
&\qquad \to 0 \quad \text{as } k \to \infty,
\end{aligned}$$

which implies the assertion (iii).

Put $\Lambda = cl\{f_n(s) \mid \tau + \delta \le s \le t, n \in N\}$. Then, it follows from the assumption (2) that

$$\begin{aligned}
&\alpha\big(\big\{[T_n(\theta - s) - T(\theta - s)]f_n(s) \,\big|\, n \ge k\big\}\big) \\
&\qquad \le \alpha\big(\big\{[T_n(\theta - s) - T(\theta - s)]z \,\big|\, z \in \Lambda, n \ge k\big\}\big) \\
&\qquad \le 2 \sup_{n \ge k} \sup_{z \in \Lambda} |[T_n(\theta - s) - T(\theta - s)]z| \to 0 \quad \text{as } k \to \infty.
\end{aligned}$$

Hence we have, by Lamma 1.1 and Lebesgue's convergence theorem,

$$\alpha\left(\left\{\int_{\tau+\delta}^{\theta} [T_n(\theta - s) - T(\theta - s)]f_n(s)ds \,\middle|\, n \in N\right\}\right) = 0 \quad \text{for } \theta \in [\tau + \delta, t].$$

Put

$$v^n(\theta) = \int_{\tau+\delta}^{\theta} [T_n(\theta - s) - T(\theta - s)]f_n(s)ds$$

for $\theta \in [\tau + \delta, t]$. Then we have

$$\begin{aligned}
|v^n(\theta)| &\le \int_{\tau+\delta}^{t} \sup_{\tau+\delta \le s \le t} \sup_{z \in \Lambda} |[T_n(t - s) - T(t - s)]z|ds \\
&\to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

and hence $\{v^n(\theta)\}$ is equicontinuous on $[\tau + \delta, t]$. These facts imply the assertion (iv).

Furthermore, by assertions (iii) and (iv), Theorem 1 and Lemma 1.1, we get

$$\begin{aligned}
&\alpha\left(\left\{\int_{\tau+\delta}^{\cdot} T_n(\cdot - s)f_n(s)ds \mid [\tau+\delta, t] \,\middle|\, n \ge k\right\}\right) \\
&\qquad = \alpha\left(\left\{\int_{\tau+\delta}^{\cdot} T(\cdot - s)f_n^0(s)ds \mid [\tau+\delta, t] \,\middle|\, n \ge k\right\}\right) \\
&\qquad \le \gamma_T \sup_{\tau+\delta \le \theta \le t} \alpha\left(\left\{\int_{\tau+\delta}^{\theta} T(\theta - s)f_n^0(s)ds \,\middle|\, n \ge k\right\}\right) \qquad (4.4) \\
&\qquad \le 2\gamma_T \sup_{\tau+\delta \le \theta \le t} \int_{\tau+\delta}^{\theta} \alpha\big(\big\{T(\theta - s)f_n^0(s) \,\big|\, n \ge k\big\}\big)ds \\
&\qquad \le 2\gamma_T \sup_{0 < s \le a} \hat{\alpha}(T(s)) \int_{\tau+\delta}^{t} \alpha\big(\big\{F(s, x_s^n) \,\big|\, n \ge k\big\}\big)ds,
\end{aligned}$$

provided $\hat{\alpha}(T(s)) > 0$ for all $s \in (0, a]$.

Next, we will prove that Condition (C) is satisfied. From (4.3) and (4.4) we get

$$\alpha\big(\{y^n \mid [\tau, t] \,|\, n \in N\}\big) \le 2C(H+1)\delta + 2\gamma_T \sup_{0<s\le a} \hat{\alpha}(T(s)) \int_{\tau+\delta}^{t} \alpha\big(\{F(s, x_s^n) \,|\, n \ge k\}\big) ds .$$

Since ε is arbitrary, we obtain the following relation:

$$\begin{aligned}\alpha\big(\{y^n \mid [\tau, t] \,|\, n \in N\}\big) &\le 2\gamma_T \sup_{0<s\le a} \hat{\alpha}(T(s)) \lim_{\varepsilon\to 0+} \int_{\tau+\delta}^{t} \alpha\big(\{F(s, x_s^n) \,|\, n \ge k_\varepsilon\}\big) ds \\ &:= v(t) .\end{aligned}$$

If we set $v(\tau) = 0$, then $v(t)$ is continuous on $[\tau, \gamma]$. Hence, in view of the continuity of F together with (ii) in Step 2, we can easily see that $\lim_{t\to\tau+} v(t)/(t-\tau) = v(\tau) = 0$. Using (1.1) and (ii) in Step 2 again, we have

$$\begin{aligned}\alpha\big(\{x_t^n \,|\, n \in N\}\big) &= \alpha\big(\{x_t^n \,|\, n \ge k_\varepsilon\}\big) \\ &\le K(t-\tau-\delta)\alpha\big(\{x^n \mid [\tau+\delta, t] \,|\, n \ge k_\varepsilon\}\big) \\ &\quad + M(t-\tau-\delta)\alpha\big(\{x_{\tau+\delta}^n - \psi \,|\, n \ge k_\varepsilon\}\big) \\ &\le K(t-\tau-\delta)\alpha\big(\{\hat{x}^n \mid [\tau, t] \,|\, n \ge k_\varepsilon\}\big) + M(t-\tau-\delta)\varepsilon \\ &\le K(t-\tau-\delta)\alpha\big(\{y^n \mid [\tau, t] \,|\, n \in N\}\big) + M(t-\tau-\delta)\varepsilon ,\end{aligned}$$

from which we get

$$\alpha\big(\{x_t^n \,|\, n \in N\}\big) \le K(t-\tau)v(t) .$$

Hence, we obtain, together with (2) in Hypothesis (B),

$$\alpha\big(\{F(t, x_t^n) \,|\, n \in N\}\big) \le \omega\big(t, \alpha\big(\{x_t^n \,|\, n \in N\}\big)\big) \le \omega(t, K(t-\tau)v(t)) \quad \text{a.e.}$$

and therefore,

$$v(t) \le 2\gamma_T \sup_{0<s\le a} \hat{\alpha}(T(s)) \int_{\tau+}^{t} \omega(s, K(s-\tau)v(s)) ds ,$$

because of the condition (3) in Hypothesis (B). Repeating the same argument as in the proof of Theorem 2 and using Hypothesis (B), we have $\alpha\big(\{y^n \mid [\tau, \gamma] \,|\, n \in N\}\big) = 0$, and hence Condition (C) is satisfied.

Step 3. We will prove the conclusion of the theorem. Let $\mathcal{F}$ be the collection of all solutions x of IP(T, F, σ, φ) such that the conditions (i) and (ii) in Theorem 3 hold for some subsequence of N. Clearly, $\mathcal{F}$ is nonempty. Using the standard order relation in $\mathcal{F}$, we see that $\mathcal{F}$ is an inductively ordered set. Therefore there exists a maximal element $\tilde{x} : [\sigma, \beta) \to E$ in $\mathcal{F}$ by Zorn's lemma. It is not difficult to show that $\tilde{x}$ is in NS(T, F, σ, φ). This proves the theorem (for detail, refer to [19]). q.e.d.

REMARK 4.2. We note that the conditions (C3) and (C4) and the assumption (3) in Theorem 3 hold if the following conditions are satisfied:

(H1-1) $F : D \to E$ is continuous and takes closed bounded sets into bounded sets.

(C-7) F_n, $n \in N$, is continuous and $F_n \to F$ uniformly on Ω as $n \to \infty$ for each closed bounded set Ω of D.

REMARK 4.3. Theorem 3 is based on Theorem 2. However, if $A = 0$, then we can prove the theorem based on Proposition 2.6 instead of Theorem 2. We note that a similar result can be found in [19] if $A = 0$, $g_n = 0$ and F is uniformly continuous.

REMARK 4.4. If $F(t, \psi) = F(t, \psi(0))$ and $F_n(t, \psi) = F_n(t, \psi(0))$, $n \in N$, then the condition 2) in Theorem 3 can be replaced by the following condition:

(2′) $T_n(t)$, $n \in N$, is a C_0-semigroup on E, and for each closed bounded set $\Lambda \subset E$ and for each $t \in R^+$,

$$\alpha\left(\left\{[T_n(t) - T(t)]z \,\middle|\, z \in \Lambda, n \geq k\right\}\right) \to 0 \quad \text{as } k \to \infty .$$

Set $\tau(\sigma, \varphi) = \inf\left\{\tau_y \,\middle|\, y \in \mathrm{NS}(T, F, \sigma, \varphi)\right\}$. Then the following result is easily proved by using Theorem 3.

LEMMA 4.5. *Let Hypotheses* (H1-0), (H2) *and* (B) *be satisfied. Then*

(1) $\sigma < \tau(\sigma, \varphi) \leq \infty$ *and there exists an* $x_0 \in \mathrm{NS}(T, F, \sigma, \varphi)$ *such that* $\tau_{x_0} = \tau(\sigma, \varphi)$.

(2) $\tau(\sigma, \varphi)$ *is lower semicontinuous on D.*

We will make use of the following hypothesis in place of Hypothesis (B):

(B-l) The equation (4.2) in Hypothesis (B) is replaced by the equation

$$\frac{d}{dt}u(t) = 2\gamma_T \sup_{0<s\leq a} \hat{\alpha}(T(s))[\omega(t, K(t - \tau)u(t)) + lK(t - \tau)u(t)]$$

for a.a. $t \in (\tau, \tau + a)$.

Clearly, Hypothesis (B) is derived from Hypothesis (B-l).

Set $\mathcal{M}(k) = \{G : D \to E \,|\, G$ is a continuous function on $D \subset R \times \mathcal{B}$ such that $\alpha\left(\left\{G(t, \eta) - F(t, \eta) \,\middle|\, \eta \in B\right\}\right) \leq k\alpha(B)$ for each $(t, B) \subset D$, where B is a bounded subset of $\mathcal{B}\}$, where $k > 0$ is constant.

LEMMA 4.6. *Suppose that Hypotheses* (H1-0), (H2) *and* (B-l) *are satisfied. If* $G \in \mathcal{M}(l)$ *and* $g^1_{loc}(\mathrm{pr}\, D)$, *then for any* $(\tau, \psi) \in D$, $\mathrm{NS}(T, G + g, \tau, \psi)$ *is nonempty.*

PROOF. Let any (τ, ψ) be fixed in D. Then there are positive numbers a and r such that $[\tau - a, \tau + a] \times \mathcal{B}(\psi, r) \subset D$ and that G and F are bounded over there. Since $G \in \mathcal{M}(l)$, we have $\alpha(G(t, B)) \leq \alpha(F(t, B)) + \alpha\left(\left\{G(t, \eta) - F(t, \eta) \,\middle|\, \eta \in B\right\}\right) \leq \omega(t, \alpha(B)) + l\alpha(B)$ for all $B \subset \mathcal{B}(\psi, r)$ and for a.a. $t \in [\tau, \tau + a]$. On the other hand, using the same argument as in the proof of [22, Lemma 5.3], we can construct approximate solutions for $\mathrm{IP}(T, G + g, \tau, \psi)$. In view of Hypothesis (B-l), it is not difficult to show that $\mathrm{IP}(T, G + g, \tau, \psi)$ has a local solution. Hence, it follows from Zorn's lemma that $\mathrm{NS}(T, G + g, \tau, \psi)$ is nonempty. q.e.d.

We give a result on the continuous dependence of solutions for $\mathrm{IP}(T, F, \sigma, \varphi)$.

PROPOSITION 4.7. *Suppose that Hypotheses* (H1-1), (H2) *and* (B-l) *are satisfied. Let d be a number in* $(\sigma, \tau(\sigma, \varphi))$. *Then for any* $\varepsilon > 0$ *there exists a* $\delta(\varepsilon) > 0$ *such that if* $|\alpha - \sigma| < \delta$, $|\psi - \varphi|_{\mathcal{B}} < \delta$, $(\alpha, \psi) \in D$, $|G(t, \eta) - F(t, \eta)| < \delta$, $(t, \eta) \in D$, *and* $\int_J |g(t)|dt < \delta$, *where* $J = [\min\{\alpha, \sigma\}, d]$, $G \in \mathcal{M}(l)$ *and* $g \in \mathcal{L}^1_{loc}(\mathrm{pr}\, D)$ *then every* $y(t, \alpha, \psi, G + g)$ *in*

$\mathrm{NS}(T, G+g, \alpha, \psi)$ *is always continued beyond* $t = d$ *and satisfies*

$$|y_t(\alpha, \psi, G+g) - x_t(\sigma, \varphi)|_{\mathcal{B}} < \varepsilon \quad \text{for } t \in [\max\{\sigma, \alpha\}, d]\,, \tag{4.5}$$

for some $x(t; \sigma, \varphi)$ *in* $\mathrm{NS}(T, F, \sigma, \varphi)$ *which may depend on* $y(t; \alpha, \psi, G+g)$.

PROOF. First of all, we shall show that for any d in $(\sigma, \tau(\sigma, \varphi))$ there exists a $\delta > 0$ such that if $|\alpha - \sigma| < \delta$, $|\psi - \varphi|_{\mathcal{B}} < \delta$, $(\alpha, \psi) \in D$, $|G(t, \eta) - F(t, \eta)| < \delta$, $(t, \eta) \in D$, and $\int_J |g(t)|dt < \delta$, $J = [\min\{\alpha, \sigma\}, d]$, then $\tau_y > d$ for all $y \in \mathrm{NS}(T, G+g, \alpha, \psi)$. It follows from Lemma 4.6 that $\mathrm{NS}(T, G+g, \alpha, \psi)$ is nonempty. For a contradiction, we assume that there exist some d_0 in $(\sigma, \tau(\sigma, \varphi))$ and sequences $\{(\sigma_n, \varphi_n)\}$, $\{G_n\}$, $\{g_n\}$ and $\{y^n\}$, $y^n \in \mathrm{NS}(T, G_n + g_n, \sigma_n, \varphi_n)$, such that $(\sigma_n, \varphi_n) \to (\sigma, \varphi) \in D$, $\sup\big\{|G_n(t, \eta) - F(t, \eta)| \,\big|\, (t, \eta) \in D\big\} \to 0$ and $\int_{\alpha_n}^{d_0} |g_n(t)|dt \to 0$ as $n \to \infty$, where $\alpha_n = \min\{\sigma_n, \sigma\}$, and that $\sigma_n < \tau_n \le d_0$, where $\tau_n = \tau_{y^n}$. Since $\sigma_n \to \sigma$ as $n \to \infty$, we may assume that $\tau_n \to \tau_0$, $\sigma \le \tau_0 \le d_0$, as $n \to \infty$. Thus it follows easily from Theorem 3 that there exists a $z \in \mathrm{NS}(T, F, \sigma, \varphi)$ such that $\sigma < \tau_z \le \tau_0$. Hence $\tau_z \le d_0 < \tau(\sigma, \varphi)$, which contradicts the definition of $\tau(\sigma, \varphi)$.

Next, we shall show that the inequality (4.5) holds. For a contradiction, suppose that the conclusion is false. Then there exist some $\varepsilon_0 > 0$ and sequences $\{(\sigma_n, \varphi_n)\}$, $\{G_n\}$, $\{g_n\}$ and $\{y^n\}$, $y^n \in \mathrm{NS}(T, G_n + g_n, \sigma_n, \varphi_n)$, such that $(\sigma_n, \varphi_n) \to (\sigma, \varphi) \in D$, $\sup\big\{|G_n(t, \eta) - F(t, \eta)| \,\big|\, (t, \eta) \in D\big\} \to 0$ and $\int_{\alpha_n}^{d} |g_n(t)|dt \to 0$ as $n \to \infty$, where $\alpha_n = \min\{\sigma_n, \sigma\}$, and that for all $x \in \mathrm{NS}(T, F, \sigma, \varphi)$,

$$|y^n_{t_n}(\sigma_n, \varphi_n, G_n + g_n) - x_{t_n}(\sigma, \varphi)|_{\mathcal{B}} \ge \varepsilon_0 \quad \text{for some } t_n \in [s_n, d]\,,$$

where $s_n = \max\{\sigma_n, \sigma\}$, t_n may depend on $x(t, \sigma, \varphi)$ and $y^n \in \mathrm{NS}(T, G_n + g_n, \sigma_n, \varphi_n)$. From Theorem 3 we may assume that $|y^n - z|_{[s_n, d]} \to 0$ as $n \to \infty$ for some $z \in \mathrm{NS}(T, F, \sigma, \varphi)$. Hence we have, together with the axiom (B_2),

$$\begin{aligned}
\varepsilon_0 &\le |y^n_{t_n} - z_{t_n}|_{\mathcal{B}} \\
&\le K(t_n - s_n)|y^n - z|_{[s_n, d]} + M(t_n - s_n)\{|y^n_{s_n} - \varphi|_{\mathcal{B}} + |z_{s_n} - \varphi|_{\mathcal{B}}\} \\
&\to 0 \quad \text{as } n \to \infty\,,
\end{aligned}$$

which is a contradiction. q.e.d.

Using Theorem 3, we have the following result, which is proved by using the same argument as in the proof of Proposition 4.7.

COROLLARY 4.8. *Suppose that Hypotheses* (H1-1), (H2) *and* (B) *are satisfied. Let* d *be a number in* $(\sigma, \tau(\sigma, \varphi))$. *Then for any* $\varepsilon > 0$ *there exists a* $\delta(\varepsilon) > 0$ *such that if* $|\alpha - \sigma| < \delta$, $|\psi - \varphi|_{\mathcal{B}} < \delta$, $(\alpha, \psi) \in D$, *and* $\int_J |g(t)|dt < \delta$, *where* $J = [\min\{\alpha, \sigma\}, d]$, *and* $g \in \mathcal{L}^1_{loc}(\mathrm{pr}\, D)$, *then every* $y(t, \alpha, \psi, F+g)$ *in* $\mathrm{NS}(T, F+g, \alpha, \psi)$ *satisfies*

$$|y_t(\alpha, \psi, F+g) - x_t(\sigma, \varphi)|_{\mathcal{B}} < \varepsilon \quad \text{for } t \in [\max\{\sigma, \alpha\}, d]\,,$$

for some $x(t; \sigma, \varphi)$ *in* $\mathrm{NS}(T, F, \sigma, \varphi)$ *which may depend on* $y(t; \alpha, \psi, F+g)$.

The above fact extends a well-known result in ordinary differential equations in finite dimensional spaces (refer to [25]).

REFERENCES

[1] D. BOTHE, Multivalued differential equations on graphs and applications, Dissertation, Gesamthochschule Paderborn, 1992.

[2] K. DEIMLING, Multivalued Differential Equations, Walter de Gruyter, Berlin, New York, 1992.

[3] J. K. HALE AND J. KATO, Phase spaces for retarded equation with infinite delay, Funkcial. Ekvac. 21 (1978), 11–41.

[4] Y. HINO, S. MURAKAMI AND T. NAITO, Functional Differential Equations with Infinite Delay, Lecture Notes in Math. 1473, Springer-Verlag, 1991.

[5] H. P. HEINZ, Theorems of Ascoli type involving measures of noncompactness, Nonlinear Anal. 5 (1981), 277–286.

[6] H. P. HEINZ, On the behavior of measures of noncompactness with respect to differentiation and integration of vector-valued function, Nonlinear Anal. 7 (1983), 1351–1371.

[7] H. R. HENRIQUEZ, Periodic solutions of quasi-linear partial functional differential equations with unbounded delay, Funkcial. Ekvac. 37 (1994), 329–344.

[8] T. IWAMIYA, Global existence of mild solutions to semilinear differential equations in Banach spaces, Hiroshima Math. J. 16 (1986), 499–530.

[9] M. A. KRASNOSELSKII, S. G. KREIN AND P. E. SOBOLEVSKII, On differential equations with unbounded operators in Banach spaces, Dokl. Akad. Nauk SSSR 111 (1956), 19–22.

[10] V. LAKSHMIKANTHAM AND S. LEELA, Nonlinear Differential Equations in Abstract Spaces, International Series in Nonlinear Mathematics Theory, Methods and Applications 2, Pergamon Press, Oxford-New York, 1981.

[11] R. H. MARTIN, JR., Nonlinear Operators and Differential Equations in Banach Spaces, Wiley-Intersci., New York, 1976.

[12] R. D. NUSSBAUM, A generalization of the Ascoli theorem and an application to functional differential equations, J. Math. Anal. Appl. 35 (1971), 600–610.

[13] H. MÖNCH AND G. F. HARTEN, On the Cauchy problem for ordinary differential equations in Banach spaces, Arch. Math. 39 (1982), 153–160.

[14] A. PAZY, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer, 1983.

[15] K. SCHUMACHER, Existence and continuous dependence for functional differential equations with infinite delay, Arch. Rational Mech. Anal. 7 (1978), 315–334.

[16] K. SCHUMACHER, Remarks on semilinear partial functional differential equations with infinite delay, J. Math. Anal. Appl. 80 (1981), 261–290.

[17] J. S. SHIN, An existence theorem of functional differential equations with infinite delay in a Banach space, Funkcial. Ekvac. 30 (1987), 19–29.

[18] J. S. SHIN, On the uniqueness of solutions for functional differential equations with infinite delay, Funkcial. Ekvac. 30 (1987), 225–236.

[19] J. S. SHIN, Existence of solutions and Kamke's theorem for functional differential equations in Banach spaces, J. Differential Equations 81 (1989), 294–312.

[20] J. S. SHIN, Kneser type theorems for functional differential equations in a Banach space, Funkcial. Ekavc. 35 (1992), 451–466.

[21] J. S. SHIN, Uniqueness of mild solutions to semilinear functional differential equations in Banach spaces, Proc. International Symp., World Scientific, Singapore, Funct. Differ. Equ. (1991), 334–338.

[22] J. S. SHIN, Comparison theorems and uniqueness of mild solutions to semilinear functional differential equations in Banach spaces, Nonlinear Anal. 23 (1994), 825–847.

[23] J. S. SHIN AND T. NAITO, Semi-Fredholm operators and periodic solutions for linear functional differential equations in Banach spaces, J. Differential Equations 153 (1999), 407–441.

[24] C. C. TRAVIS AND G. F. WEBB, Existence and stability for partial functional differential equations, Trans. Amer. Math. Soc. 200 (1974), 395–418.

[25] T. YOSHIZAWA, Stability Theory by Liapunove's Second Method, Publications of the Mathematical Society of Japan, No. 9, The Mathematical Society of Japan, Tokyo, 1966.

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