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UNIQUENESS, EXISTENCE AND NONEXISTENCE OF NORMAL SOLUTIONS TO A PROBLEM IN BOUNDARY LAYER THEORY

JUNYU WANG* AND WENJIE GAO*

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Abstract. A rigorous mathematical analysis is given for a well-known problem for a third-order nonlinear ordinary differential equation, which arises in boundary layer theory in fluid mechanics and was firstly deduced by Falkner and Skan in 1930. It is proved that the problem is equivalent to a singular nonlinear two-point boundary value problem of second order. For the singular nonlinear boundary value problem, uniqueness, existence and nonexistence of positive solutions are established by utilizing a priori estimates, comparison principles and a modified shooting type method. These results are easily turned over to the original problem.

1. Introduction.

The third-order nonlinear ordinary differential equation for $f(\eta)$

$$f''' + ff'' + \lambda(1 - f'^2) = 0\,, \quad 0 < \eta < +\infty\,, \tag{1.1}$$

with boundary conditions

$$f(0) = 0\,, \quad f'(0) = 0\,, \quad f'(+\infty) = 1\,, \tag{1.2}$$

is of great importance in the boundary layer theory in fluid mechanics. The equation (1.1) was firstly deduced by Falkner and Skan [4] in 1930. However, the cases $\lambda = 0$ and $\lambda = 1/2$ of (1.1) are often called the Blasius and Homann differential equations, respectively. Many authors have investigated the boundary value problem (1.1)–(1.2) by using numerical and analytical methods. For details, see [5, pp. 519–537; 7, pp. 149–151] and the references therein. In this paper, we present a new approach to study the problem (1.1)–(1.2) and provide some new information about a normal solution to the problem.

A function $f(\eta, \lambda)$ is called a normal solution of the problem (1.1)–(1.2) if f satisfies (1.1)–(1.2) and

$$0 < f'(\eta, \lambda) < 1\,, \quad f''(\eta, \lambda) > 0 \quad \text{for all } \eta > 0\,.$$

Here and henceforth a prime denotes differentiation with respect to the variable η or the variable t in case of no confusion.

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The introduction of a new independent variable t and a new unknown function $w(t)$ as what we do in the following can transform the problem of finding a normal solution of (1.1)–(1.2) into a problem of finding a positive solution to a singular nonlinear boundary value problem of second order.

Assume that $f(\eta)$ is a normal solution to the boundary value problem (1.1)–(1.2). Then $\eta = g(t)$, the inverse function to $t = f'(\eta)$, exists and is strictly increasing in $(0, 1)$, $g(0) = 0$, $g(1-0) = \infty$, and

$$t \equiv f'(g(t)) \quad \text{for all } t \in (0, 1)\,. \tag{1.3}$$

Differentiating (1.3) with respect to t yields

$$w(t) := f''(g(t)) = \frac{1}{g'(t)}\,, \quad 0 < t < 1\,. \tag{1.4}$$

Substituting $\eta = g(t)$ into (1.1), we obtain

$$w'(t)w(t) + f(g(t))w(t) + \lambda(1 - t^2) = 0\,, \quad 0 \le t < 1\,. \tag{1.5}$$

Here we have used (1.3), (1.4) and the fact that

$$w'(t) = f'''(g(t))g'(t) = \frac{f'''(g(t))}{w(t)}\,, \quad 0 < t < 1\,.$$

Dividing (1.5) by $w(t)$ and then differentiating it with respect to t, we get

$$w''(t) = -\lambda\left(\frac{1-t^2}{w(t)}\right)' - \frac{t}{w(t)}\,, \quad 0 < t < 1\,. \tag{1.6}$$

From (1.2), (1.5) and $g(1-0) = +\infty$, it follows that

$$w'(0)w(0) = -\lambda\,, \quad w(1) = 0\,. \tag{1.7}$$

Problem (1.6)–(1.7) is singular at $t = 1$ and possibly at $t = 0$ if $w(0) = 0$. The particular case $\lambda = 0$ of the problem has been studied by many authors. For detailed analysis of the case or a general form of the case, one is referred to [1–4, 6, 8, 9]. As far as we know, the singular nonlinear boundary value problem (1.6)–(1.7) has not been studied yet when $\lambda \neq 0$.

Problem (1.6)–(1.7) can be used as a model example to study a singular nonlinear boundary value problem.

In Section 2 we will prove that there is a $\lambda_* \in (-1/2, 0)$ such that for each fixed $\lambda \ge \lambda_*$ the problem (1.6)–(1.7) has a unique positive solution $w(t, \lambda)$ and it has no positive solution for any $\lambda < \lambda_*$. Our arguments for the existence and uniqueness of the $w(t, \lambda)$ involve a modified shooting type argument, a priori estimates and comparison principles.

The results for the normal solutions of (1.1)–(1.2) will be stated in Section 3 by using the results in Section 2.

2. The results for the Problem with Singularities. In this section, we concentrate our attention upon the singular nonlinear problem (1.6)–(1.7). We adopt a modified shooting type method to prove the existence of positive solution to the problem (1.6)–(1.7). The uniqueness and nonexistence results will also be proved.

A function $w(t)$ is said to be a positive solution to the problem (1.6)–(1.7), if

(i) $w(t) \in C[0,1] \cap C^2(0,1)$, $w(t) > 0$ for all $t \in (0,1)$,

(ii) $w(t) \in C^1[0,1)$ when $w(0) > 0$, and

(iii) $w(t)$ satisfies (1.6) and (1.7).

It is easily seen that the solution of (1.6) and (1.7) with $w(0) \geq 0$ can be represented by

$$w(t) = \int_t^1 \frac{(1-s)(\lambda+\lambda s+s)ds}{w(s)} + (1-t)\int_0^t \frac{sds}{w(s)}, \quad 0 \leq t \leq 1. \tag{2.1}$$

Consequently, we have

$$w'(t) = \frac{-\lambda(1-t^2)}{w(t)} - \int_0^t \frac{sds}{w(s)}, \quad 0 < t < 1. \tag{2.2}$$

It is clear that $w'(0,\lambda) = -\lambda/w(0,\lambda)$ when $w(0,\lambda) > 0$, $w'(t,\lambda) < 0$ in $(0,1)$ when $\lambda \geq 0$, and there must be one and only one point $t_M \in (0,1)$ such that $w'(t,\lambda) > 0$ in $(0, t_M)$ and $w'(t,\lambda) < 0$ in $(t_M, 1)$ when $\lambda < 0$.

On the basis of (2.1) and (2.2), we can establish the following

PROPOSITION 1. *Let $w(t,\lambda)$ be a positive solution to the problem* (1.6)–(1.7). *Then the following four statements hold.*

(i) *The problem* (1.6)–(1.7) *has no positive solution for any $\lambda \leq -1/2$.*

(ii) *For each fixed $\lambda \geq 0$, we have*

$$\sqrt{\frac{4\lambda+1}{6}} < w(0,\lambda) < \sqrt{\frac{4\lambda+1}{3}}. \tag{2.3}$$

(iii) *For each fixed $\lambda > -1/2$, we have*

$$w(t,\lambda) \sim \sqrt{2}(1-t)|\log(1-t)|^{1/2} \quad \textit{as } t \to 1. \tag{2.4}$$

(iv) *For some $\lambda_* \in (-1/2, 0)$* such that $w(0,\lambda_*) = 0$, *we have*

$$w(t,\lambda) \sim \sqrt{2|\lambda_*|}\,t^{1/2} \quad \textit{as } t \to 0. \tag{2.5}$$

PROOF. (i) When $\lambda \leq -1/2$, (2.1) implies that $w(0,\lambda) < 0$ and hence the problem (1.6)–(1.7) has no positive solution.

(ii) We know that $w(t,\lambda)$ is strictly decreasing on $[0,1]$ for each fixed $\lambda \geq 0$. Then we have by (2.2) that

$$-w'(t,\lambda) < \frac{\lambda(1-t^2)}{w(t,\lambda)} + \frac{1}{w(t,\lambda)}\int_0^t sds, \quad 0 < t < 1,$$

i.e.,

$$-w'(t,\lambda)w(t,\lambda) < \lambda(1-t^2) + \frac{1}{2}t^2, \quad 0 < t < 1.$$

Integrating the above from 0 to 1, we obtain

$$\frac{1}{2}w^2(0,\lambda) < \frac{2\lambda}{3} + \frac{1}{6}.$$

This proves the right hand side inequality of (2.3).

On the other hand, we have by (2.1) that

$$\begin{aligned} w(0,\lambda) &= \int_0^1 \frac{(1-s)(\lambda+\lambda s+s)}{w(s,\lambda)}ds \\ &> \frac{1}{w(0,\lambda)}\int_0^1 (1-s)(\lambda+\lambda s+s)ds \\ &= \frac{1}{w(0,\lambda)}\frac{4\lambda+1}{6}. \end{aligned}$$

The other part of (2.3) is thus proved.

(iii) Let

$$\alpha(t) := Ay(t), \quad \beta(t) := By(t), \quad y(t) := (1-t)|\log(1-t)|^{1/2},$$

where $A = \sqrt{2-\varepsilon}$, $B = \sqrt{2+\varepsilon}$, and $\varepsilon \in (0, 1/4)$ is arbitrary. Let us define a mapping $\Phi : D \mapsto D$ by

$$(\Phi u)(t) := \int_t^1 \frac{(1-s)(\lambda+\lambda s+s)ds}{u(s)} + Q(1-t) + (1-t)\int_{1-\delta}^t \frac{sds}{u(s)}$$

for any $u \in D$, where $D := \{u \in C[1-\delta, 1]$; there is a $k > 0$ such that $u(t) \geq ky(t)$ on $[1-\delta, 1]\}$, δ is a sufficiently small positive number which is to be determined, and $Q := \int_0^{1-\delta}(s/w(s,\lambda))ds$ is a fixed positive number for given $\lambda > -1/2$.

Then we use the L'Hospital rule to obtain

$$\begin{aligned} \lim_{t\to 1}\frac{(\Phi\beta)(t)}{y(t)} &= \lim_{t\to 1}\frac{[-(1-t)(\lambda+\lambda t+t)]/\beta(t)}{y'(t)} + \lim_{t\to 1}\frac{Q}{|\log(1-t)|^{1/2}} \\ &\quad + \lim_{t\to 1}\frac{t/\beta(t)}{(|\log(1-t)|^{1/2})'} \\ &= \lim_{t\to 1}\frac{2y(t)}{\beta(t)} = \frac{2}{B}. \end{aligned}$$

Since $A = \sqrt{2-\varepsilon} < 2/\sqrt{2+\varepsilon} = 2/B < \sqrt{2+\varepsilon} = B$, we can choose a sufficiently small positive number δ_1 such that

$$\alpha(t) < (\Phi\beta)(t) < \beta(t) \quad \text{for all } t \in [1-\delta_1, 1). \tag{2.6}$$

Similarly, we obtain

$$\lim_{t\to 1}\frac{(\Phi\alpha)(t)}{y(t)} = \frac{2}{A} = \frac{2}{\sqrt{2-\varepsilon}}.$$

Since $B < 2/\sqrt{2-\varepsilon} < \sqrt{2+2\varepsilon}$, we can find a sufficiently small positive number δ_2 such that

$$\alpha(t) < \beta(t) < (\Phi\alpha)(t) < \sqrt{2+2\varepsilon}\,y(t) \quad \text{for all } t \in [1-\delta_2, 1). \tag{2.7}$$

Let $\delta := \min\{\delta_1, \delta_2\}$. Then (2.6) and (2.7) hold for all $t \in [1-\delta, 1)$.

A simple calculation shows us that the $\alpha(t)$ is a strictly lower solution of (1.6) on $[1-\delta, 1)$ and the $\beta(t)$ is a strictly upper solution of (1.6) on $[1-\delta, 1)$. By utilizing the method of lower and upper solution, we know that for each $\xi \in [\alpha(1-\delta), \beta(1-\delta)]$ there is a solution $w(t)$ of (1.6) such that

$$\alpha(t) \le w(t) \le \beta(t) \quad \text{on } [0,1] \quad \text{and} \quad w(1-\delta) = \xi\,.$$

Note that the Φ is monotonically decreasing on D, i.e., $(\Phi u_1)(t) \ge (\Phi u_2)(t)$ on $[1-\delta, 1]$ for any $u_1, u_2 \in D$, $u_1(t) \le u_2(t)$ on $[1-\delta, 1]$. Hence, we conclude that each $u(t)$, which satisfies $u(t) \ge \beta(t)$ on $[1-\delta, 1]$, cannot be a fixed point of Φ on $[1-\delta, 1]$, since

$$(\Phi u)(t) \le (\Phi\beta)(t) < \beta(t) \le u(t) \quad \text{on } [1-\delta, 1)\,.$$

On the other hand, for each $u(t)$ satisfying $u(t) \le \alpha(t)$ on $[1-\delta, 1]$ and $u \in D$, we have

$$(\Phi u)(t) \ge (\Phi\alpha)(t) > \alpha(t) \ge u(t) \quad \text{on } [1-\delta, 1)\,,$$

and hence deduce that such a $u(t)$ cannot be a fixed point of Φ on $[1-\delta, 1]$. This implies that

$$\alpha(t) \le w(t,\lambda) \le \beta(t) \quad \text{on } [1-\delta, 1]\,, \tag{2.8}$$

since $w(t,\lambda)$ is certainly a fixed point of Φ on $[1-\delta, 1]$. It can be easily concluded from (2.8) that $\lim_{t\to 1} w(t,\lambda)/y(t) = \sqrt{2}$, since ε is arbitrary.

(iv) Statements (i) and (ii) imply that $\lambda_* \in (-1/2, 0)$ when $w(0,\lambda_*) = 0$. Let $u(t) := w^2(t,\lambda_*)$. Then we have by (2.2) that

$$u'(t) = 2|\lambda_*|(1-t^2) - 2w(t,\lambda_*) \int_0^t \frac{s\,ds}{w(s,\lambda_*)}\,, \quad 0 < t < 1\,.$$

Letting $t \to 0$ in the above gives $u'(0) = 2|\lambda_*|$, which implies (2.5).

To demonstrate the uniqueness and existence of the $w(t,\lambda)$, we need to consider the following initial value problem

$$\begin{cases} w''(t) = \dfrac{(2\lambda-1)t}{w(t)} + \dfrac{\lambda(1-t^2)w'(t)}{w^2(t)} =: F(t, w(t), w'(t))\,, & 0 \le t\,, \\ w(0) = x > 0\,, \\ w'(0) = -\dfrac{\lambda}{x}\,. \end{cases} \tag{2.9}$$

It is obvious that $F \in C^\infty(R \times R_+ \times R)$ with $R = (-\infty, +\infty)$ and $R_+ = (0, +\infty)$.

A standard theorem in ordinary differential equation theory shows that the initial value problem (2.9) has locally a unique positive solution, denoted by $w(t,x,\lambda)$. Furthermore, there exists a $\gamma = \gamma(x,\lambda) > 0$ such that $[0,\gamma)$ is a right maximal interval of existence for the $w(t,x,\lambda)$. From now on, we denote $\rho = \rho(x,\lambda) =: \min\{\gamma, 1\}$.

A simple calculation proves that the solution of (2.9) can be represented as a solution of the initial value problem

$$\begin{cases} w'(t) = \dfrac{-\lambda(1-t^2)}{w(t)} - \displaystyle\int_0^t \frac{s\,ds}{w(s)}\,, & 0 < t\,, \\ w(0) = x\,. \end{cases} \tag{2.10}$$

Here x is allowed to be zero.

Concerning the initial value problem (2.10), we can prove the following three lemmas.

LEMMA 1 *Let $w(t, x, \lambda)$ be the unique positive solution of the problem* (2.10). *Then*

(i) *if $x_1 > x_2 > 0$, then $w(t, x_1, \lambda) > w(t, x_2, \lambda)$ for all $t \in [0, \rho(x_2, \lambda))$ and hence $\rho(x_1, \lambda) \geq \rho(x_2, \lambda)$;*

(ii) *if $\lambda_1 > \lambda_2$, $x > 0$, then $w(t, x, \lambda_1) < w(t, x, \lambda_2)$ for all $t \in (0, \rho(x, \lambda_1))$ and hence $\rho(x, \lambda_1) \leq \rho(x, \lambda_2)$.*

LEMMA 2. *Let $x \in R_+$ and $-\infty < \alpha < \beta < +\infty$ such that $\gamma(x, \beta) < 1 < \gamma(x, \alpha)$. Then for each fixed $y \in [0, w(1, x, \alpha))$, then must be one and only one $\lambda \in (\alpha, \beta)$ such that $w(1, x, \lambda) = y$.*

LEMMA 3. *Let $\lambda > \lambda_*$ and $0 < a < b < +\infty$ such that $\gamma(a, \lambda) < 1 < \gamma(b, \lambda)$. Here $\lambda_* \in (-1/2, 0)$ such that $w(0, \lambda_*) = 0$ and it will be defined by Proposition 5. Then there must be one and only one $x \in (a, b)$ such that $w(1, x, \lambda) = 0$.*

PROOF OF LEMMA 1. (i) If $w(t, x_1, \lambda) \leq w(t, x_2, \lambda)$ for some $t \in (0, \rho(x_2, \lambda))$, then there must exist a $b \in (0, \rho(x_2, \lambda))$ such that $w(t, x_1, \lambda) > w(t, x_2, \lambda)$ for $t \in [0, b)$, $w(b, x_1, \lambda) = w(b, x_2, \lambda)$ and hence $w'(b, x_1, \lambda) \leq w'(b, x_2, \lambda)$. Therefore, by (2.10), it leads to

$$\begin{aligned} 0 \geq & w'(b, x_1, \lambda) - w'(b, x_2, \lambda) = \frac{-\lambda(1 - b^2)}{w(b, x_1, \lambda)} + \frac{\lambda(1 - b^2)}{w(b, x_2, \lambda)} \\ & + \int_0^b s \left(\frac{1}{w(s, x_2, \lambda)} - \frac{1}{w(s, x_1, \lambda)} \right) ds > 0, \end{aligned}$$

which is a contradiction. Hence (i) is proven.

(ii) Suppose that for some $t \in (0, \rho(x, \lambda_1))$, $w(t, x, \lambda_1) \geq w(t, x, \lambda_2)$. Then there must be a point $b \in (0, \rho(x, \lambda_1))$ such that $w(t, x, \lambda_2) > w(t, x, \lambda_1)$ for $t \in (0, b)$, $w(b, x, \lambda_1) = w(b, x, \lambda_2)$ and hence $w'(b, x_1, \lambda) \geq w'(b, x_2, \lambda)$, since $w(0, x, \lambda_1) = w(0, x, \lambda_2)$ and $w'(0, x, \lambda_1) = -\lambda_1/x < -\lambda_2/x = w'(0, x, \lambda_2)$. A similar argument as above shows us a contradiction, and hence the conclusion follows.

PROOF OF LEMMA 2. We prove this lemma by employing a modified shooting type argument. We first consider the case $y = 0$. Let x, α and β satisfy the assumption in Lemma 2 and let

$$\lambda_S =: \sup\{\lambda; \gamma(x, \lambda) > 1\} \quad \text{and} \quad \lambda_I =: \inf\{\lambda; \gamma(x, \lambda) \leq 1\}.$$

The assumption in Lemma 2 and the results in Lemma 1 show us that both λ_I and λ_S are well defined and $\alpha \leq \lambda_S \leq \lambda_I \leq \beta$. Then the lower semicontinuity of $\gamma(x, \lambda)$ in [5, p. 94] implies that $\gamma(x, \lambda_I) \leq 1 \leq \gamma(x, \lambda_S)$.

If $\gamma(x, \lambda_I) < \gamma(x, \lambda_S)$, Lemma 1 would imply that $\lambda_S < \lambda_I$. Hence, for any $\lambda_0 \in (\lambda_S, \lambda_I)$, we must have $\gamma(x, \lambda_0) \leq 1$ since $\lambda_0 > \lambda_S$, and $\gamma(x, \lambda_0) \geq 1$ since $\lambda_0 < \lambda_I$ by the definition of λ_I and λ_S. Therefore, $\gamma(x, \lambda_0) = 1$, which contradicts to the assumption $\lambda_0 < \lambda_I$. Hence, we must have $\gamma(x, \lambda_I) = \gamma(x, \lambda_S) = 1$. The same argument proves that $\lambda_S = \lambda_I$. Therefore, $w(t, x, \lambda_S)$ is a solution of Problem (1.6)–(1.7).

It remains to prove that such a λ is unique. Assume that $w(1, x, \lambda_1) = w(1, x, \lambda_2) = 0$ for some $\lambda_1 \neq \lambda_2$. Without loss of generality, we may assume $\lambda_1 > \lambda_2$. Lemma 1 tells us that

$$w(t, x, \lambda_1) =: w_1(t) < w_2(t) := w(t, x, \lambda_2) \quad \text{for all } t \in (0, 1).$$

Since $w_j(t)$, $j = 1, 2$, is a positive solution to the problem (1.6)–(1.7) with $\lambda = \lambda_j$, we have by (2.1) that

$$\begin{aligned} 0 >& w_1(t) - w_2(t) = (1-t) \int_0^t s \left(\frac{1}{w_1(s)} - \frac{1}{w_2(s)} \right) ds \\ &+ \int_t^1 (1-s)(s + \lambda_1 s + \lambda_1) \left(\frac{1}{w_1(s)} - \frac{1}{w_2(s)} \right) ds \\ &+ \int_t^1 \frac{(1-s)}{w_2(s)} (s + \lambda_1 s + \lambda_1 - s - \lambda_2 s - \lambda_2) ds > 0 \end{aligned}$$

for all $t \in [\bar{t}, 1)$ where $\bar{t} := \max\{0, -\lambda_1/(1+\lambda_1)\}$. Since the second term is positive for $\lambda_1 \geq 0$ or for $\lambda_1 \in (-1/2, 0)$ and $|\lambda_1|/(1 - |\lambda_1|) < t < 1$, we obtain a contraction. This shows that the λ is uniquely determined by $x > 0$.

We now consider the case $y \in (0, w(1, x, \alpha))$. In this case, a standard shooting type argument proves that there must be a unique $\lambda = \lambda(x, y) \in (\alpha, \lambda_S)$ such that $w(1, x, \lambda) = y$.

We claim that such a $\lambda = \lambda(x, y)$ is unique. In fact, if $\lambda_1 > \lambda_2$ and $w(1, x, \lambda_1) = w(1, x, \lambda_2) = y > 0$, then Lemma 1 implies that $y = w(1, x, \lambda_1) < w(1, x, \lambda_2) = y$, a contradiction. The Lemma is thus proved.

PROOF OF LEMMA 3. The proof is similar to that of Lemma 2 and hence is omitted here.

We are now ready to prove the uniqueness of positive solutions to the problem (1.6)–(1.7).

PROPOSITION 2. *The problem* (1.6)–(1.7) *has at most one positive solution for any* $\lambda > -1/2$.

PROOF. Let $w_1(t)$ and $w_2(t)$ be two positive solutions to the problem (1.6)–(1.7).

If $w_1(0) = w_2(0) = 0$, we, in the proof of Theorem 3, will see that there is a unique $\lambda_* \in (-1/2, 0)$ such that (1.6)–(1.7) has a solution with initial value 0. Hence, $w_1(t)$ and $w_2(t)$ are two solutions of (1.6)–(1.7) with the same initial value and the same $\lambda = \lambda_*$. It suffices to discuss only the following three cases.

Case (i). $w_1(t) \geq w_2(t)$ for all $t \in [0, 1]$. Then

$$\begin{aligned} 0 \leq& w_1(t) - w_2(t) = \int_t^1 (1-s)(s - |\lambda_*| - |\lambda_*| s) \left(\frac{1}{w_1(s)} - \frac{1}{w_2(s)} \right) ds \\ &+ (1-t) \int_0^t s \left(\frac{1}{w_1(s)} - \frac{1}{w_2(s)} \right) ds \leq 0 \end{aligned}$$

if $|\lambda_*|/(1 - |\lambda_*|) < t < 1$. It is clear that at least one of the two terms should be negative if $w_1(t) \neq w_2(t)$. Therefore, $w_1 \equiv w_2$.

Case (ii). There is a $b \in (0, 1)$ such that $w_1(t) \geq w_2(t)$ for $t \in [0, b]$ but $w_1(t) < w_2(t)$ for t in $(b, b+\delta)$ with $\delta > 0$ sufficiently small. Then $w_1'(b) \leq w_2'(b)$. The uniqueness result to the initial value problem would result $w_1(t) \equiv w_2(t)$ on $[0, 1)$ if $w_1'(b) = w_2'(b)$. We now assume $w_1'(b) < w_2'(b)$. Then, by (2.2),

$$\begin{aligned} 0 &> w_1'(b) - w_2'(b) \\ &= \frac{|\lambda_*|(1-b^2)}{w_1(b)} - \frac{|\lambda_*|(1-b^2)}{w_2(b)} + \int_0^b s\left(\frac{1}{w_2(s)} - \frac{1}{w_1(s)}\right) ds \geq 0, \end{aligned}$$

which is a contradiction.

Case (iii). There exists a sequence of points $t_j \in (0, 1)$ such that

$$t_1 > t_2 > \cdots > t_j > t_{j+1} > \cdots$$

with $t_j \to 0$ as $j \to \infty$ and $(-1)^j(w_1(t_j) - w_2(t_j)) > 0$. Then, by the continuity of $w_1(t) - w_2(t)$, there would be positive local maximum points near $t = 0$. Let $t^* \in (0, 1)$ be one such point, i.e., $w_1(t^*) - w_2(t^*) > 0$, $w_1'(t^*) = w_2'(t^*) > 0$ and

$$\begin{aligned} 0 &\geq w_1''(t^*) - w_2''(t^*) \\ &= -\frac{t^*(2|\lambda_*|+1)}{w_1(t^*)} + \frac{t^*(2|\lambda_*|+1)}{w_2(t^*)} \\ &\quad + |\lambda_*|(1-(t^*)^2)w'(t^*)\left(\frac{1}{w_2^2(t^*)} - \frac{1}{w_1^2(t^*)}\right) > 0, \end{aligned}$$

which is again a contradiction.

If $w_1(0) = w_2(0) > 0$, the uniqueness of the initial value problem (2.9) or (2.10) shows that $w_1(t) \equiv w_2(t)$ for all $t \in [0, 1)$. Without loss of generality, we may assume $w_1(0) > w_2(0) \geq 0$. Then an argument similar to that in proving the case (ii) as above shows that $w_1(t) > w_2(t)$ for all $t \in [0, 1)$. Then, an inequality similar to that in proving case (i) will give us a contradiction. The proposition is thus proved.

We are now in a position to prove the existence of a positive solution to (1.6)–(1.7).

PROPOSITION 3. *For each fixed* $x > 0$, *there must be a unique* $\lambda = \lambda(x) \in (-1/2, +\infty)$ *such that the problem* (1.6)–(1.7) *has a positive solution* $w(t, \lambda)$ *with* $w(0, \lambda) = x$.

PROOF. We first claim that for each fixed $x > 0$,

(i) $\rho(x, -1/2) = 1$ and

(ii) $\rho(x, 3x^2) < 1$.

Claim (i) follows from the fact that

$$\begin{aligned} w(t,x,-1/2) &= x + \frac{1}{2}\int_0^t \frac{(1-s^2)ds}{w(s,x,-1/2)} - \int_0^t \frac{s(t-s)ds}{w(s,x,-1/2)} \\ &> \frac{1}{2}\int_0^t \frac{(1-s^2)ds}{w(s,x,-1/2)} - \int_0^t \frac{s(1-s)ds}{w(s,x,-1/2)} \\ &= \frac{1}{2}\int_0^t \frac{(1-s)^2 ds}{w(s,x,-1/2)} > 0 \end{aligned}$$

holds for all $t \in [0,1]$, where we used the equation in (2.10).

As for claim (ii), noticing that $w(t,x,3x^2)$ is strictly decreasing with respect to t, we have

$$\begin{aligned} w(t,x,3x^2) &= x - \int_0^t \frac{3x^2(1-s^2)+(t-s)s}{w(s,x,3x^2)}ds \\ &< x - \frac{1}{x}\int_0^t \{3x^2(1-s^2)+(t-s)s\}ds \\ &= x - \frac{1}{x}\left[3x^2\left(t-\frac{t^3}{3}\right)+\frac{t^3}{6}\right] \end{aligned}$$

for $0 < t < \rho(x,3x^2)$. Hence if $\rho(x,3x^2) = 1$, we would have

$$\lim_{t\to 1} w(t,x,3x^2) \le -x - \frac{1}{6x} < 0,$$

which contradicts the fact that $w(t,x,\lambda)$ is a positive solution to (2.9).

Then the results in Lemma 2 implies that there exists a unique $\lambda = \lambda(x) \in (-1/2, 3x^2)$ such that $w(t,\lambda) := w(t,x,\lambda(x))$ is a positive solution to the problem (1.6)–(1.7).

Concerning the function $w(t,\lambda) := w(t,x,\lambda(x))$, the following comparison principle holds.

PROPOSITION 4. *If $\bar{x} > x > 0$, then $\lambda(\bar{x}) > \lambda(x)$ and $w(t,\lambda(\bar{x})) \ge w(t,\lambda(x))$ for $t \in [0,1]$.*

PROOF. Set $y_n = w(1,x,-1/2)/2^n$, where $n = 1,2,\ldots$. By Lemma 2, there exists a sequence of numbers

$$\lambda_0 = -\frac{1}{2} < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots < \lambda(x)$$

such that $w(1,x,\lambda_n) = y_n$. Since $\bar{x} > x > 0$, we know that $w(1,\bar{x},-1/2) > w(1,x,-1/2)$. Therefore, we may find $\bar{\lambda}_0 > \lambda_0$ such that $w(1,\bar{x},\bar{\lambda}_0) = y_0$. It is obvious that

$$w(t,\bar{x},\bar{\lambda}_0) > w(t,x,\lambda_0) \quad \text{for } 0 \le t < 1\,.$$

The same argument can be applied to find $\bar{\lambda}_n > \lambda_n$ for each n such that $w(1,\bar{x},\bar{\lambda}_n) = y_n$ and

$$w(t,\bar{x},\bar{\lambda}_n) > w(t,x,\lambda_n) \quad \text{for all } 0 \le t < 1\,.$$

Since $\{\bar{\lambda}_n\}$ and $\{\lambda_n\}$ are all strictly increasing sequences and hence the limits exist, we have

$$\bar{\lambda} := \lambda(\bar{x}) =: \lim \bar{\lambda}_n \geq \lim \lambda_n =: \lambda(x) =: \lambda\,,$$

and

$$w(t, \bar{\lambda}) = \lim w(t, \bar{x}, \bar{\lambda}_n) \geq \lim w(t, x, \lambda_n) = w(t, \lambda) \quad \text{on } [0, 1]\,.$$

If $\bar{\lambda} = \lambda$, then by the uniqueness result, we get $w(t, \bar{\lambda}) \equiv w(t, \lambda)$ on $[0, 1]$. But $w(0, \bar{\lambda}) = \bar{x} > x = w(0, \lambda)$ which is impossible and hence $\bar{\lambda} > \lambda$. The proof is thus complete.

PROPOSITION 5. *There exists a* $\lambda_* \in (-1/2, 0)$ *such that the problem* (1.6)–(1.7) *has a unique positive solution* $w(t, \lambda_*)$ *with* $w(0, \lambda_*) = 0$.

PROOF. Let $\{x_n\}$ be a strictly decreasing sequence and

$$0 < x_1 < \frac{1}{\sqrt{6}}, \quad \lim_{n\to\infty} x_n = 0\,.$$

Proposition 3, Proposition 4 and (2.3) imply that there exists a unique $\lambda_n \in (-1/2, 0)$ such that the problem (1.6)–(1.7) has a positive solution $w(t, \lambda_n) := w(t, x_n, \lambda_n)$ with $\lambda_n > \lambda_{n+1}$ and $w(t, \lambda_n) \geq w(t, \lambda_{n+1})$ on $[0, 1]$.

Notice that

$$w(t, \lambda_n) = \int_t^1 \frac{(1-s)(s + \lambda_n s + \lambda_n)}{w(s, \lambda_n)} ds + (1-t) \int_0^t \frac{sds}{w(t, \lambda_n)}\,, \quad 0 \leq t \leq 1\,, \tag{2.11}$$

$$w'(t, \lambda_n) = \frac{|\lambda_n|(1-t^2)}{w(t, \lambda_n)} - \int_0^t \frac{sds}{w(s, \lambda_n)}\,, \quad 0 \leq t < 1\,. \tag{2.12}$$

Then we have by (2.11) that

$$w(t, \lambda_n) \geq (1-t) \int_0^t \frac{sds}{w(s, \lambda_n)} \geq (1-t) \int_0^t \frac{sds}{w(s, \lambda_1)} =: v(t)\,, \quad 0 \leq t \leq 1\,.$$

From the proof of (2.4), we know that

$$v(t) \sim \sqrt{2}(1-t)|\log(1-t)|^{1/2} \quad \text{as } t \to 1\,.$$

Also, we know that there exists a $t_n \in (0, 1)$ such that $w'(t, \lambda_n) > 0$ in $[0, t_n)$, $w'(t, \lambda_n) < 0$ in $(t_n, 1)$ and $w'(t_n, \lambda_n) = 0$. Since $w(t, \lambda_n)$ is concave on $[0, t_n]$, we have

$$w(t, \lambda_n) \geq w(t_n, \lambda_n) \frac{t}{t_n} \quad \text{on } [0, t_n]\,.$$

From (2.12) and the fact that $w'(t_n, \lambda_n) = 0$, we know that

$$\frac{|\lambda_n|(1-t_n^2)}{w(t_n, \lambda_n)} = \int_0^{t_n} \frac{sds}{w(s, \lambda_n)} < \int_0^{t_n} \frac{t_n sds}{sw(t_n, \lambda_n)} = \frac{t_n^2}{w(t_n, \lambda_n)}\,,$$

which implies that

$$t_n > \sqrt{\frac{|\lambda_n|}{1 + |\lambda_n|}} > \sqrt{\frac{|\lambda_1|}{2}}\,.$$

Let $u_n(t) := w^2(t, \lambda_n)$. Then we have by (2.11) that $u_n'(0) = 2|\lambda_n| > 2|\lambda_1|$, which implies that there exists a $\delta \in (0, \sqrt{|\lambda_1|/2})$ such that

$$w(t, \lambda_n) \geq \sqrt{|\lambda_1| t} \quad \text{for all } t \in [0, \delta].$$

Therefore, we have

$$w(t, \lambda_n) \geq w_0(t) \quad \text{for all } t \in [0, 1],$$

where

$$w_0(t) := \begin{cases} \sqrt{|\lambda_1| t}, & 0 \leq t \leq \delta, \\ v(t), & \delta < t \leq 1, \end{cases} \quad \text{and} \quad \int_0^t \frac{ds}{w_0(s)} < +\infty.$$

Set $\lambda_* := \lim_{n\to\infty} \lambda_n \geq -1/2$. Then the sequence $\{w(t, \lambda_n)\}$ converges uniformly in any closed subinterval of (0, 1) and $w(t, \lambda_*) := \lim_{n\to\infty} w(t, \lambda_n) \geq w_0(t)$. Letting $n \to \infty$ in (2.11) and then applying the Dominated Convergence Theorem, we get

$$\begin{aligned} w(t, \lambda_*) &= \int_t^1 \frac{(1-s)(s - |\lambda_*|s - |\lambda_*|)ds}{w(s, \lambda_*)} + (1-t)\int_0^t \frac{sds}{w(s, \lambda_*)} \\ &\geq w_0(t) \end{aligned} \tag{2.13}$$

for $0 \leq t \leq 1$. Since $w_0(t) \leq w(t, \lambda_*) \leq w(t, \lambda_1)$ on [0, 1], $w(t, \lambda_*)$ is continuous near $t = 1$ and hence $w(1, \lambda_*) = 0$.

On the other hand, we have by (2.12) that

$$w'(t, \lambda_n) \leq |\lambda_n|/w(t, \lambda_n) \quad \text{for all } t \in [0, 1],$$

i.e.,

$$w(t, \lambda_n) \leq \sqrt{2|\lambda_n| t + x_n^2}, \quad 0 \leq t \leq 1.$$

Hence, we get by letting $n \to \infty$

$$w(t, \lambda_*) \leq \sqrt{2|\lambda_*| t}, \quad 0 \leq t \leq 1.$$

This shows that $w(t, \lambda_*)$ is also continuous near $t = 0$ and hence $w(0, \lambda_*) = 0$.

Therefore, we conclude by (2.13) that the $w(t, \lambda_*)$ is a positive solution to the problem (1.6)–(1.7).

Proposition 1 implies that $\lambda_* > -1/2$ and Proposition 2 tells us that $w(t, \lambda_*)$ is unique.

REMARK. Numerical analysis shows us that $\lambda_* \doteq -0.199$ (see [7, p. 150]).

With all the above propositions and lemmas, we can now establish the following three main theorems.

THEOREM 1. *There exists a* $\lambda_* \in (-1/2, 0)$ *such that the problem* (1.6)–(1.7) *has a unique positive solution* $w(t, \lambda)$ *for each fixed* $\lambda \geq \lambda_*$.

THEOREM 2. *Let* $w(t, \lambda)$ *be the unique positive solution to the problem* (1.6)–(1.7) *with* $\lambda \geq \lambda_*$. *Then the following two statements hold.*

i) *If* $\lambda_1 \geq \lambda_2$, *then* $w(t, \lambda_1) \geq w(t, \lambda_2)$ *for all* $t \in [0, 1]$.

ii) *If* $\lambda_1 > \lambda_2$, *then* $w(0, \lambda_1) > w(0, \lambda_2)$.

THEOREM 3. *The problem* (1.6)–(1.7) *has no positive solution for any* $\lambda < \lambda_*$.

PROOF OF THEOREM 1. Let λ_* be defined by Proposition 5.

Proposition 3, Proposition 4 and Proposition 5 show us that for each fixed $x \geq 0$ there exists a unique $\lambda = \lambda(x)$ such that the problem (1.6)–(1.7) has a unique positive solution $w(t, \lambda)$ with $w(0, \lambda) = x$ and that the $\lambda(x)$, as a function of x, is strictly increasing on $[0, +\infty)$ and

$$\lim_{x\to 0} \lambda(x) = \lambda_* \qquad \lim_{x\to+\infty} \lambda(x) = +\infty, \tag{2.14}$$

since by (2.3), if $x > \sqrt{1/3}$.

$$\sqrt{\frac{4\lambda(x)+1}{6}} < x < \sqrt{\frac{4\lambda(x)+1}{3}}.$$

But it has not been pointed out yet that whether the problem (1.6)–(1.7) has a positive solution for each fixed $\lambda > \lambda_*$. Therefore, we need only to prove that the problem (1.6)–(1.7) has a positive solution for each fixed $\lambda > \lambda_*$, since the solution $w(t, \lambda_*)$ has been proved to exist. Here we will employ a modified shooting type argument again.

Let $\lambda > \lambda_*$ be fixed. By (2.14), we can choose a sufficiently small positive number a such that $\lambda(a) \in (\lambda_*, \lambda)$. We claim that $\rho(a, \lambda) < 1$. In fact, $1 = \rho(a, \lambda(a)) \geq \rho(a, \lambda)$, by Lemma 1, and if $\rho(a, \lambda) = 1$ then $w(t, \lambda) = w(t, a, \lambda)$ is also a positive solution to (1.6)–(1.7). This is impossible by the uniqueness result.

By (2.14) again, we can choose a sufficiently large positive number b such that $\lambda(b) > \lambda$ and hence by Lemma 1 $\rho(b, \lambda) \geq \rho(b, \lambda(b)) = 1$. Lemma 3 shows us that there must be a unique $x = x(\lambda) \in (a, b)$ such that $w(t, \lambda) = w(t, x(\lambda), \lambda)$ is a positive solution to the problem (1.6)–(1.7). Theorem 1 is thus proved.

PROOF OF THEOREM 2. Theorem 2 follows from the results in Proposition 4 and Proposition 2.

PROOF OF THEOREM 3. Proposition 1 shows us that the problem (1.6)–(1.7) has no positive solution for $\lambda \leq -1/2$ and hence we have only to prove that for any $\lambda \in (-1/2, \lambda_*)$, the problem (1.6)–(1.7) has no positive solution. Proposition 2 and Proposition 3 imply that if $w(t, \lambda)$ is a positive solution to the problem (1.6)–(1.7) with $\lambda \in (-1/2, \lambda_*)$, then $w(0, \lambda) = 0$. By (2.2), we get

$$\begin{aligned}\frac{1}{2}(w^2(t,\lambda) - w^2(t,\lambda_*))' =& (|\lambda| - |\lambda_*|)(1-t^2) \\ &- w(t,\lambda)\int_0^t \frac{sds}{w(s,\lambda)} + w(t,\lambda_*)\int_0^t \frac{sds}{w(s,\lambda_*)}\end{aligned}$$

for $0 < t < 1$. Therefore,

$$\lim_{t\to 0} \frac{1}{2}(w^2(t,\lambda) - w^2(t,\lambda_*))' = |\lambda| - |\lambda_*| > 0.$$

Hence, there is a $\delta \in (0, 1)$ such that $w(t, \lambda) > w(t, \lambda_*)$ in $(0, \delta)$. Assume that there exists a $b \in (0, 1)$ such that $w(t, \lambda) > w(t, \lambda_*)$ for $t \in (0, b)$, $w(b, \lambda) = w(b, \lambda_*)$, and hence

$w'(b,\lambda) \le w'(b,\lambda_*)$. Then, we are lead to

$$0 \ge w'(b,\lambda) - w'(b,\lambda_*) = \frac{|\lambda|(1-b^2)}{w(b,\lambda)} - \frac{|\lambda_*|(1-b^2)}{w(b,\lambda_*)} - \int_0^b s\left(\frac{1}{w(s,\lambda)} - \frac{1}{w(s,\lambda_*)}\right) ds > 0,$$

which is a contradiction. Hence, $w(t,\lambda) > w(t,\lambda_*)$ for all $t \in (0,1)$. Then,

$$\begin{aligned} 0 &< w(t,\lambda) - w(t,\lambda_*) \\ &= \int_t^1 \frac{(1-s)(-|\lambda| - |\lambda|s + s)ds}{w(s,\lambda)} - \int_t^1 \frac{(1-s)(s - |\lambda_*| - |\lambda_*|s)ds}{w(s,\lambda_*)} \\ &\quad + (1-t)\int_0^t s\left(\frac{1}{w(s,\lambda)} - \frac{1}{w(s,\lambda_*)}\right) ds < 0 \end{aligned}$$

for $t \in (|\lambda|/(1-|\lambda|), 1)$, which is again a contradiction. This proves Theorem 3.

3. The results to the original problem. In this section, we will state and prove the uniqueness, existence and nonexistence results to the problem (1.1)–(1.2) by utilizing the unique positive solution $w(t,\lambda)$ to the problem (1.6)–(1.7). In fact, we have

THEOREM 4. *Let $\lambda_* \in (-1/2, 0)$ be the number defined by Proposition 5. Then, the boundary value problem (1.1)–(1.2) has a unique normal solution $f(\eta,\lambda)$ for each fixed $\lambda \ge \lambda_*$.*

THEOREM 5. *Let $f(\eta,\lambda)$ be the normal solution, where $\lambda \ge \lambda_*$. Then*

$$f'(\eta,\lambda_1) > f'(\eta,\lambda_2) \quad \textit{for all } \eta > 0, \quad \textit{whenever} \quad \lambda_1 > \lambda_2.$$

THEOREM 6. *The problem (1.1)–(1.2) has no normal solution for $\lambda < \lambda_*$.*

Some analogous results to the problem (1.1)–(1.2) have been achieved by many other authors. One may find them in [1–7] and the references therein. But most of the results were obtained by numerical analysis. To our knowledge, the uniqueness result for $\lambda \in [\lambda_*, 0)$ has not been seen and the comparison principle in Theorem 5 has not been proved theoretically in other references, although the numerical results in [7] provided us some hints. Also, the result of nonexistence to normal solutions has not been proved by others yet.

PROOF OF THEOREM 4. Let $w(t,\lambda)$ be the unique positive solution to the problem (1.6)–(1.7) with $\lambda \ge \lambda_*$. We define

$$g(t,\lambda) := \int_0^t \frac{ds}{w(s,\lambda)}, \quad \text{for } t \ge 0.$$

From (2.4) and (2.5), it follows that the integral converges at $t = 0$ in the case of $\lambda = \lambda_*$ (i.e., $w(0,\lambda_*) = 0$) and that

$$g(1-0,\lambda) = \int_0^1 \frac{ds}{w(s,\lambda)} = +\infty.$$

Therefore, $\eta = g(t, \lambda)$ is a strictly increasing function defined on $[0, 1]$ and hence has an inverse function. We denote the inverse function as $t = f'(\eta, \lambda)$ and

$$f(\eta, \lambda) = \int_0^{\eta} f'(s, \lambda) ds \quad \text{for all } \eta \geq 0 .$$

Then (1.2) is automatically satisfied by $f(\eta)$.

Plugging $t = f'(\eta, \lambda)$ into the definition of $g(t, \lambda)$, we get

$$\eta \equiv g(f'(\eta, \lambda), \lambda) = \int_0^{f'(\eta,\lambda)} \frac{ds}{w(s, \lambda)} \quad \text{for all } \eta \geq 0 . \tag{3.1}$$

Differentiating (3.1) with respect to η yields

$$f''(\eta, \lambda) = w(f'(\eta, \lambda), \lambda) > 0 \quad \text{for all } \eta > 0 , \tag{3.2}$$

and hence $0 < f'(\eta, \lambda) < 1$ for all $\eta > 0$. Notice that

$$\frac{d}{d\eta}\left(\int_0^{f'(\eta,\lambda)} \frac{sds}{w(s, \lambda)}\right) = \frac{f'(\eta, \lambda)}{w(f'(\eta, \lambda), \lambda)} f''(\eta, \lambda) = f'(\eta, \lambda) .$$

Combining the fact that $f'(0, \lambda) = 0$, we get

$$f(\eta, \lambda) = \int_0^{f'(\eta,\lambda)} \frac{sds}{w(s, \lambda)} \quad \text{for all } \eta \geq 0 .$$

Differentiating (3.2) with respect to η, we get

$$f'''(\eta, \lambda) = w'(f'(\eta, \lambda), \lambda) f''(\eta, \lambda) \quad \text{for all } \eta > 0 .$$

Plugging $t = f'(\eta, \lambda)$ into (2.2), we get

$$w'(f'(\eta, \lambda), \lambda) + \int_0^{f'(\eta,\lambda)} \frac{sds}{w(s, \lambda)} + \frac{\lambda(1 - (f'(\eta, \lambda))^2)}{w(f'(\eta, \lambda), \lambda)} = 0, \quad \eta > 0 .$$

That is,

$$\frac{f'''(\eta, \lambda)}{f''(\eta, \lambda)} + f(\eta, \lambda) + \frac{\lambda(1 - (f'(\eta, \lambda))^2)}{f''(\eta, \lambda)} = 0, \quad \eta > 0 .$$

This shows that such a function $f(\eta, \lambda)$, defined by using $w(t, \lambda)$, is a solution of (1.1) which satisfies the conditions to a normal solution. The proof is complete.

Proof of Theorem 5. Let $\lambda_1 > \lambda_2 \geq \lambda_*$. We know by (3.1) that

$$\eta = \int_0^{f'(\eta,\lambda_1)} \frac{ds}{w(s, \lambda_1)} = \int_0^{f'(\eta,\lambda_2)} \frac{ds}{w(s, \lambda_2)} \quad \text{for all } \eta \geq 0 .$$

We rewrite the above equality in the form

$$\int_{f'(\eta,\lambda_2)}^{f'(\eta,\lambda_1)} \frac{ds}{w(s, \lambda_1)} = \int_0^{f'(\eta,\lambda_2)} \left(\frac{1}{w(s, \lambda_2)} - \frac{1}{w(s, \lambda_1)}\right) ds > 0 \tag{3.3}$$

whenever $\eta > 0$. Here we have used Theorem 2. The conclusion of Theorem 5 follows from (3.3).

Proof of Theorem 6. Theorem 3 implies Theorem 6.

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DEPARTMENT OF MATHEMATICS
JILIN UNIVERSITY
CHANGCHUN 130023
PEOPLE'S REPUBLIC OF CHINA

E-mail address: math@mail.jlu.edu.cn