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LAGRANGIAN SURFACES OF CONSTANT CURVATURE IN COMPLEX EUCLIDEAN PLANE

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Abstract. In this article we completely classify Lagrangian H-umbilical surfaces of constant curvature in complex Euclidean plane.

1. Lagrangian H-umbilical surfaces with $\lambda = 2\mu$. We follow the notation and definitions given in [1, 2]. Professors I. Castro and F. Urbano kindly pointed out that Theorem 3.1 in [1] holds only for submanifolds of dimension ≥ 3.

For $n = 2$, we have the following result instead.

THEOREM 1. *The following two statements hold*:

(i) *Let b be a nonzero number, $\theta(t)$ a function of one variable defined on an open interval $(\alpha, \beta) \ni 0$ and $z(t)$ a $\boldsymbol{C}^2$-valued solution of the ordinary differential equation*:

$$z''(t) - i\theta'(t)z'(t) + b^2 z(t) = 0\,, \tag{1.1}$$

satisfying the two conditions: $|z(t)|^2 = 1/(4b^2)$ *and* $|z'(t)|^2 = 1/4$. *Then the map*

$$L(s,t) = e^{2ibs}z(t) + \int_0^t z'(t)e^{-2i\theta(t)}dt \tag{1.2}$$

defines a Lagrangian H-umbilical surface of constant curvature b^2 in $\boldsymbol{C}^2$ whose induced metric is

$$g = ds^2 + \cos^2(bs + \theta(t))dt^2\,, \tag{1.3}$$

and whose second fundamental form satisfies

$$h(e_1, e_1) = 2bJe_1, \quad h(e_1, e_2) = bJe_2, \quad h(e_2, e_2) = bJe_1 \tag{1.4}$$

with respect to $e_1 = \partial/\partial s$ and $e_2 = \sec(bs + \theta(t))\partial/\partial t$.

(ii) *Conversely, if $\tilde{L} : M \to \boldsymbol{C}^2$ is a Lagrangian H-umbilical surface whose second fundamental form satisfies* (1.4) *for some function $b \neq 0$ with respect to some orthonormal local frame field $\{e_1, e_2\}$, then we have*:

(ii-1) *b is a constant*,

(ii-2) *there exist a function $\theta(t)$ and a local coordinate system $\{s, t\}$ on M with $e_1 = \partial/\partial s$ such that the metric tensor is given by* (1.3), *and*

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(ii-3) *the immersion $\tilde{L}$ is congruent to the L given in statement* (i).

PROOF. Let b be a nonzero number, $\theta(t)$ a function of one variable defined on $(\alpha, \beta) \ni 0$ and z a $\boldsymbol{C}^2$-valued solution of (1.1) satisfying $|z|^2 = 1/(4b^2)$ and $|z'|^2 = 1/4$. Define a map L by (1.2). Then we have

$$\begin{aligned} &L_s = 2ibe^{2ibs}z(t)\,, \quad L_t = (e^{2ibs} + e^{-2i\theta})z'(t)\,,\\ &L_{ss} = -4b^2e^{2ibs}z(t)\,, \quad L_{st} = 2ibe^{2ibs}z'(t)\,,\\ &L_{tt} = (e^{2ibs} + e^{-2i\theta})z''(t) - 2ie^{-2i\theta}\theta'(t)z'(t)\,. \end{aligned} \tag{1.5}$$

It follows from $|z|^2 = 1/(4b^2)$, $|z'|^2 = 1/4$ and (1.5) that the metric induced by L is given by (1.3). Moreover, a direct computation shows that L satisfies

$$\begin{aligned} &L_{ss} = 2ibL_s\,,\\ &L_{st} = ib\sec(bs+\theta)e^{i(bs+\theta)}L_t\,,\\ &L_{tt} = ib\cos(bs+\theta)e^{-i(bs+\theta)}L_s - \theta'\tan(bs+\theta)L_t, \end{aligned} \tag{1.6}$$

by virtue of (1.1). Thus, L is a Lagrangian H-umbilical immersion whose second fundamental form satisfies (1.4) with respect to $e_1 = \partial/\partial s$, $e_2 = \sec(bs+\theta(t))\partial/\partial t$.

Conversely, assume that $\tilde{L} : M \to \boldsymbol{C}^2$ is a Lagrangian H-umbilical immersion satisfying (1.4) for some function $b \neq 0$ with respect to some orthonormal frame field. Then (1.4) and the equation of Codazzi imply that b is constant and the integral curves of e_1 are geodesic in M. Thus, there exists a local coordinate system $\{s, u\}$ with $e_1 = \partial/\partial s$ so that the metric tensor of M is $g = ds^2 + f^2(s,u)du^2$. Since the Gaussian curvature K of M satisfies $K = -f_{ss}/f$, (1.4) implies that $f_{ss} = -b^2 f$. Solving this equation yields $f = A(u)\cos(bs) + B(u)\sin(bs)$ for some functions $A(u)$ and $B(u)$. Thus we get

$$g = ds^2 + r^2(u)\cos^2(bs + \varphi(u))du^2\,, \tag{1.7}$$

where $A = r(u)\cos\varphi(u)$, $B = r(u)\sin\varphi(u)$ and $r(u) = \sqrt{A^2 + B^2}$. If we denote by $t = t(u)$ an antiderivative of $r(u)$, then (1.7) becomes

$$g = ds^2 + \cos^2(bs + \theta(t))dt^2 \tag{1.8}$$

for some function $\theta(t)$. From (1.8) we find

$$\begin{aligned} &\nabla_{\partial/\partial s}\frac{\partial}{\partial s} = 0\,, \quad \nabla_{\partial/\partial s}\frac{\partial}{\partial t} = -b\tan(bs+\theta(t))\frac{\partial}{\partial t}\,,\\ &\nabla_{\partial/\partial t}\frac{\partial}{\partial t} = \frac{b}{2}\sin(2bs+2\theta(t))\frac{\partial}{\partial s} - \theta'(t)\tan(bs+\theta(t))\frac{\partial}{\partial t}\,. \end{aligned} \tag{1.9}$$

Using (1.4), (1.8), (1.9) and the formula of Gauss, we see that the immersion satisfies the following system of PDEs:

$$\begin{aligned} &\tilde{L}_{ss} = 2ib\tilde{L}_s\,,\\ &\tilde{L}_{st} = ib\sec(bs+\theta)e^{i(bs+\theta)}\tilde{L}_t\,,\\ &\tilde{L}_{tt} = ib\cos(bs+\theta)e^{-i(bs+\theta)}\tilde{L}_s - \theta'\tan(bs+\theta)\tilde{L}_t\,. \end{aligned} \tag{1.10}$$

After solving the first and the second equations in (1.10), we obtain

$$\tilde{L} = A(t)e^{2ibs} + B(t) \tag{1.11}$$

for some $\boldsymbol{C}^2$-valued functions $A(t)$, $B(t)$. Differentiating (1.11) gives

$$\tilde{L}_t = A'(t)e^{2ibs} + B'(t)\,. \tag{1.12}$$

Substituting this into the second equation in (1.10) gives

$$B'(t) = e^{-2i\theta}A'(t)\,. \tag{1.13}$$

Combining (1.12) and (1.13) yields

$$\tilde{L}_t = A'(t)(e^{2ibs} + e^{-2i\theta})\,. \tag{1.14}$$

Thus, after applying a suitable translation, we discover that

$$\tilde{L} = (A(t) + C)e^{2ibs} + \int_0^t A'(t)e^{-2i\theta(t)}dt \tag{1.15}$$

for some constant vector C. Therefore, if we put $z(t) = A(t) + C$, we obtain (1.2).

Using (1.2), we find

$$\tilde{L}_s = 2ibe^{2ibs}z(t)\,, \quad \tilde{L}_t = (e^{2ibs} + e^{-2i\theta})z'(t)\,, \tag{1.16}$$

which implies that $|L_s|^2 = 4b^2|z(t)|^2$ and $|L_t|^2 = 4\cos^2(bs + \theta)|z'(t)|^2$. Comparing these with (1.8) gives $|z|^2 = 1/(4b^2)$ and $|z'|^2 = 1/4$. Therefore, the immersion is congruent to the immersion L defined in statement (i). □

Concerning statement (i) of Theorem 1 we provide the following theorem which insures the existence of the solution of the equation (1.1) subject to the two conditions. The proof uses the same idea as that of Theorem 4.1 in [3].

THEOREM 2. *For any nonzero real number b and any differentiable function θ of one variable defined on an open interval I, there exists a $\boldsymbol{C}^2$-valued solution of the differential equation*:

$$z''(t) - i\theta'(t)z'(t) + b^2z(t) = 0\,, \tag{1.17}$$

that also satisfies the two conditions: $|z|^2 = 1/(4b^2)$ and $|z'|^2 = 1/4$.

PROOF. Let b be a nonzero number and $\theta(t)$ a differentiable function of one variable defined on an open interval I. Let D be a simply-connected open domain in $\boldsymbol{R}^2 = \{(s, t) : s, t \in \boldsymbol{R}\}$ on which the function $\cos(bs + \theta(t))$ is nowhere zero.

Let $M = (D, g)$ denote the Riemannian 2-manifold equipped with metric tensor g given by (1.8). Then its Levi-Civita connection satisfies (1.9). Let σ denote the TM-valued symmetric bilinear form so that

$$\begin{aligned} &\sigma\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}\right) = 2b\frac{\partial}{\partial s}\,, \quad \sigma\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right) = b\frac{\partial}{\partial t}\,, \\ &\sigma\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = b\cos^2(bs + \theta(t))\frac{\partial}{\partial s}\,. \end{aligned} \tag{1.18}$$

Then $\langle\sigma(X,Y),Z\rangle$ and $(\nabla\sigma)(X,Y,Z) := \nabla_X\sigma(U,Z)-\sigma(\nabla_X Y,Z)-\sigma(Y,\nabla_X Z)$ are totally symmetric. Moreover, a direct computation shows that the curvature tensor of M satisfies $R(X,Y)Z = \sigma(\sigma(Y,Z),X)-\sigma(\sigma(X,Z),Y)$. Hence, by applying Theorem A in [1], we see that there exists a Lagrangian isometric immersion $\tilde{L} : M \to \boldsymbol{C}^2$ whose second fundamental form $h = J\sigma$ satisfies (1.4). As in the proof of Theorem 1, we also see that the immersion satisfies (1.10).

After solving the first and the second equations of (1.10) in the same way as in the proof of Theorem 1, we obtain that

$$\tilde{L} = e^{2ibs}z(t) + \int_0^t z'(t)e^{-2i\theta(t)}dt + c \tag{1.19}$$

for some $\boldsymbol{C}^2$-valued function $z(t)$ and constant vector c. By substituting (1.19) into the last equation in (1.10), we discover

$$z''(t) - i\theta'(t)z'(t) + b^2 z(t) = 0\,. \tag{1.20}$$

Thus, $z(t)$ is a solution of (1.17). Hence, by applying (1.8) and (1.19), we see that $z(t)$ satisfies the two side conditions: $|z|^2 = 1/(4b^2)$ and $|z'|^2 = 1/4$ as well. □

REMARK 1. Theorems 1 and 2 imply that the class of Lagrangian H-umbilical surfaces in $\boldsymbol{C}^2$ satisfying (1.4) is very large.

REMARK 2. After the reparametrization $t = 2s$, the $\boldsymbol{C}^2$-valued solution $z(s)$ given in Theorem 2 is nothing but a Legendre curve in the hypersphere $S^3(r) \subset \boldsymbol{C}^2$ with radius $r = 1/(2b)$.

REMARK 3. If $\theta = 0$, the immersion defined by (1.2) is nothing but a Lagrangian pseudo-sphere as introduced in [1]. However, the immersion (1.2) is not a complex extensor in general.

EXAMPLE. Let a, b be two real numbers with $8a \neq b^2$ and $b \neq 0$. If $\theta = 2at$, then the solution of (1.1) is given by

$$z(t) = e^{iat}(c_1 e^{i\gamma t} + c_2 e^{-i\gamma t})\,, \quad \gamma = \sqrt{a^2+b^2} \tag{1.21}$$

for some constant vectors $c_1, c_2 \in \boldsymbol{C}^2$. Using (1.21) and the condition $|z|^2 = 1/(4b^2)$, we get

$$|c_1|^2 + |c_2|^2 = \frac{1}{4b^2}\,, \quad \langle c_1, c_2\rangle = \langle ic_1, c_2\rangle = 0\,. \tag{1.22}$$

Moreover, from (1.21) and the condition $|z'|^2 = 1/4$, we find

$$(a+\gamma)^2|c_1|^2 + (a-\gamma)^2|c_2|^2 = \frac{1}{4}\,. \tag{1.23}$$

Solving (1.22) and (1.23) for $|c_1|^2$ and $|c_2|^2$ yields

$$|c_1|^2 = \frac{\gamma - a}{8b^2\gamma}\,, \quad |c_2|^2 = \frac{\gamma + a}{8b^2\gamma}\,. \tag{1.24}$$

In view of (1.21) and (1.22), we may choose

$$c_1 = \left(\frac{\sqrt{\gamma - a}}{2b\sqrt{2\gamma}},\, 0\right), \quad c_2 = \left(0,\, \frac{\sqrt{\gamma + a}}{2b\sqrt{2\gamma}}\right).$$

Hence, the corresponding Lagrangian H-umbilical immersion is given by

$$\tilde{L} = \frac{e^{i(\gamma - 3a)t}}{2\sqrt{2}b\sqrt{\gamma}(b^2 - 8a^2)}(\sqrt{\gamma - a}(4a^2 + b^2 + 4a\gamma + (b^2 - 8a^2)e^{2i(bs+2at)}),$$
$$\sqrt{\gamma + a}(4a^2 + b^2 - 4a\gamma + (b^2 - 8a^2)e^{2i(bs+2at)})e^{-2i\gamma t}), \quad \gamma = \sqrt{a^2 + b^2}.$$

2. Lagrangian H-umbilical surfaces of constant curvature. Due to the error, we shall replace statement (ii-1) of Theorem 4.3 in [1] by the following result which completely classifies Lagrangian H-umbilical surfaces of constant curvature in $\boldsymbol{C}^2$.

THEOREM 3. *Let $L : M \to \boldsymbol{C}^2$ be a Lagrangian H-umbilical immersion of a surface of constant curvature K into $\boldsymbol{C}^2$. Then one of the following six statements holds*:

(1) *$K = 0$ and M is a Lagrangian cylinder over a curve.*

(2) *M is a flat Lagrangian surface whose immersion is congruent to*

$$L(t, u) = uz(t) + \int_0^t b(t)z'(t)dt, \tag{2.1}$$

where $b = b(t)$ is a real-valued function defined on an open interval $I \ni 0$ and $z : I \to S^3(1) \subset \boldsymbol{C}^2$ is a unit speed Legendre curve in $S^3(1)$.

(3) *$K = b^2 > 0$ and L is congruent to the Lagrangian immersion*

$$L(s, t) = e^{2ibs}z(t) + \int_0^t z'(t)e^{-2i\theta(t)}dt, \tag{2.2}$$

where b is a positive number, θ is a function defined on an open interval $I \ni 0$ and $z : I \to S^3(1/(2b)) \subset \boldsymbol{C}^2$ is a Legendre curve with constant speed $1/2$ in $S^3(1/(2b))$ of radius $1/(2b)$ satisfying (1.1).

(4) *$K = b^2 > 1$ and the immersion is congruent to*

$$\begin{aligned} L(s, t) = {} & \frac{\cos(bs)}{\sqrt{b^2 - 1}} \exp\left(i \sin^{-1}\left(\frac{b \sin bs}{\sqrt{b^2 - 1}}\right) - \frac{i}{b}\tan^{-1}\left(\frac{\sin bs}{\sqrt{b^2 \cos^2 bs - 1}}\right)\right) \\ & \times \left(ic_1 + \cos\left(\sqrt{b^2 - 1}\, t\right), ic_2 + \sin\left(\sqrt{b^2 - 1}\, t\right)\right) \\ & + \left(\int_0^s \exp\left\{2i \sin^{-1}\left(\frac{b \sin bs}{\sqrt{b^2 - 1}}\right)\right.\right. \\ & \left.\left. - \frac{i}{b}\tan^{-1}\left(\frac{\sin bs}{\sqrt{b^2 \cos^2 bs - 1}}\right)\right\} ds\right)(c_1, c_2) \end{aligned} \tag{2.3}$$

for some constants c_1, c_2.

(5) $K = b^2 > 0$ *and the immersion is congruent to*

$$
\begin{aligned}
L(s,t) = {} & \frac{\cos(bs)}{\sqrt{b^2+1}} \exp\left(i \sin^{-1}\left(\frac{b\sin bs}{\sqrt{b^2+1}}\right) + \frac{i}{b}\tanh^{-1}\left(\frac{\sin bs}{\sqrt{b^2\cos^2 bs+1}}\right)\right) \\
& \times \left(ic_1 + \cos\left(\sqrt{b^2+1}\,t\right), ic_2 + \sin\left(\sqrt{b^2+1}\,t\right)\right) \\
& + \left(\int_0^s \exp\left\{ 2i \sin^{-1}\left(\frac{b\sin bs}{\sqrt{b^2+1}}\right)\right.\right. \\
& \left.\left. + \frac{i}{b}\tanh^{-1}\left(\frac{\sin bs}{\sqrt{b^2\cos^2 bs+1}}\right)\right\} ds\right)(c_1,c_2)
\end{aligned} \tag{2.4}
$$

for some constants c_1, c_2.

(6) $K = -b^2 < 0$ *and the immersion is congruent to*

$$
\begin{aligned}
L(s,t) = {} & \frac{\cosh(bs)}{\sqrt{1-b^2}} \exp\left(i \sin^{-1}\left(\frac{b\sinh bs}{\sqrt{1-b^2}}\right) + \frac{i}{b}\tan^{-1}\left(\frac{\sinh bs}{\sqrt{1-b^2\cosh^2 bs}}\right)\right) \\
& \times \left(ic_1 + \cos\left(\sqrt{1-b^2}\,t\right), ic_2 + \sin\left(\sqrt{1-b^2}\,t\right)\right) \\
& + \left(\int_0^s \exp\left\{ 2i \sin^{-1}\left(\frac{b\sinh bs}{\sqrt{1-b^2}}\right)\right.\right. \\
& \left.\left. + \frac{i}{b}\tan^{-1}\left(\frac{\sinh(bs)}{\sqrt{1-b^2\cosh^2(bs)}}\right)\right\} ds\right)(c_1,c_2)
\end{aligned} \tag{2.5}
$$

for some constants c_1, c_2.

PROOF. Let $L : M \to \boldsymbol{C}^2$ be a Lagrangian H-umbilical immersion of a surface of constant curvature K into $\boldsymbol{C}^2$. Then, by definition, the second fundamental form of L satisfies

$$
h(e_1,e_1) = \lambda J e_1\,, \quad h(e_1,e_2) = \mu J e_2\,, \quad h(e_2,e_2) = \mu J e_1 \tag{2.6}
$$

for some functions λ, μ with respect to some orthonormal frame field e_1, e_2. From (2.6) and the equation of Codazzi, we find

$$
\begin{aligned}
& e_1\mu = (\lambda - 2\mu)\omega_1^2(e_2)\,, \quad e_2\mu = 3\mu\omega_1^2(e_1)\,, \\
& e_2\lambda = (\lambda - 2\mu)\omega_1^2(e_1)\,,
\end{aligned} \tag{2.7}
$$

where $\nabla_X e_1 = \omega_1^2(X)e_2$. Using (2.6) and the equation of Gauss, we find

$$
\lambda\mu - \mu^2 = K = \text{constant}\,. \tag{2.8}
$$

Differentiating (2.8) with respect to e_2 and applying (2.7) yield $\mu(\lambda - 2\mu)\omega_2^1(e_1) = 0$. Thus, we have either $\mu(\lambda - 2\mu) = 0$ or $\omega_1^2(e_1) = 0$.

If $\mu(\lambda - 2\mu) = 0$, then we obtain $K = 0$ or $\lambda = 2\mu$. Thus, by applying the Main Theorem of [2] and Theorem 1, we have statements (1), (2) or (3).

If $\mu(\lambda - 2\mu) \neq 0$, we get $\omega_1^2(e_1) = 0$. So, from (2.7), we find that $e_2\lambda = e_2\mu = 0$. Since $\nabla_{e_1} e_1 = 0$, there is a local coordinate system $\{s, u\}$ on M such that the metric tensor is

given by

$$g = ds^2 + G^2(s,u)du^2 \tag{2.9}$$

for some function G with $\partial/\partial s = e_1$, $\partial/\partial u = Ge_2$. From $e_2\lambda = e_2\mu = 0$, we get $\lambda = \lambda(s)$ and $\mu = \mu(s)$. Using (2.9), we find

$$\nabla_{\partial/\partial u}\frac{\partial}{\partial s} = (\ln G)_s\frac{\partial}{\partial u}\,, \quad \omega_1^2(e_2) = \frac{G_s}{G}\,. \tag{2.10}$$

Now, let us assume that $\lambda \neq 2\mu$. Then, by (2.7)–(2.10), we obtain

$$(\ln G)_s = \frac{\mu'}{\lambda - 2\mu} = \frac{\mu\mu'}{K - \mu^2}\,.$$

Solving this equation gives $G = F(u)/\sqrt{|K-\mu^2|}$ for some function F. Thus, (2.9) becomes

$$g = ds^2 + \frac{F^2(u)}{|K-\mu^2(s)|}du^2\,. \tag{2.11}$$

If t denotes an antiderivative of $F(u)$, then we obtain from (2.11) that

$$g = ds^2 + \frac{dt^2}{|K-\mu^2(s)|}\,, \quad G^2(s) = \frac{1}{|K-\mu^2(s)|}\,. \tag{2.12}$$

CASE (a) $K = b^2 > 0$. Since G satisfies $G_{ss} + b^2G = 0$ in this case, we have $G = c_1\cos(bs) + c_2\sin(bs)$ for some constants c_1, c_2, not both zero. Thus, we obtain $G = r\cos(bs + c)$ for some constants $r \neq 0$ and c. After applying a suitable translation in s and a suitable dilation in t, we obtain $G(s) = \cos(bs)$. Therefore, we find

$$\begin{aligned} &g = ds^2 + \cos^2(bs)dt^2\,,\\ &\nabla_{\partial/\partial s}\frac{\partial}{\partial s} = 0\,, \quad \nabla_{\partial/\partial s}\frac{\partial}{\partial t} = -b\tan(bs)\frac{\partial}{\partial t}\,,\\ &\nabla_{\partial/\partial t}\frac{\partial}{\partial t} = \frac{b}{2}\sin(2bs)\frac{\partial}{\partial s}\,. \end{aligned} \tag{2.13}$$

CASE (a-1) $K = b^2 > \mu^2$ and $b > 0$. In this case, we have $b^2 = \mu^2 + \sec^2(bs) \geq 1$. Without loss of generality, we may assume that

$$\lambda = \frac{2b^2 - \sec^2(bs)}{\sqrt{b^2 - \sec^2(bs)}}\,, \quad \mu = \sqrt{b^2 - \sec^2(bs)}\,. \tag{2.14}$$

By applying (2.6), (2.13) and (2.14), we see that the immersion satisfies

$$\begin{aligned} L_{ss} &= i\frac{2b^2 - \sec^2(bs)}{\sqrt{b^2 - \sec^2(bs)}}L_s,\\ L_{st} &= \big(i\sqrt{b^2 - \sec^2(bs)} - b\tan(bs)\big)L_t\,,\\ L_{tt} &= \big(i\sqrt{b^2\cos^2(bs) - 1} + b\sin(bs)\big)\cos(bs)L_s\,. \end{aligned} \tag{2.15}$$

After solving the second equation of (2.15), we get

$$L_t = F(t)\cos(bs)\exp\left(i\sin^{-1}\left(\frac{b\sin bs}{\sqrt{b^2-1}}\right)-\frac{i}{b}\tan^{-1}\left(\frac{\sin bs}{\sqrt{b^2\cos^2 bs-1}}\right)\right) \tag{2.16}$$

for some $\boldsymbol{C}^2$-valued function $F(t)$. Thus, we have

$$\begin{aligned} L = A(s) + B(t)\cos(bs)\exp\Bigg(i\sin^{-1}\left(\frac{b\sin bs}{\sqrt{b^2-1}}\right) \\ -\frac{i}{b}\tan^{-1}\left(\frac{\sin bs}{\sqrt{b^2\cos^2 bs-1}}\right)\Bigg), \end{aligned} \tag{2.17}$$

where $B(t)$ is an antiderivative of $F(t)$ and $A(s)$ is a $\boldsymbol{C}^2$-valued function. From (2.17) we find

$$\begin{aligned} L_s = {} & A'(s) - B(t)\big(b\sin bs - i\sqrt{b^2\cos^2 bs-1}\big) \\ & \times\exp\left(i\sin^{-1}\left(\frac{b\sin bs}{\sqrt{b^2-1}}\right)-\frac{i}{b}\tan^{-1}\left(\frac{\sin bs}{\sqrt{b^2\cos^2 bs-1}}\right)\right), \end{aligned} \tag{2.18}$$

$$\begin{aligned} L_{ss} = {} & A''(s) - B(t)\frac{(\sqrt{b^2-\sec^2 bs}+ib\tan bs)(2b^2\cos^2 bs-1)}{\sqrt{b^2\cos^2 bs-1}} \\ & \times\exp\left(i\sin^{-1}\left(\frac{b\sin bs}{\sqrt{b^2-1}}\right)-\frac{i}{b}\tan^{-1}\left(\frac{\sin bs}{\sqrt{b^2\cos^2 bs-1}}\right)\right). \end{aligned} \tag{2.19}$$

Substituting (2.18) and (2.19) into the first equation in (2.15), we find

$$A''(s) = i\frac{2b^2-\sec^2(bs)}{\sqrt{b^2-\sec^2(bs)}}A'(s)\,. \tag{2.20}$$

Solving this equation gives

$$A'(s) = C\exp\left\{2i\sin^{-1}\left(\frac{b\sin bs}{\sqrt{b^2-1}}\right)-\frac{i}{b}\tan^{-1}\left(\frac{\sin bs}{\sqrt{b^2\cos^2 bs-1}}\right)\right\} \tag{2.21}$$

for some vector C in $\boldsymbol{C}^2$. Hence, we obtain

$$A(s) = C\int_0^s \exp\left\{2i\sin^{-1}\left(\frac{b\sin bs}{\sqrt{b^2-1}}\right)-\frac{i}{b}\tan^{-1}\left(\frac{\sin bs}{\sqrt{b^2\cos^2 bs-1}}\right)\right\}ds + E \tag{2.22}$$

for vector E in $\boldsymbol{C}^2$. We may assume $E = 0$ by applying a suitable translation on $\boldsymbol{C}^2$. Hence, we obtain from (2.17) that

$$\begin{aligned} L = {} & C\int_0^s \exp\left\{2i\sin^{-1}\left(\frac{b\sin bs}{\sqrt{b^2-1}}\right)-\frac{i}{b}\tan^{-1}\left(\frac{\sin bs}{\sqrt{b^2\cos^2 bs-1}}\right)\right\}ds \\ & + B(t)\cos(bs)\exp\left(i\sin^{-1}\left(\frac{b\sin bs}{\sqrt{b^2-1}}\right)-\frac{i}{b}\tan^{-1}\left(\frac{\sin bs}{\sqrt{b^2\cos^2 bs-1}}\right)\right). \end{aligned} \tag{2.23}$$

Thus we have

$$L_s = (C + i\sqrt{b^2-1}B)\exp\left(2i\sin^{-1}\left(\frac{b\sin bs}{\sqrt{b^2-1}}\right)\right.$$
$$\left. - \frac{i}{b}\tan^{-1}\left(\frac{\sin bs}{\sqrt{b^2\cos^2 bs - 1}}\right)\right), \tag{2.24}$$

$$L_t = B'(t)\cos(bs)\exp\left(i\sin^{-1}\left(\frac{b\sin bs}{\sqrt{b^2-1}}\right) - \frac{i}{b}\tan^{-1}\left(\frac{\sin bs}{\sqrt{b^2\cos^2 bs-1}}\right)\right). \tag{2.25}$$

Substituting (2.24) and (2.25) into the last equation of (2.15) yields

$$B''(t)+(b^2-1)B(t) = i\sqrt{b^2-1}C\,, \tag{2.26}$$

which implies that

$$B(t) = C_1\cos\sqrt{b^2-1}\,t + C_2\sin\sqrt{b^2-1}\,t + \frac{i}{\sqrt{b^2-1}}C\,.$$

Substituting these into (2.23) gives

$$L = C\int_0^s \exp\left\{2i\sin^{-1}\left(\frac{b\sin bs}{\sqrt{b^2-1}}\right) - \frac{i}{b}\tan^{-1}\left(\frac{\sin bs}{\sqrt{b^2\cos^2 bs-1}}\right)\right\}ds$$
$$+\left(C_1\cos\sqrt{b^2-1}\,t + C_2\sin\sqrt{b^2-1}\,t + \frac{i}{\sqrt{b^2-1}}C\right)\cos(bs) \tag{2.27}$$
$$\times\exp\left(i\,\sin^{-1}\left(\frac{b\sin bs}{\sqrt{b^2-1}}\right) - \frac{i}{b}\tan^{-1}\left(\frac{\sin bs}{\sqrt{b^2\cos^2 bs-1}}\right)\right).$$

Hence, we may obtain (2.3) by choosing suitable initial conditions.

CASE (a-2) $K = b^2 < \mu^2$ and $b > 0$. In this case, we have $\mu^2 - b^2 = \sec^2(bs)$. Hence, without loss of generality, we may assume that

$$\lambda = \frac{2b^2+\sec^2(bs)}{\sqrt{b^2+\sec^2(bs)}}\,, \quad \mu = \sqrt{b^2+\sec^2(bs)}\,. \tag{2.28}$$

By applying (2.6), (2.13) and (2.28), we obtain

$$L_{ss} = i\frac{2b^2+\sec^2(bs)}{\sqrt{b^2+\sec^2(bs)}}L_s\,,$$
$$L_{st} = \left(i\sqrt{b^2+\sec^2(bs)} - b\tan(bs)\right)L_t\,, \tag{2.29}$$
$$L_{tt} = \left(i\sqrt{b^2\cos^2(bs)+1} + b\sin(bs)\right)\cos(bs)L_s\,.$$

After solving this system of PDEs in the same way as in Case (a-1) and after choosing suitable initial conditions, we conclude that the immersion is congruent to (2.4).

CASE (b) $K = -b^2 < 0$ and $b > 0$. Since G satisfies $G_{ss} - b^2G = 0$ in this case, we get $G = c_1\cosh(bs) + c_2\sinh(bs)$ for some constants c_1, c_2, not both zero. Thus, by applying

a suitable translation in s and a suitable dilation in t, we obtain

$$\begin{aligned} &g = ds^2 + \cosh^2(bs)dt^2\,,\\ &\nabla_{\partial/\partial s}\frac{\partial}{\partial s} = 0\,, \quad \nabla_{\partial/\partial s}\frac{\partial}{\partial t} = b\tanh(bs)\frac{\partial}{\partial t}\,,\\ &\nabla_{\partial/\partial t}\frac{\partial}{\partial t} = -\frac{b}{2}\sinh(2bs)\frac{\partial}{\partial s}\,. \end{aligned} \tag{2.30}$$

From (2.12) and the first equation in (2.30), we find $\operatorname{sech}^2 bs = b^2 + \mu^2$. Thus we obtain $1 \geq \operatorname{sech}^2 bs \geq b^2$ and $\mu = \pm\sqrt{\operatorname{sech}^2 bs - b^2}$. Without loss of generality, we may assume that

$$\lambda = \frac{\operatorname{sech}^2(bs) - 2b^2}{\sqrt{\operatorname{sech}^2(bs) - b^2}}\,, \quad \mu = \sqrt{\operatorname{sech}^2(bs) - b^2}\,. \tag{2.31}$$

By applying (2.6), (2.30) and (2.31), we see that the immersion satisfies

$$\begin{aligned} &L_{ss} = i\frac{\operatorname{sech}^2(bs) - 2b^2}{\sqrt{\operatorname{sech}^2(bs) - b^2}}L_s\,,\\ &L_{st} = \big(i\sqrt{\operatorname{sech}^2(bs) - b^2} + b\tanh(bs)\big)L_t\,,\\ &L_{tt} = \big(i\sqrt{1 - b^2\cosh^2(bs)} - b\sinh(bs)\big)\cosh(bs)L_s\,. \end{aligned} \tag{2.32}$$

After solving this system of PDEs in the same way as in Case (a-1) and after choosing suitable initial conditions, we conclude that the immersion L is congruent to (2.5). □

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