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QUASI-EINSTEIN TOTALLY REAL SUBMANIFOLDS OF THE NEARLY KÄHLER 6-SPHERE

RYSZARD DESZCZ, FRANKI DILLEN, LEOPOLD VERSTRAELEN AND LUC VRANCKEN

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Abstract. We investigate Lagrangian submanifolds of the nearly Kähler 6-sphere. In particular we investigate Lagrangian quasi-Einstein submanifolds of the 6-sphere. We relate this class of submanifolds to certain tubes around almost complex curves in the 6-sphere.

1. Introduction. In this paper, we investigate 3-dimensional totally real submanifolds M^3 of the nearly Kähler 6-sphere S^6. A submanifold M^3 of S^6 is called totally real if the almost complex structure J on S^6 interchanges the tangent and the normal space. It has been proven by Ejiri ([E1]) that such submanifolds are always minimal and orientable. In the same paper, he also classified those totally real submanifolds with constant sectional curvature. Note that 3-dimensional Einstein manifolds have constant sectional curvature. Here, we will investigate the totally real submanifolds of S^6 for which the Ricci tensor has an eigenvalue with multiplicity at least 2. In general, a manifold M^n whose Ricci tensor has an eigenvalue of multiplicity at least $n-1$ is called quasi-Einstein.

The paper is organized as follows. In Section 2, we recall the basic formulas about the vector cross product on $\boldsymbol{R}^7$ and the almost complex structure on S^6. We also relate the standard Sasakian structure on S^5 with the almost complex structure on S^6. Then, in Section 3, we derive a necessary and sufficient condition for a totally real submanifold of S^6 to be quasi-Einstein. Using this condition, we deduce from [C], see also [CDVV1] and [DV], that totally real submanifolds M with $\delta_M = 2$ are quasi-Einstein. Here, δ_M is the Riemannian invariant defined by

$$\delta_M(p) = \tau(p) - \inf K(p)\,,$$

where $\inf K$ is the function assigning to each $p \in M$ the infimum of $K(\pi)$, $K(\pi)$ denoting the sectional curvature of a 2-plane π of T_pM, where π runs over all 2-planes in T_pM and τ is the scalar curvature of M defined by $\tau = \sum_{i<j} K(e_i \wedge e_j)$. Totally real submanifolds of S^6 with $\delta_M = 2$ have been classified in [DV]. Essentially, these submanifolds are either local lifts of holomorphic curves in $\boldsymbol{C}P^2$ or tubes with radius $\pi/2$ in the direction of NN^2, where N^2 is a non-totally geodesic almost complex curve and NN^2 denotes the vector bundle whose fibres are planes orthogonal to the first osculating space of N^2.

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In Section 4, we then construct some other examples of 3-dimensional quasi-Einstein totally real submanifolds by considering also tubes with different radii. More specifically, we prove

THEOREM 1. *Let $\phi : N^2 \to S^6$ be an almost complex curve in $S^6(1)$ without totally geodesic points. Denote by UN^2 the unit tangent bundle of N^2. Define*

$$\psi_\gamma : UN^2 \to S^6 : v \mapsto \cos\gamma\phi + \sin\gamma v \times \frac{\alpha(v, v)}{\|\alpha(v, v)\|},$$

where α denotes the second fundamental form of the surface N^2. Then ψ is an immersion on an open dense subset of UN^2. Moreover ψ is totally real if and only if either

(1) $\gamma = \pi/2$, *or*

(2) $\cos^2\gamma = 5/9$ *and N^2 is a superminimal surface.*

Further, in both cases the immersion defines a quasi-Einstein metric on UN^2 and if (1) *holds, then with respect to this metric $\delta_{UN^2} = 2$ and if* (2) *holds, then $\delta_{UN^2} < 2$.*

The above theorem also generalizes results obtained by Ejiri [E2], who only considered tubes around superminimal almost complex curves and who omitted Case (1).

Next, in Section 5, we prove the following converse:

THEOREM 2. *Let $F : M^3 \to S^6$ be a totally real immersion of a 3-dimensional quasi-Einstein manifold. Then either $\delta_{M^3} = 2$ or there exists an open dense subset W of M such that each point p of W has a neighborhood W_1 such that either*

(1) $F(W_1) = \psi_\gamma(UN^2)$, *where N^2 is a superminimal linearly full almost complex curve in S^6, and ψ_γ with $\cos^2\gamma = 5/9$ is as defined in Theorem* 1, *or*

(2) $F(W_1)$ *is an open subset of $\tilde{\psi}(S^3)$, where $\tilde{\psi}$ is as defined in Section* 4.

Case (2) can be considered as a limit case of the previous one, by taking for N^2 a totally geodesic almost complex curve. Note also that Theorem 2 together with the classification theorems of [DV] provides a complete classification of the totally real quasi-Einstein submanifolds of S^6.

2. The vector cross product and the almost complex structure on S^6. We give a brief exposition of how the standard nearly Kähler structure on S^6 arises in a natural manner from the Cayley multiplication. For further details about the Cayley numbers and their automorphism group G_2, we refer the reader to [W] and [HL].

The multiplication on the Cayley numbers $\mathcal{O}$ may be used to define a vector cross product on the purely imaginary Cayley numbers $\boldsymbol{R}^7$ using the formula

$$u \times v = (1/2)(uv - vu), \tag{2.1}$$

while the standard inner product on $\boldsymbol{R}^7$ is given by

$$(u, v) = -(1/2)(uv + vu). \tag{2.2}$$

It is now elementary to show that

$$u \times (v \times w) + (u \times v) \times w = 2(u, w)v - (u, v)w - (w, v)u, \tag{2.3}$$

and that the triple scalar product $(u \times v, w)$ is skew symmetric in u, v, w, see [HL] for proofs.

Conversely, denoting by $\mathrm{Re}(\mathcal{O})$ (respectively $\mathrm{Im}(\mathcal{O})$) the real (respectively imaginary) Cayley numbers, the Cayley multiplication on $\mathcal{O}$ is given in terms of the vector cross product and the inner product by

$$(2.4)\qquad \begin{aligned} &(r+u)(s+v) = rs - (u, v) + rv + su + (u \times v)\,, \\ &r, s \in \mathrm{Re}(\mathcal{O})\,, \quad u, v \in \mathrm{Im}(\mathcal{O})\,. \end{aligned}$$

In view of (2.1), (2.2) and (2.4), it is clear that the group G_2 of automorphisms of $\mathcal{O}$ is precisely the group of isometries of $\boldsymbol{R}^7$ preserving the vector cross product.

An ordered orthonormal basis $e_1, \dots, e_7$ is said to be a *G_2-frame* if

$$(2.5)\qquad e_3 = e_1 \times e_2\,, \quad e_5 = e_1 \times e_4\,, \quad e_6 = e_2 \times e_4\,, \quad e_7 = e_3 \times e_4\,.$$

For example, the standard basis of $\boldsymbol{R}^7$ is a G_2-frame. Moreover, if e_1, e_2, e_4 are mutually orthogonal unit vectors with e_4 orthogonal to $e_1 \times e_2$, then e_1, e_2, e_4 determine a unique G_2-frame $e_1, \dots, e_7$ and $(\boldsymbol{R}^7, \times)$ is generated by e_1, e_2, e_4 subject to the relations

$$(2.6)\qquad e_i \times (e_j \times e_k) + (e_i \times e_j) \times e_k = 2\delta_{ik} e_j - \delta_{ij} e_k - \delta_{jk} e_i\,.$$

Therefore, for any G_2-frame, we have the following multiplication table:

$\times$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	0	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	$-e_3$	0	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	0	e_7	$-e_6$	e_5	$-e_4$
e_4	$-e_5$	$-e_6$	$-e_7$	0	e_1	e_2	e_3
e_5	e_4	$-e_7$	e_6	$-e_1$	0	$-e_3$	e_2
e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	0	$-e_1$
e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	0

The standard nearly Kähler structure on S^6 is then obtained as follows.

$$Ju = x \times u\,, \quad u \in T_x S^6\,,\ x \in S^6\,.$$

It is clear that J is an orthogonal almost complex structure on S^6. In fact, J is a nearly Kähler structure in the sense that the (2,1)-tensor field G on $S^6(1)$ defined by

$$G(X, Y) = (\tilde{\nabla}_X J)(Y)\,,$$

where $\tilde{\nabla}$ is the Levi-Civita connection on $S^6(1)$, is skew-symmetric. A straightforward computation also shows that

$$G(X, Y) = X \times Y - \langle x \times X, Y \rangle x\,.$$

For more information on the properties of the Cayley multiplication, J and G, we refer to [Ca2], [BVW] and [DVV].

Finally, we recall the explicit construction of a Sasakian structure on $S^5(1)$ starting from $\boldsymbol{C}^3$ and its relation with the nearly Kähler structure on S^6. For more details about Sasakian structures we refer the reader to [B]. We consider S^5 as the hypersphere in $S^6 \subset \boldsymbol{R}^7$ given by the equation $x_4 = 0$ and define

$$j : S^5(1) \to \boldsymbol{C}^3 : (x_1, x_2, x_3, 0, x_5, x_6, x_7) \mapsto (x_1 + ix_5, x_2 + ix_6, x_3 + ix_7)\,.$$

Then at a point p the structure vector field is given by

$$\xi(p) = (x_5, x_6, x_7, 0, -x_1, -x_2, -x_3) = e_4 \times p\,,$$

and for a tangent vector $v = (v_1, v_2, v_3, 0, v_5, v_6, v_7)$, orthogonal to ξ, we have

$$\phi(v) = (-v_5, -v_6, -v_7, 0, v_1, v_2, v_3) = v \times e_4\,.$$

Also, $\phi\xi(p) = 0 = (e_4 \times p) \times e_4 - \langle (e_4 \times p) \times e_4, p \rangle p$, from which we deduce for any tangent vector w to S^5 that

$$\phi(w) = w \times e_4 - \langle w \times e_4, p \rangle p\,. \tag{2.7}$$

3. A pointwise characterization. Let M^3 be a totally real submanifold of S^6. From [E1], we know that M^3 is minimal and that for tangent vector fields X and Y, $G(X, Y)$ is a normal vector field on M^n. Moreover, $\langle h(X, Y), JZ \rangle$ is symmetric in X, Y and Z. Denote by S the Ricci tensor of M^3 defined by

$$S(Y, Z) = \text{trace}\{X \mapsto R(X, Y)Z\}\,,$$

and denote by Ric the associated 1-1 tensor field, i.e.,

$$\langle \text{Ric}(Y), Z \rangle = S(Y, Z)\,.$$

Let $p \in M$ and assume that p is not a totally geodesic point of M^3. Then, we know from [V] that there exists an orthonormal basis $\{e_1, e_2, e_3\}$ at the point p such that

$$\begin{aligned} h(e_1, e_1) &= \lambda_1 Je_1\,, & h(e_2, e_2) &= \lambda_2 Je_1 + aJe_2 + bJe_3\,, \\ h(e_1, e_2) &= \lambda_2 Je_2\,, & h(e_2, e_3) &= bJe_2 - aJe_3\,, \\ h(e_1, e_3) &= \lambda_3 Je_3\,, & h(e_3, e_3) &= \lambda_3 Je_1 - aJe_2 - bJe_3\,, \end{aligned}$$

where $\lambda_1 + \lambda_2 + \lambda_3 = 0$, $\lambda_1 > 0$, $\lambda_1 - 2\lambda_2 \geq 0$ and $\lambda_1 - 2\lambda_3 \geq 0$. If $\lambda_2 = \lambda_3$, we can choose e_2 and e_3 such that $b = 0$. Then by a straightforward computation, we have the following lemma:

LEMMA 3.1. *Let $\{e_1, e_2, e_3\}$ be the basis constructed above. Then*

$$(S(e_i, e_j)) = \begin{pmatrix} 2 - \lambda_1^2 - \lambda_2^2 - \lambda_3^2 & -(\lambda_2 - \lambda_3)a & -(\lambda_2 - \lambda_3)b \\ -(\lambda_2 - \lambda_3)a & 2 - 2\lambda_2^2 - 2a^2 - 2b^2 & 0 \\ -(\lambda_2 - \lambda_3)b & 0 & 2 - 2\lambda_3^2 - 2a^2 - 2b^2 \end{pmatrix}.$$

Remark that if $\lambda_2 = \lambda_3$, it immediately follows from Lemma 3.1 that M is quasi-Einstein.

LEMMA 3.2. *Let M^3 be a 3-dimensional totally real submanifold of S^6 with the second fundamental form h. Then the Ricci tensor S has a double eigenvalue at a point p of M^3*

if and only if p is a totally geodesic point or there exists an orthonormal basis $\{e_1, e_2, e_3\}$ *of* T_pM *such that either*

$$
\begin{array}{lll}
(1) & h(e_1,e_1) = \lambda J e_1\,, & h(e_2,e_2) = -\lambda J e_1\,, \\
 & h(e_1,e_2) = -\lambda J e_2\,, & h(e_2,e_3) = 0\,, \\
 & h(e_1,e_3) = 0\,, & h(e_3,e_3) = 0\,, \\
(2) & h(e_1,e_1) = 2\lambda J e_1\,, & h(e_2,e_2) = -\lambda J e_1 + aJe_2\,, \\
 & h(e_1,e_2) = -\lambda J e_2\,, & h(e_2,e_3) = -aJe_3\,, \\
 & h(e_1,e_3) = -\lambda J e_3\,, & h(e_3,e_3) = -\lambda J e_1 - aJe_2\,,
\end{array}
$$

where λ *is a non-zero number.*

PROOF. If p is a totally geodesic point of M^3, there is nothing to prove. Hence, we may assume that p is not totally geodesic and we can use the basis $\{e_1, e_2, e_3\}$ constructed above. So we see that

$$
\begin{aligned}
\mathrm{Ric}(e_2) &= -(\lambda_2 - \lambda_3)ae_1 + 2(1 - \lambda_2^2 - a^2 - b^2)e_2\,, \\
\langle \mathrm{Ric}(e_1), e_3\rangle &= -(\lambda_2 - \lambda_3)b\,.
\end{aligned}
$$

Since M^3 is quasi-Einstein, we know that e_2, $\mathrm{Ric}(e_2)$ and $\mathrm{Ric}(\mathrm{Ric}(e_2))$ have to be linearly dependent. Hence the above formulas imply that

$$ab(\lambda_2 - \lambda_3)^2 = 0\,.$$

If $\lambda_2 = \lambda_3$, we see that $\{e_1, e_2, e_3\}$ satisfies Case (2) of Lemma 3.2 by rechoosing e_2 and e_3 if necessary. Therefore, we may assume that $\lambda_2 \neq \lambda_3$. Then, if necessary by interchanging e_2 and e_3, we may assume that $b = 0$.

Suppose now that $a = 0$. Hence e_1, e_2 and e_3 are eigenvectors of Ric. Since we assumed that $\lambda_2 \neq \lambda_3$, we see that (if necessary after interchanging e_2 and e_3, which is allowed in this case since a and b both vanish) M^3 is quasi-Einstein if and only if

$$2 - \lambda_1^2 - \lambda_2^2 - \lambda_3^2 = 2 - 2\lambda_2^2\,,$$

which reduces to

$$-2\lambda_1^2 - 2\lambda_1\lambda_2 = 0\,.$$

Hence, since $\lambda_1 \neq 0$, we see that $\lambda_2 = -\lambda_1$ and $\lambda_3 = 0$. Thus $\{e_1, e_2, e_3\}$ is a basis as described in Case (1) of Lemma 3.2.

Finally, we consider the case that $\lambda_2 \neq \lambda_3$ and $a \neq 0$. Since $a \neq 0$, we see that M^3 is quasi-Einstein if and only if $2 - 2\lambda_3^2 - 2a^2$ is a double eigenvalue of S. This is the case if and only if

$$\det\begin{pmatrix} \lambda_3^2 - \lambda_1^2 - \lambda_2^2 + 2a^2 & (\lambda_3 - \lambda_2)a \\ (\lambda_3 - \lambda_2)a & 2(\lambda_3^2 - \lambda_2^2) \end{pmatrix} = 0\,.$$

Since $\lambda_2 \neq \lambda_3$ and $\lambda_3 = -\lambda_1 - \lambda_2$, this is the case if and only if

$$\det\begin{pmatrix} 2\lambda_1\lambda_2 + 2a^2 & -(\lambda_1 + 2\lambda_2)a \\ a & -2\lambda_1 \end{pmatrix} = 0\,,$$

i.e., if and only if

$$4\lambda_1^2\lambda_2 = -3\lambda_1 a^2 + 2\lambda_2 a^2 . \tag{3.1}$$

Now, we consider the following change of basis

$$\begin{aligned} u_1 &= \frac{1}{\sqrt{a^2+4\lambda_1^2}}(ae_1 - 2\lambda_1 e_2) , \\ u_2 &= \frac{1}{\sqrt{a^2+4\lambda_1^2}}(2\lambda_1 e_1 + ae_2) , \\ u_3 &= e_3 . \end{aligned}$$

Then, using (3.1), we have

$$\begin{aligned} h(ae_1 - 2\lambda_1 e_2, ae_1 - 2\lambda_1 e_2) &= (a^2\lambda_1 + 4\lambda_1^2\lambda_2)Je_1 + (-4a\lambda_1\lambda_2 + 4a\lambda_1^2)Je_2 \\ &= -2(\lambda_1 - \lambda_2)a(aJe_1 - 2\lambda_1 Je_2) , \\ h(ae_1 - 2\lambda_1 e_2, e_3) &= a(\lambda_1 - \lambda_2)Je_3 , \\ h(2\lambda_1 e_1 + ae_2, e_3) &= (2\lambda_1\lambda_3 - a^2)Je_3 , \\ h(ae_1 - 2\lambda_1 e_2, 2\lambda_1 e_1 + ae_2) &= (2a\lambda_1^2 - 2a\lambda_1\lambda_2)Je_1 + (a^2\lambda_2 - 2a^2\lambda_1 - 4\lambda_1^2\lambda_2)Je_2 \\ &= a(\lambda_1 - \lambda_2)(2\lambda_1 Je_1 + aJe_2) . \end{aligned}$$

Using now the minimality of M, together with the fact that $\langle h(X,Y), JZ\rangle$ is totally symmetric it follows that the basis $\{u_1, u_2, u_3\}$ satisfies Case (2) of Lemma 3.2.

REMARK 3.3. An elementary computation shows that if Case (1) of Lemma 3.2 is satisfied, the Ricci tensor has eigenvalues $2(1-\lambda^2)$, $2(1-\lambda^2)$ and 2, while if Case (2) is satisfied its eigenvalues are $2-6\lambda^2$, $2-2\lambda^2-2a^2$ and $2-2\lambda^2-2a^2$.

REMARK 3.4. Submanifolds satisfying Case (1) of Lemma 3.2 are exactly those totally real submanifolds of S^6 which satisfy Chen's equality (see [CDVV1], [CDVV2] and [DV]). A complete classification of these submanifolds was obtained in [DV].

4. Examples of totally real submanifolds.

EXAMPLE 4.1. We recall from [DVV] the following example: Consider the unit sphere

$$S^3 = \left\{(y_1, y_2, y_3, y_4) \in \boldsymbol{R}^4 \mid y_1^2 + y_2^2 + y_3^2 + y_4^2 = 1\right\}$$

in $\boldsymbol{R}^4$. Let X_1, X_2 and X_3 be the vector fields defined by

$$\begin{aligned} X_1(y_1, y_2, y_3, y_4) &= (y_2, -y_1, y_4, -y_3) , \\ X_2(y_1, y_2, y_3, y_4) &= (y_3, -y_4, -y_1, y_2) , \\ X_3(y_1, y_2, y_3, y_4) &= (y_4, y_3, -y_2, -y_1) . \end{aligned}$$

Then X_1, X_2 and X_3 form a basis of tangent vector fields to S^3. Moreover, we have $[X_1, X_2] = 2X_3$, $[X_2, X_3] = 2X_1$ and $[X_3, X_1] = 2X_2$. Inspired by [M], we define a metric $\langle .,. \rangle_1$ on S^3 such that X_1, X_2 and X_3 are orthogonal and such that $\langle X_2, X_2\rangle_1 = \langle X_3, X_3\rangle_1 = 8/3$ and

$\langle X_1, X_1\rangle_1 = 4/9$. Then $E_1 = (3/2)X_1$, $E_2 = (\sqrt{3}/2\sqrt{2})X_2$ and $E_3 = -(\sqrt{3}/2\sqrt{2})X_3$ form an orthonormal basis on S^3. We denote the Levi-Civita connection of $\langle .,.\rangle_1$ by ∇. We recall from [DVV] that there exists an isometric totally real immersion ψ from $(S^3, \langle .,.\rangle_1)$ given by

$$\psi : S^3(1) \to S^6(1) : (y_1, y_2, y_3, y_4) \mapsto (x_1, x_2, x_3, x_4, x_5, x_6, x_7),$$

where

$$\begin{aligned}
x_1 &= (1/9)(5y_1^2 + 5y_2^2 - 5y_3^2 - 5y_4^2 + 4y_1),\\
x_2 &= -(2/3)y_2,\\
x_3 &= (2\sqrt{5}/9)(y_1^2 + y_2^2 - y_3^2 - y_4^2 - y_1),\\
x_4 &= (\sqrt{3}/9\sqrt{2})(-10y_3y_1 - 2y_3 - 10y_2y_4),\\
x_5 &= (\sqrt{3}\sqrt{5}/9\sqrt{2})(2y_1y_4 - 2y_4 - 2y_2y_3),\\
x_6 &= (\sqrt{3}\sqrt{5}/9\sqrt{2})(2y_1y_3 - 2y_3 + 2y_2y_4),\\
x_7 &= (\sqrt{3}/9\sqrt{2})(10y_1y_4 + 2y_4 - 10y_2y_3).
\end{aligned}$$

Its connection is given by

$$\begin{aligned}
&\nabla_{E_1}E_1 = 0, && \nabla_{E_2}E_2 = 0, && \nabla_{E_3}E_3 = 0,\\
&\nabla_{E_1}E_2 = -(11/4)E_3, && \nabla_{E_1}E_3 = (11/4)E_2, && \nabla_{E_2}E_3 = -(1/4)E_1,\\
&\nabla_{E_2}E_1 = (1/4)E_3, && \nabla_{E_3}E_1 = -(1/4)E_2, && \nabla_{E_3}E_2 = (1/4)E_1,
\end{aligned}$$

and its second fundamental form satisfies

$$\begin{aligned}
&h(E_1, E_1) = (\sqrt{5}/2)JE_1, && h(E_3, E_1) = -(\sqrt{5}/4)JE_3,\\
&h(E_1, E_2) = -(\sqrt{5}/4)JE_2, && h(E_3, E_2) = 0,\\
&h(E_2, E_2) = -(\sqrt{5}/4)JE_1, && h(E_3, E_3) = -(\sqrt{5}/4)JE_1.
\end{aligned}$$

Hence $\tilde{\psi}$ is quasi-Einstein.

EXAMPLE 4.2. Here, we will consider tubes in the direction of the orthogonal complement of the first osculating space on an almost complex curve. In [E2], N. Ejiri already showed that a tube with radius $\cos^2\gamma = 5/9$ on a superminimal almost complex curve defines a totally real submanifold of S^6, and in [DV] it was shown that a tube with radius $\pi/2$ on any almost complex curve defines a totally real submanifold M with $\delta_M = 2$.

An immersion $\bar{\phi} : N \to S^6(1)$ is called almost complex if J preserves the tangent space, i.e., $J_p\bar{\phi}_\star(T_pN) = \bar{\phi}_\star(T_pN)$. It is well-known that such immersions are always minimal, and as indicated in [BVW] there are essentially 4 types of almost complex immersions in $S^6(1)$, namely, those which are

(I) linearly full in $S^6(1)$ and superminimal,
(II) linearly full in $S^6(1)$ but not superminimal,
(III) linearly full in some totally geodesic $S^5(1)$ in $S^6(1)$ (and thus by [Ca1] necessarily not superminimal),
(IV) totally geodesic.

Now, let $\bar{\phi} : N \to S^6(1)$ be an almost complex curve. We denote its position vector in $\boldsymbol{R}^7$ also by $\bar{\phi}$. For the proof of elementary properties of such surfaces, we refer to [S]. Here, we simply recall that for tangent vector fields X and Y to N and for a normal vector field η, we have

$$\alpha(X, JY) = J\alpha(X, Y)\,, \tag{4.1}$$

$$A_{J\eta} = JA_\eta = -A_\eta J\,, \tag{4.2}$$

$$\nabla^\perp_X J\eta = G(X, \eta) + J\nabla^\perp_X \eta\,, \tag{4.3}$$

$$(\nabla\alpha)(X, Y, JZ) = J(\nabla\alpha)(X, Y, Z) + G(\bar{\phi}_\star X, \alpha(Y, Z))\,, \tag{4.4}$$

where α denotes the second fundamental form of the immersion and the pull-back of J to N is also denoted by J.

Next, if necessary, by restricting ourselves to an open dense subset of N, we may assume that N does not contain any totally geodesic points. Let $p \in N$ and V be an arbitrary unit tangent vector field defined on a neighborhood W of p. We define a local non vanishing function $\mu = \|\alpha(V, V)\|$ and an orthogonal tangent vector field U such that $\bar{\phi}_\star U = J\bar{\phi}_\star V = \bar{\phi} \times \bar{\phi}_\star V$. Then, using the properties of the vector cross product, it is easy to see that $F_1 = \bar{\phi}$, $F_2 = \bar{\phi}_\star V$, $F_3 = J\bar{\phi}_\star V$, $F_4 = \alpha(V, V)/\mu$, $F_5 = \alpha(V, JV)/\mu = J\alpha(V, V)/\mu = F_1 \times F_4$, $F_6 = F_2 \times \alpha(V, V)/\mu$ and $F_7 = F_3 \times \alpha(V, V)/\mu$ form a G_2-frame and hence satisfy the multiplication table as defined in Section 2.

Since $F_4, \dots, F_7$ form a basis for the normal space along N, it is clear that we can write any normal vector field as a linear combination of these basis vector fields. Thus there exist functions $a_1, \dots, a_4$ such that

$$(\nabla\alpha)(V, V, V) = \mu(a_1F_4 + a_2F_5 + a_3F_6 + a_4F_7)\,. \tag{4.5}$$

Then using (4.4) and the multiplication table, we get that

$$(\nabla\alpha)(V, V, U) = \mu(-a_2F_4 + a_1F_5 + (1 + a_4)F_6 - a_3F_7)\,. \tag{4.6}$$

From (4.5) and (4.6), it is immediately clear that

(1) N is an almost complex curve of Type (I) if and only if $a_3 = 0$ and $a_4 = -1/2$.

(2) N is an almost complex curve of Type (III) if and only if $a_4 + a_3^2 + a_4^2 = 0$.

Introducing local functions μ_1 and μ_2 on N by

$$\nabla_V V = \mu_1 U\,, \quad \nabla_U U = \mu_2 V\,, \quad \nabla_V U = -\mu_1 V\,, \quad \nabla_U V = -\mu_2 U\,,$$

it follows from (4.5) and (4.6) that $a_1 = V(\mu)/\mu$ and $a_2 = -U(\mu)/\mu$.

Now, in order to construct explicitly the totally real immersion from the unit tangent bundle, we recall a technical lemma from [DV].

LEMMA 4.1. *Denote by D the standard connection on $\boldsymbol{R}^7$. Then, we have*

$$\begin{aligned}
D_V(\mu F_4) &= \mu(-\mu F_2 + a_1 F_4 + (a_2 + 2\mu_1)F_5 + a_3 F_6 + a_4 F_7)\,,\\
D_U(\mu F_4) &= \mu(\mu F_3 - a_2 F_4 + (a_1 - 2\mu_2)F_5 + (1 + a_4)F_6 - a_3 F_7)\,,\\
D_V(\mu F_5) &= \mu(-\mu F_3 - (a_2 + 2\mu_1)F_4 + a_1 F_5 + (1 + a_4)F_6 - a_3 F_7)\,,\\
D_U(\mu F_5) &= \mu(-\mu F_2 - (a_1 - 2\mu_2)F_4 - a_2 F_5 - a_3 F_6 - a_4 F_7)\,,\\
D_V(\mu F_6) &= \mu(-a_3 F_4 - (a_4 + 1)F_5 + a_1 F_6 + (a_2 + 3\mu_1)F_7)\,,\\
D_U(\mu F_6) &= \mu(-(a_4 + 1)F_4 + a_3 F_5 - a_2 F_6 + (a_1 - 3\mu_2)F_7)\,,\\
D_V(\mu F_7) &= \mu(-a_4 F_4 + a_3 F_5 - (a_2 + 3\mu_1)F_6 + a_1 F_7)\,,\\
D_U(\mu F_7) &= \mu(a_3 F_4 + a_4 F_5 + (3\mu_2 - a_1)F_6 - a_2 F_7)\,.
\end{aligned}$$

PROOF OF THEOREM 1. We define a map

$$\bar{\psi} : UN \to S^6(1) : v_p \mapsto \cos\gamma\bar{\phi}(p) + \sin\gamma\bar{\phi}_\star(v) \times \frac{\alpha(v, v)}{\|\alpha(v, v)\|}\,.$$

Using the above vector fields, we can write $v_p = \cos(t/3)V + \sin(t/3)U$; and an easy computation shows that the map $\bar{\psi}$ can be locally parameterized by

$$\bar{\psi}(q, t) = \cos\gamma F_1(q) + \sin\gamma(\cos t F_6(q) + \sin t F_7(q))\,, \tag{4.7}$$

where $q \in W$ and $t \in \boldsymbol{R}$. Since the case with $\gamma = \pi/2$ was already treated in [DV], we restrict ourselves here to the case that $\cos\gamma \neq 0$. We immediately see that

$$\bar{\psi}_\star\left(\frac{\partial}{\partial t}\right) = \sin\gamma(-\sin t F_6 + \cos t F_7)\,. \tag{4.8}$$

Using Lemma 4.1, we then obtain that

$$\begin{aligned}
\bar{\psi}_\star &= \cos\gamma D_V F_1 + \sin\gamma(\cos t D_V F_6 + \sin t D_V F_7)\\
&= \cos\gamma F_2 + \sin\gamma(-\cos t(V(\mu)/\mu)F_6)\\
&\quad + \cos t(-a_3 F_4 - (a_4 + 1)F_5 + a_1 F_6 + (a_2 + 3\mu_1)F_7) - \sin t(V(\mu)/\mu)F_7\\
&\quad + \sin t(-a_4 F_4 + a_3 F_5 - (a_2 + 3\mu_1)F_6 + a_1 F_7)\\
&= \cos\gamma F_2 + \sin\gamma((-a_3\cos t - a_4\sin t)F_4\\
&\quad + (a_3\sin t - (a_4 + 1)\cos t)F_5) + (3\mu_1 - (U(\mu)/\mu))\bar{\psi}_\star\left(\frac{\partial}{\partial t}\right)\,.
\end{aligned} \tag{4.9}$$

Using similar computations, we also get that

$$\begin{aligned}
\bar{\psi}_\star(U) &= \cos\gamma F_3 + \sin\gamma((a_3\sin t - (1 + a_4)\cos t)F_4\\
&\qquad + (a_3\cos t + a_4\sin t)F_5) + (-3\mu_2 + (V(\mu)/\mu))\bar{\psi}_\star\left(\frac{\partial}{\partial t}\right)\,.
\end{aligned} \tag{4.10}$$

From (4.8), (4.9) and (4.10), we see that $\bar{\psi}$ is an immersion at every point (q, t).

Now, we put

$$X = V - (3\mu_1 - (U(\mu)/\mu))\frac{\partial}{\partial t},$$

$$Y = U - (-3\mu_2 + (V(\mu)/\mu))\frac{\partial}{\partial t}.$$

A straightforward computation, using the multiplication table of Section 2, then shows that

$$\begin{aligned}\bar{\psi} \times \bar{\psi}_\star\left(\frac{\partial}{\partial t}\right) &= -\sin^2\gamma F_1 + \cos\gamma\sin\gamma(\cos t F_6 + \sin t F_7),\\ \bar{\psi} \times \bar{\psi}_\star(X) &= \sin^2\gamma(a_3\cos 2t + a_4\sin 2t + (1/2)\sin 2t)F_2\\ &\quad + (\cos^2\gamma + \sin^2\gamma(a_3\sin 2t - a_4\cos 2t - \cos^2 t))F_3\\ &\quad + \cos\gamma\sin\gamma(-a_3\sin t + (a_4+2)\cos t)F_4\\ &\quad + \cos\gamma\sin\gamma(-a_3\cos t - (a_4-1)\sin t)F_5.\end{aligned}$$

Consequently, $\bar{\psi}$ is a totally real immersion if and only if

$$\langle\bar{\psi} \times \bar{\psi}_\star(X), \bar{\psi}_\star(Y)\rangle = 0,$$

i.e., if and only if

$$\begin{aligned}\cos\gamma(\cos^2\gamma + \sin^2\gamma(3a_3\sin 2t - a_4\cos 2t - \cos^2 t - a_3^2\\ - (a_4+2)(1+a_4)\cos^2 t - a_4(a_4-1)\sin^2 t)) = 0.\end{aligned}$$

Hence, since we assumed $\cos\gamma \neq 0$, we find that

$$3a_3\sin^2\gamma\sin 2t - 3(a_4+1/2)\sin^2\gamma\cos 2t + \cos^2\gamma - \sin^2\gamma(a_4 + a_3^2 + a_4^2 + 3/2) = 0.$$

Since the above formula has to be satisfied for every value of t, we deduce that $a_3 = 0$, $a_4 = -1/2$ and $\cos^2\gamma = 5/9$. Hence N^2 is a superminimal almost complex curve in S^6 and the radius of the tube satisfies $\cos^2\gamma = 5/9$. Using the above values for a_3 and a_4, we then obtain that

$$\begin{aligned}\bar{\psi}_\star(X) &= \cos\gamma F_2 + (1/2)\sin\gamma(\sin t F_4 - \cos t F_5),\\ \bar{\psi}_\star(Y) &= \cos\gamma F_3 - (1/2)\sin\gamma(\cos t F_4 + \sin t F_5),\\ J\bar{\psi}_\star\left(\frac{\partial}{\partial t}\right) &= -\sin^2\gamma F_1 + \cos\gamma\sin\gamma(\cos t F_6 + \sin t F_7),\\ J\bar{\psi}_\star(X) &= (1/3)F_3 + (3/2)\cos\gamma\sin\gamma(\cos t F_4 + \sin t F_5),\\ J\bar{\psi}_\star(Y) &= -(1/3)F_2 + (3/2)\cos\gamma\sin\gamma(\sin t F_4 - \cos t F_5).\end{aligned}$$

Therefore, by a straightforward computation, we obtain that

$$\begin{aligned}D_{\frac{\partial}{\partial t}}\bar{\psi}_\star(X) &= (1/2)\sin\gamma(\cos t F_4 + \sin t F_5) = -(1/2)((1/3)\bar{\psi}_\star(Y) - \cos\gamma J\bar{\psi}_\star(X)),\\ D_{\frac{\partial}{\partial t}}\bar{\psi}_\star(Y) &= -(1/2)\sin\gamma(-\sin t F_4 + \cos t F_5) = (1/2)((1/3)\bar{\psi}_\star(X) + \cos\gamma J\bar{\psi}_\star(Y)),\\ D_{\frac{\partial}{\partial t}}\bar{\psi}_\star\left(\frac{\partial}{\partial t}\right) &= -\sin\gamma(\cos t F_6 + \sin t F_7) = -\left((4/9)\bar{\psi} + \cos\gamma J\bar{\psi}_\star\left(\frac{\partial}{\partial t}\right)\right).\end{aligned}$$

So, if we put $E_1 = (3/2)(\partial/\partial t)$, $E_2 = (\sqrt{3}/\sqrt{2})X$ and $E_3 = (\sqrt{3}/\sqrt{2})Y$ we see that E_1, E_2 and E_3 form an orthonormal basis of the tangent space to UN and

$$\begin{aligned} h(E_1, E_1) &= -(3/2)\cos\gamma\, J\bar{\psi}_\star(E_1)\,, \\ h(E_1, E_2) &= (3/4)\cos\gamma\, J\bar{\psi}_\star(E_2)\,, \\ h(E_1, E_3) &= (3/4)\cos\gamma\, J\bar{\psi}_\star(E_3)\,. \end{aligned}$$

Since UN is totally real (and thus minimal) and $\langle h(X,Y), JZ\rangle$ is totally symmetric in X, Y and Z, the above formulas and Lemma 3.1 imply that $\bar{\psi}$ is quasi-Einstein. Since the first normal space is 3-dimensional, with respect to the induced metric we have $\delta_{UN} < 2$ (see [C]). Hence UN satisfies Case (2) of Lemma 3.2.

5. Proof of Theorem 2. Throughout this section we will assume that $F : M^3 \to S^6$ is a totally real immersion which is quasi-Einstein. Unless otherwise indicated, we will identify M^3 with its image in S^6.

First, we remark that if $\delta_M = 2$, there is nothing to prove. Next, we assume that M^3 is Einstein. Since a 3-dimensional Einstein manifold has constant sectional curvatures, it follows from [E1] that a neighborhood of p is G_2-congruent with an open part of the image of the totally real immersion of $S^3(1/16)$ in $S^6(1)$ as described in [E1] (see also [DVV]). From [E2], we also know that we can consider this image as the tube with radius γ, with $\cos^2\gamma = 5/9$ on the almost complex curve with constant Gaussian curvature 1/6. This completes the proof in this case.

Next assume that $p \in M$ such that Ric has an eigenvalue with multiplicity 2 and $\delta_M(p) \neq 2$. Since in a neighborhood of p, M is quasi-Einstein, but not Einstein, there exist local orthonormal vector fields $\{E_1, E_2, E_3\}$ such that E_1 spans the 1-dimensional eigenspace and $\{E_2, E_3\}$ span the 2-dimensional eigenspace. Hence, applying Lemma 3.2, we see that there exist local functions λ, a and b such that

$$\begin{aligned} h(E_1, E_1) &= 2\lambda J E_1\,, & h(E_2, E_2) &= -\lambda J E_1 + aJE_2\,, \\ h(E_1, E_2) &= -\lambda J E_2\,, & h(E_2, E_3) &= -aJE_3\,, \\ h(E_1, E_3) &= -\lambda J E_3\,, & h(E_3, E_3) &= -\lambda J E_1 - aJE_2\,. \end{aligned}$$

If necessary by changing the sign of E_3, we may assume that $G(E_1, E_2) = JE_3$, $G(E_2, E_3) = JE_1$ and $G(E_3, E_1) = JE_2$. We now introduce local functions $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$ by

$$\begin{aligned} \nabla_{E_1}E_1 &= a_1E_2 + a_2E_3\,, & \nabla_{E_1}E_2 &= -a_1E_1 + a_3E_3\,, & \nabla_{E_1}E_3 &= -a_2E_1 - a_3E_2\,, \\ \nabla_{E_2}E_1 &= b_1E_2 + b_2E_3\,, & \nabla_{E_2}E_2 &= -b_1E_1 + b_3E_3\,, & \nabla_{E_2}E_3 &= -b_2E_1 - b_3E_2\,, \\ \nabla_{E_3}E_1 &= c_1E_2 + c_2E_3\,, & \nabla_{E_3}E_2 &= -c_1E_1 + c_3E_3\,, & \nabla_{E_3}E_3 &= -c_2E_1 - c_3E_2\,. \end{aligned}$$

LEMMA 5.1. *The function λ satisfies $\lambda = \sqrt{5}/4$. Moreover, $a_1 = a_2 = c_2 = b_1 = 0$ and $b_2 = -c_1 = 1/4$.*

PROOF. A straightforward computation shows that

$$\begin{aligned}(\nabla h)(E_2, E_1, E_1) &= 2E_2(\lambda)JE_1 - 2\lambda JE_3 + 4\lambda(b_1 JE_2 + b_2 JE_3)\,,\\ (\nabla h)(E_1, E_2, E_1) &= -E_1(\lambda)JE_2 - \lambda JE_3 - \lambda(-a_1 JE_1)\\ &\qquad + 2\lambda a_1 JE_1 - a_1(-\lambda JE_1 + aJE_2) - a_2(-aJE_3)\,.\end{aligned}$$

Hence, it follows from the Codazzi equation $(\nabla h)(E_2, E_1, E_1) = (\nabla h)(E_1, E_2, E_1)$ that

$$E_2(\lambda) = 2\lambda a_1\,, \tag{5.1}$$

$$E_1(\lambda) = -4\lambda b_1 - aa_1\,, \tag{5.2}$$

$$4b_2 = 1 + (a/\lambda)a_2\,. \tag{5.3}$$

Similarly, we obtain from the Codazzi equation $(\nabla h)(E_3, E_1, E_1) = (\nabla h)(E_1, E_3, E_1)$ that

$$E_3(\lambda) = 2\lambda a_2\,, \tag{5.4}$$

$$E_1(\lambda) = -4\lambda c_2 + aa_1\,, \tag{5.5}$$

$$4c_1 = -1 + (a/\lambda)a_2\,. \tag{5.6}$$

Comparing (5.5) and (5.2), we get that

$$c_2 - b_1 = (a/2\lambda)a_1\,. \tag{5.7}$$

A straightforward computation, using (5.1) and (5.4), then shows that

$$(\nabla h)(E_1, E_2, E_3) = aa_2 JE_1 + (3aa_3 + a - a_2\lambda)JE_2 - (a_1\lambda + E_1(a))JE_3\,, \tag{5.8}$$

$$(\nabla h)(E_2, E_1, E_3) = (4\lambda b_2 - \lambda)JE_1 + b_2 aJE_2 + (b_1 a - 2\lambda a_1)JE_3\,, \tag{5.9}$$

$$(\nabla h)(E_3, E_1, E_2) = (4\lambda c_1 + \lambda)JE_1 - (ac_1 + 2\lambda a_2)JE_2 + ac_2 JE_3\,. \tag{5.10}$$

Therefore, using the Codazzi equations and (5.3), (5.6) and (5.7), we get that

$$((a^2/2\lambda) + 2\lambda)a_2 = 0\,, \quad ((a^2/2\lambda) + 2\lambda)a_1 = 0\,.$$

Hence $a_1 = a_2 = 0$ and we deduce from the previous equations that

$$c_2 = b_1\,, \quad c_1 = -1/4\,, \quad b_2 = 1/4\,.$$

This implies that the function λ is a solution of the following system of differential equations:

$$E_1(\lambda) = -4\lambda b_1\,, \quad E_2(\lambda) = 0\,, \quad E_3(\lambda) = 0\,,$$

Since $[E_2, E_3] = -(1/2)E_1 - b_3 E_2 - c_3 E_3$, it immediately follows from the integrability conditions that $b_1 = 0$ and hence λ is a constant.

To compute the actual value of λ we use the Gauss equation. We have

$$R(E_1, E_2)E_1 = -E_2 + \lambda^2 E_2 + 2\lambda^2 E_2 = (3\lambda^2 - 1)E_2\,.$$

On the other hand, we have

$$\begin{aligned}R(E_1, E_2)E_1 &= \nabla_{E_1}\nabla_{E_2}E_1 - \nabla_{E_2}\nabla_{E_1}E_1 - \nabla_{[E_1,E_2]}E_1\\ &= \nabla_{E_1}((1/4)E_3) - (a_3 - 1/4)\nabla_{E_3}E_1\\ &= -(1/4)a_3 E_2 + (1/4)(a_3 - 1/4)E_2 = -(1/16)E_2\,.\end{aligned}$$

Hence

$$\lambda^2 = 5/16.$$

Since λ is positive, this completes the proof of the lemma.

Now, in order to complete the proof of the theorem, we have to make a distinction between M and its image under F in S^6. First, we consider the case that a is identically zero in a neighborhood of the point p. Then, we have the following lemma:

LEMMA 5.2. *There exists an orthonormal basis $\{E_1, E_2, E_3\}$ with $G(E_1, E_2) = JE_3$, $G(E_2, E_3) = JE_1$ and $G(E_3, E_1) = JE_2$, defined on a neighborhood of the point p such that*

$$\begin{aligned} &h(E_1, E_1) = (\sqrt{5}/2)JE_1, && h(E_2, E_2) = -(\sqrt{5}/4)JE_1, \\ &h(E_1, E_2) = -(\sqrt{5}/4)JE_2, && h(E_2, E_3) = 0, \\ &h(E_1, E_3) = -(\sqrt{5}/4)JE_3, && h(E_3, E_3) = -(\sqrt{5}/4)JE_1. \end{aligned}$$

Moreover, they satisfy

$$\begin{aligned} &\nabla_{E_1}E_1 = 0, && \nabla_{E_1}E_2 = -(11/4)E_3, && \nabla_{E_1}E_3 = (11/4)E_2, \\ &\nabla_{E_2}E_1 = (1/4)E_3, && \nabla_{E_2}E_2 = 0, && \nabla_{E_2}E_3 = -(1/4)E_1, \\ &\nabla_{E_3}E_1 = -(1/4)E_2, && \nabla_{E_3}E_2 = (1/4)E_1, && \nabla_{E_3}E_3 = 0. \end{aligned}$$

PROOF. We take the local orthonormal basis $\{E_1, E_2.E_3\}$ constructed in the previous lemma. Clearly, this basis already satisfies the first condition. Since $a_1 = a_2 = c_2 = b_1 = 0$ and $b_2 = -c_1 = 1/4$, the Gauss equations $\langle R(E_1, E_2)E_2, E_3\rangle = 0$, $\langle R(E_1, E_3)E_3, E_2\rangle = 0$ and $\langle R(E_2, E_3)E_3, E_2\rangle = 21/16$ reduce to

$$E_1(b_3) - E_2(a_3) - c_3(a_3 - 1/4) = 0, \tag{5.11}$$

$$-E_1(c_3) + E_3(a_3) - b_3(a_3 - 1/4) = 0, \tag{5.12}$$

$$E_3(b_3) - E_2(c_3) - (1/2)a_3 - b_3^2 - c_3^2 = 11/8. \tag{5.13}$$

Now, we use the following transformation of the local frame $\{E_1, E_2, E_3\}$:

$$\begin{aligned} U_1 &= E_1, \\ U_2 &= \cos\theta E_2 + \sin\theta E_3, \\ U_3 &= -\sin\theta E_2 + \cos\theta E_3. \end{aligned}$$

where θ is an arbitrary locally defined function on M. It is immediately clear that $\{U_1, U_2, U_3\}$ satisfies the conditions of the lemma if and only if the function θ satisfies the following system of differential equations:

$$\begin{aligned} &d\theta(E_1) + a_3 + 11/4 = 0, \\ &d\theta(E_2) + b_3 = 0, \\ &d\theta(E_3) + c_3 = 0, \end{aligned}$$

i.e., $d\theta = -(a_3 + 11/4)\theta_1 - b_3\theta_2 - c_3\theta_3$, where $\{\theta_1, \theta_2, \theta_3\}$ is that dual basis of $\{E_1, E_2, E_3\}$. Now, this system locally has a solution if and only if the 1-form $\omega = (a_3 + 11/4)\theta_1 + b_3\theta_2 +$

$c_3\theta_3$ is closed. One can easily verify that $d\omega = 0$ is equivalent with (5.11), (5.12) and (5.13). □

The proof now follows from the Cartan-Ambrose-Hicks Theorem and the uniqueness theorem for totally real immersions.

Finally, we deal with the case that $a(p) \neq 0$. Then, we have the following lemma:

LEMMA 5.3. *Let $\{E_1, E_2, E_3\}$ be the local orthonormal frame constructed before. Then we have*

$$E_1(a) = 0 \quad \textit{and} \quad a_3 = -1/4\,.$$

PROOF. We look again at the proof of Lemma 5.2. From (5.8), (5.9), (5.10) and the Codazzi equation, we get $E_1(a) = 0$ and $3a(1/4 + a_3) = 0$. Since $a(p) \neq 0$, this completes the proof.

From now on we will make a distinction between M and its image under F in S^6. We will also write explicitly $J_p v$ as $p \times v$, since we will be using the almost complex structure at different points of S^6. Let us recall that we have a local basis $\{E_1, E_2, E_3\}$ on U such that

$$\begin{aligned}
&\nabla_{E_1} E_1 = 0\,, && \nabla_{E_1} E_2 = -(1/4)E_3\,, && \nabla_{E_1} E_3 = (1/4)E_2\,,\\
&\nabla_{E_2} E_1 = (1/4)E_3\,, && \nabla_{E_2} E_2 = b_3 E_3\,, && \nabla_{E_2} E_3 = -(1/4)E_1 - b_3 E_2\,,\\
&\nabla_{E_3} E_1 = -(1/4)E_2\,, && \nabla_{E_3} E_2 = (1/4)E_1 + c_3 E_3\,, && \nabla_{E_3} E_3 = -c_3 E_2\,;
\end{aligned}$$

$$\begin{aligned}
&h(E_1, E_1) = (\sqrt{5}/2)F \times F_\star E_1\,, && h(E_2, E_2) = -(\sqrt{5}/4)F \times F_\star E_1 + aF \times F_\star E_2\,,\\
&h(E_1, E_2) = -(\sqrt{5}/4)F \times F_\star E_2\,, && h(E_2, E_3) = -aF \times F_\star E_3\,,\\
&h(E_1, E_3) = -(\sqrt{5}/4)F \times F_\star E_3\,, && h(E_3, E_3) = -(\sqrt{5}/4)F \times F_\star E_1 - aF \times F_\star E_2\,;
\end{aligned}$$

$$\begin{aligned}
F_\star E_1 \times F_\star E_2 &= F \times F_\star E_3\,,\\
F_\star E_2 \times F_\star E_3 &= F \times F_\star E_1\,,\\
F_\star E_3 \times F_\star E_1 &= F \times F_\star E_2\,.
\end{aligned}$$

We now define a mapping $G : U \to S^6$, where U is a neighborhood of p, by

$$G(q) = (\sqrt{5}/3)F(q) + (2/3)F \times F_\star(E_1(q))\,.$$

Then, using the above formulas, we find that

$$\begin{aligned}
D_{E_1} G &= (\sqrt{5}/3)F_\star(E_1) + (2/3)F \times h(E_1, E_1) = 0\,,\\
D_{E_2} G &= (\sqrt{5}/3)F_\star(E_2) + (2/3)(F_\star(E_2) \times F_\star(E_1) + F \times F_\star(\nabla_{E_2} E_1) + F \times h(E_2, E_1))\\
&= (\sqrt{5}/2)F_\star(E_2) - (1/2)F \times F_\star(E_3)\,,\\
D_{E_3} G &= (\sqrt{5}/3)F_\star(E_3) + (2/3)(F_\star(E_3) \times F_\star(E_1) + F \times F_\star(\nabla_{E_3} E_1) + F \times h(E_3, E_1))\\
&= (\sqrt{5}/2)F_\star(E_3) + (1/2)F \times F_\star(E_2)\,,
\end{aligned}$$

from which it follows that G is not an immersion.

Using [Sp, Vol. 1, p. 204], we can identify a neighborhood of p with a neighborhood $I \times W_1$ of the origin in $\boldsymbol{R}^3$ (with coordinates (t, u, v)) such that $p = (0, 0, 0)$ and $E_1 = \partial/\partial t$.

Then there exist functions α_1 and α_2 on W_1 such that $E_2+\alpha_1 E_1$ and $E_3+\alpha_2 E_1$ form a basis for the tangent space to $W_1 \subset U$ at the point $q=(0,u,v)$.

Now since $\nabla_{E_1} E_1 = 0$ and $h(E_1, E_1) = (\sqrt{5}/2)F \times F_\star E_1$, it follows that the integral curve of E_1 through the point $F(q)$ is a circle with radius 2/3, tangent vector $F_\star E_1(q)$ and normal vector $(\sqrt{5}/3)F \times F_\star E_1(q) - (2/3)F(q)$. From this it is clear that $F(U)$ can be reconstructed from W_1 by

$$\begin{aligned} F(t,u,v) = \;& (\sqrt{5}/3)((\sqrt{5}/3)F(0,u,v) + (2/3)JF_\star E_1(0,u,v)) \\ & +(2/3)(\sin(3t/2)E_1(0,u,v) - \cos(3t/2)((2/3)F(0,u,v) \\ & -(\sqrt{5}/3)F \times F_\star E_1(0,u,v))) \end{aligned} \tag{5.14}$$

Now, we look at the restriction of the map G to W_1. Since

$$D_{E_2+\alpha_1 E_1} G = (\sqrt{5}/2)F_\star(E_2) - (1/2)F \times F_\star(E_3),$$
$$D_{E_3+\alpha_2 E_1} G = (\sqrt{5}/2)F_\star(E_3) + (1/2)F \times F_\star(E_2),$$

we see that G is an immersion from W_1 into S^6. Moreover, since

$$\begin{aligned} ((\sqrt{5}/3)F + (2/3)F \times F_\star E_1) \times ((\sqrt{5}/2)F_\star E_2 &- (1/2)F \times F_\star E_3) \\ &= (\sqrt{5}/2)F_\star E_3 + (1/2)F \times F_\star E_2, \end{aligned}$$

we see that G is an almost complex immersion (and hence minimal). A straightforward computation now shows that

$$\begin{aligned} & D_{E_2+\alpha_1 E_1}((\sqrt{5}/2)F_\star E_2 - (1/2)F \times F_\star E_3) \\ & \quad = f((\sqrt{5}/2)F_\star E_3 + (1/2)F \times F_\star E_2) - (3/2)G + a((\sqrt{5}/2)F \times F_\star E_2 - (1/2)F_\star E_3), \end{aligned}$$

where f is some function whose precise value is not essential. So, if we put $X = E_2 + \alpha_1 E_1$ and $Y = E_3 + \alpha_2 E_1$, we see that X and Y are orthogonal with respect to the induced metric and have the same constant length $\sqrt{3/2}$. We also see that

$$h(X,X) = a((\sqrt{5}/2)F \times F_\star E_2 - (1/2)F_\star E_3).$$

Since G is an almost complex immersion, it follows that

$$h(X,Y) = h(X, G \times X) = G \times h(X,X).$$

Hence

$$h(X,Y) = a(-(1/2)F_\star E_2 - ((\sqrt{5}/2)F \times F_\star E_3)).$$

So, we see that the image of the tangent space and the first normal space to the almost complex immersion are spanned by $F_\star E_2(q)$, $F_\star E_3(q)$, $F \times F_\star(E_2)(q)$ and $F \times F_\star E_3(q)$. Therefore, we get that its orthogonal complement in S^6 is spanned by $F_\star(E_1)(q)$ and $(2/3)F - (\sqrt{5}/3)F \times F_\star(E_1)$. Hence, the tube on the almost complex immersion G with radius γ, with $\cos\gamma = \sqrt{5}/3$ in the direction of the orthogonal complement of the first osculating space is given by (5.14) and corresponds therefore to the original totally real immersion F. This completes the proof of Theorem 2.

REMARK 5.4. The above construction can also be applied to the totally real immersion of S^3 into S^6 constructed in Example 4.1. However, in that case, the resulting almost complex curve is totally geodesic, and hence it is impossible to define the first normal bundle. Taking coordinates

$$\begin{aligned} y_1 &= \cos(3t/2)z_1\,, \\ y_2 &= -\sin(3t/2)z_1\,, \\ y_3 &= \cos(3t/2)z_3 + \sin(3t/2)z_4\,, \\ y_4 &= -\sin(3t/2)z_3 + \cos(3t/2)z_4\,, \end{aligned}$$

we notice that $\partial/\partial t$ corresponds with the vector field E_1. Since

$$\tilde{\psi}_\star(E_1) = \begin{pmatrix} (2/3)y_2 \\ y_1 \\ -(\sqrt{5}/3)y_2 \\ -(\sqrt{3}/3\sqrt{2})y_4 \\ (\sqrt{15}/3\sqrt{2})y_3 \\ -(\sqrt{15}/3\sqrt{2})y_4 \\ (\sqrt{3}/3\sqrt{2})y_3 \end{pmatrix},$$

a straightforward computation shows that the resulting totally geodesic almost complex curve has components $(u_1, \ldots, u_7)$ given by

$$\begin{aligned} u_1 &= (\sqrt{5}/3)(y_1^2 + y_2^2 - y_3^2 - y_4^2)\,, \\ u_2 &= 0\,, \\ u_3 &= (2/3)(y_1^2 + y_2^2 - y_3^2 - y_4^2)\,, \\ u_4 &= -(2\sqrt{15}/3\sqrt{2})(y_1y_3 + y_2y_4)\,, \\ u_5 &= (2\sqrt{3}/3\sqrt{2})(y_1y_4 - y_2y_3)\,, \\ u_6 &= (2\sqrt{3}/3\sqrt{2})(y_1y_3 + y_2y_4)\,, \\ u_7 &= -(2\sqrt{15}/3\sqrt{2})(y_1y_4 - y_2y_3)\,. \end{aligned}$$

Therefore, $\tilde{\psi}(S^3)$ can still be considered as some tube, given by (5.14), on a totally geodesic almost complex curve.

REMARK 5.5. It is clear that the examples satisfying Case (1) and (2) of Theorem 2 do not contain any totally geodesic points (the length of the second fundamental form is strictly greater then a positive constant) and that the eigenvalues of Ric are bounded by a constant strictly smaller than 2. Therefore, it follows from Lemma 3.2 that these examples can not be put together differentiably with examples M satisfying $\delta_M = 2$.

REMARK 5.6. We recall that a Riemannian manifold M, of any dimension n, is locally symmetric when its Riemann-Christoffel curvature tensor R is parallel, i.e., $\nabla R = 0$, where ∇ is the Levi Civita connection of its metric $\langle .,. \rangle$, and that M is said to be semi-symmetric when more generally, $R \cdot R = 0$, meaning that $(R(X,Y) \cdot R)(X_1, X_2, X_3, X_4) = 0$ for all tangent X, Y, X_1, X_2, X_3, X_4, where $R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$ is the curvature operator of M. By pseudo-symmetric manifolds, we mean here the further generalisation of locally symmetric manifolds, namely those manifolds M for which $R \cdot R = fQ(\langle .,. \rangle, R)$, where $f : M \to \boldsymbol{R}$ is a differentiable function and $Q(\langle .,. \rangle, R)$ is defined by

$$Q(\langle .,. \rangle, R)(X, Y, X_1, X_2, X_3, X_4) = (X \wedge Y) \cdot R(X_1, X_2, X_3, X_4),$$

where $(X \wedge Y)Z = \langle X, Z \rangle Y - \langle Y, Z \rangle X$, for all vector fields Z. From the extrinsic as well as from the intrinsic point of view, this notion turns out to be natural generalisation of local- and semi-symmetry; for a survey on this see [Ver]. It is known that a 3-dimensional manifold is pseudo-symmetric if and only if it is quasi-Einstein. Therefore, Theorem 2 also provides a classification of all pseudo-symmetric 3-dimensional totally real submanifolds of S^6. The examples M with $\delta_M = 2$ satisfy $R \cdot R = Q(\langle .,. \rangle, R)$, while the examples of Case (1) and (2) of Theorem 2 satisfy $R \cdot R = (1/16)Q(\langle .,. \rangle, R)$. So the pseudo-symmetry conditions for these submanifolds are realised with constant functions f, being respectively $f = 1$ and $f = 1/16$ (the values of which we observe to be precisely the only possibilities for K for totally real immersions of M^3 into $S^6(1)$ with constant sectional curvatures K [E1]).

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R. DESZCZ
DEPARTMENT OF MATHEMATICS
AGRICULTURAL UNIVERSITY OF WROCLAW
UL. C.K. NORWIDA 25
50–375 WROCLAW
POLAND

F. DILLEN, L. VERSTRAELEN AND L. VRANCKEN
KATHOLIEKE UNIVERSITEIT LEUVEN
DEPARTEMENT WISKUNDE
CELESTIJNENLAAN 200 B
B-3001 LEUVEN
BELGIUM