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ALGEBRAIC INDEPENDENCE OF MODIFIED RECIPROCAL SUMS OF PRODUCTS OF FIBONACCI NUMBERS*

By

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Abstract. In this paper we establish, using Mahler's method, the algebraic independence of reciprocal sums of products of Fibonacci numbers including slowly increasing factors in their numerators (see Theorems 1, 5, and 6 below). Theorems 1 and 4 are proved by using Theorems 2 and 3 stating key formulas of this paper, which are deduced from the crucial Lemma 2. Theorems 5 and 6 are proved by using different technique. From Theorems 2 and 5 we deduce Corollary 2, the algebraic independence of the sum of a certain series and that of its subseries obtained by taking subscripts in a geometric progression.

1 Introduction

Let $\{F_n\}_{n\geq 0}$ be the sequence of Fibonacci numbers defined by

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n \quad (n \geq 0). \tag{1}$$

Brousseau [2] proved that for every $k \in \mathbf{N}$

$$\sigma_k = \sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+k}} = \frac{1}{F_k}\left(\frac{k(1-\sqrt{5})}{2} + \sum_{n=1}^{k} \frac{F_{n-1}}{F_n}\right).$$

Rabinowitz [8] proved that for every $k \in \mathbf{N}$

$$\sigma_k^* = \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2k}} = \frac{1}{F_{2k}} \sum_{n=1}^{k} \frac{1}{F_{2n-1} F_{2n}}.$$

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In this paper we consider the arithmetic nature of the sums of similarly constructed series such as

$$\sum_{n=1}^{\infty} \frac{(-1)^n [\log_d n]}{F_n F_{n+k}} \quad (d \in \mathbf{N} \setminus \{1\}, k \in \mathbf{N})$$

and

$$\sum_{n=1}^{\infty} \frac{[\log_d n]}{F_n F_{n+2k}} \quad (d \in \mathbf{N} \setminus \{1\}, k \in \mathbf{N}),$$

where $[x]$ denotes the largest integer not exceeding the real number x. These sums are not only transcendental but also algebraically independent in contrast with the sums σ_k and σ_k^* which are algebraic numbers.

In what follows, let $\{R_n\}_{n \geq 0}$ be the binary linear recurrence defined by

$$R_{n+2} = A_1 R_{n+1} + A_2 R_n \quad (n \geq 0), \tag{2}$$

where A_1, A_2 are nonzero integers with $\Delta = A_1^2 + 4A_2 > 0$ and R_0, R_1 are integers with $R_0 R_2 \neq R_1^2$ and $A_1 R_0 (A_1 R_0 - 2R_1) \leq 0$. We can express $\{R_n\}_{n \geq 0}$ as follows:

$$R_n = a\alpha^n + b\beta^n \quad (n \geq 0),$$

where α, β $(|\alpha| \geq |\beta|)$ are the roots of $\Phi(X) = X^2 - A_1 X - A_2$ and $a, b \in \mathbf{Q}(\sqrt{\Delta})$. It is easily seen that $|\alpha| > |\beta| > 0$. Since $R_0 R_2 - R_1^2 = ab\Delta$ and $A_1 R_0 (A_1 R_0 - 2R_1) = (\alpha^2 - \beta^2)(b^2 - a^2)$, we see that $|a| \geq |b| > 0$. Therefore $\{R_n\}_{n \geq 0}$ is not a geometric progression and $R_n \neq 0$ for any $n \geq 1$.

THEOREM 1. *The numbers*

$$\sum_{n=1}^{\infty} \frac{(-A_2)^n [\log_d n]}{R_n R_{n+k}} \quad (d \in \mathbf{N} \setminus \{1\}, k \in \mathbf{N})$$

are algebraically independent and so are the numbers

$$\sum_{n=1}^{\infty} \frac{A_2^n [\log_d n]}{R_n R_{n+2k}} \quad (d \in \mathbf{N} \setminus \{1\}, k \in \mathbf{N}).$$

EXAMPLE 1. Let $\{F_n\}_{n \geq 0}$ be the sequence of the Fibonacci numbers defined by (1). Then the numbers

$$\sum_{n=1}^{\infty} \frac{(-1)^n [\log_d n]}{F_n F_{n+k}} \quad (d \in \mathbf{N} \backslash \{1\}, k \in \mathbf{N})$$

are algebraically independent and so are the numbers

$$\sum_{n=1}^{\infty} \frac{[\log_d n]}{F_n F_{n+2k}} \quad (d \in \mathbf{N} \backslash \{1\}, k \in \mathbf{N}).$$

EXAMPLE 2. Let $\{L_n\}_{n \geq 0}$ be the sequence of Lucas numbers defined by

$$L_0 = 2, \quad L_1 = 1, \quad L_{n+2} = L_{n+1} + L_n \quad (n \geq 0). \tag{3}$$

Then the numbers

$$\sum_{n=1}^{\infty} \frac{(-1)^n [\log_d n]}{L_n L_{n+k}} \quad (d \in \mathbf{N} \backslash \{1\}, k \in \mathbf{N})$$

are algebraically independent and so are the numbers

$$\sum_{n=1}^{\infty} \frac{[\log_d n]}{L_n L_{n+2k}} \quad (d \in \mathbf{N} \backslash \{1\}, k \in \mathbf{N}).$$

Theorem 1 is deduced from Theorems 2 and 3 below. The proof will be given in Section 3.

Let $f(x)$ be a real-valued function on $x \geq 0$ such that $f'(x) > 0$ for any $x > 0$ and $f(\mathbf{N}) \subset \mathbf{N}$. Let $f^{-1}(x)$ be the inverse function of $f(x)$. For any $k \in \mathbf{N}$ we put

$$S_k = \sum_{n=f(1)}^{\infty} \frac{(-A_2)^n [f^{-1}(n)]}{R_n R_{n+k}}, \quad S_k^* = \sum_{n=f(1)}^{\infty} \frac{A_2^n [f^{-1}(n)]}{R_n R_{n+k}},$$

$$T_k = \sum_{n=f(1)}^{\infty} \frac{(-A_2)^n [f^{-1}(n)]}{R_{n+k-1} R_{n+k}},$$

and

$$U_k = \sum_{n=1}^{\infty} \frac{(-A_2)^{f(n)}}{R_{f(n)} R_{f(n)+k}}.$$

Let $\{F_n^*\}_{n \geq 0}$ be the Fibonacci type sequence defined by

$$F_0^* = 0, \quad F_1^* = 1, \quad F_{n+2}^* = A_1 F_{n+1}^* + A_2 F_n^* \quad (n \geq 0).$$

Theorem 2. *For any* $k \in \mathbf{N}$

$$S_k = \frac{1}{F_k^*} \sum_{l=1}^{k} (-A_2)^{l-1} T_l$$

and

$$U_k = \frac{1}{F_k^*} (T_1 - (-A_2)^k T_{k+1}).$$

Hence the sets of the numbers $\{S_1, \ldots, S_{k+1}\}$, $\{T_1, \ldots, T_{k+1}\}$, *and* $\{S_1 (= T_1), U_1, \ldots, U_k\}$ *generate the same vector space over* $\mathbf{Q}$.

Theorem 3. *If* $f(n) \equiv f(1) \pmod 2$ *for any* $n \geq 1$, *then*

$$S_{2k}^* = \frac{(-1)^{f(1)}}{F_{2k}^*} \sum_{l=1}^{2k} A_2^{l-1} T_l$$

for any $k \in \mathbf{N}$. *Hence the numbers* $\{S_{2l} \mid 1 \leq l \leq k\}$ *are expressed as linearly independent linear combinations over* $\mathbf{Q}$ *of the numbers* $\{T_l \mid 1 \leq l \leq 2k\}$.

Using Theorem 2, we prove also the following:

Theorem 4. *The numbers*

$$\sum_{n=1}^{\infty} \frac{A_2^{d^n}}{R_{d^n} R_{d^n+k}} \quad (d \in \mathbf{N} \backslash \{1\}, k \in \mathbf{N})$$

are algebraically independent.

Example 3. The numbers

$$\sum_{n=1}^{\infty} \frac{1}{F_{d^n} F_{d^n+k}} \quad (d \in \mathbf{N} \backslash \{1\}, k \in \mathbf{N})$$

are algebraically independent and so are the numbers

$$\sum_{n=1}^{\infty} \frac{1}{L_{d^n} L_{d^n+k}} \quad (d \in \mathbf{N} \backslash \{1\}, k \in \mathbf{N}).$$

Using different technique to that used in the proof of Theorem 4, we prove the following:

Theorem 5. *Let d be an integer greater than 1. Then the numbers*

$$\sum_{n=1}^{\infty} \frac{n^l \xi^n (-A_2)^{d^n}}{R_{d^n} R_{d^n+k}} \quad (\xi \in \overline{\mathbf{Q}}^{\times}, l \geq 0, k \in \mathbf{N}) \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-A_2)^n [\log_d n]}{R_n R_{n+1}} \tag{4}$$

are algebraically independent.

As a special case of Theorem 5 we have the following:

Corollary 1. *Let d be an integer greater than 1. Then the numbers*

$$\sum_{n=1}^{\infty} \frac{(-A_2)^{d^n}}{R_{d^n} R_{d^n+k}}, \quad \sum_{n=1}^{\infty} \frac{n(-A_2)^{d^n}}{R_{d^n} R_{d^n+k}} \quad (k \in \mathbf{N}), \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-A_2)^n [\log_d n]}{R_n R_{n+1}}$$

are algebraically independent.

Combining Corollary 1 and Theorem 2 with $f(x) = d^x$, we immediately have the following:

Corollary 2. *Let d be an integer greater than 1. Then the numbers*

$$\sum_{n=1}^{\infty} \frac{(-A_2)^n [\log_d n]}{R_n R_{n+k}}, \quad \sum_{n=1}^{\infty} \frac{n(-A_2)^{d^n}}{R_{d^n} R_{d^n+k}} \quad (k \in \mathbf{N})$$

are algebraically independent.

It is interesting that the second series of Corollary 2 is regarded as a subseries of the first one obtained by replacing n by d^n. It seems difficult to find in literature the results which assert the algebraic independence of the sum of a certain series and that of its subseries with subscripts taken in a geometric progression. For example, the algebraic independency of the numbers $\sum_{n=1}^{\infty} 1/F_n$ and $\sum_{n=1}^{\infty} 1/F_{d^n}$ $(d \geq 3)$ is open. On the other hand, Lucas [3] showed that $\sum_{n=1}^{\infty} 1/F_{2^n} = (5 - \sqrt{5})/2$. André-Jeannin [1] proved the irrationality of $\sum_{n=1}^{\infty} 1/F_n$, while its transcendency is open. Nishioka, Tanaka, and Toshimitsu [7] proved that the numbers $\sum_{n=1}^{\infty} 1/F_{d^n}$ $(d \geq 3)$ are algebraically independent.

Example 4. Let $\{F_n\}_{n \geq 0}$ be the sequence of the Fibonacci numbers defined by (1) and d an integer greater than 1. Then the numbers

$$\sum_{n=1}^{\infty}\frac{(-1)^n[\log_d n]}{F_nF_{n+k}},\quad \sum_{n=1}^{\infty}\frac{n}{F_{d^n}F_{d^n+k}}\quad (k\in\mathbf{N})$$

are algebraically independent.

Example 5. Let $\{L_n\}_{n\geq 0}$ be the sequence of Lucas numbers defined by (3) and d an integer greater than 1. Then the numbers

$$\sum_{n=1}^{\infty}\frac{(-1)^n[\log_d n]}{L_nL_{n+k}},\quad \sum_{n=1}^{\infty}\frac{n}{L_{d^n}L_{d^n+k}}\quad (k\in\mathbf{N})$$

are algebraically independent.

If Δ is not a perfect square, we can prove the algebraic independence of the sums of the series (4) of Theorem 5 without the factor $(-A_2)^{d^n}$ in their numerators as follows:

Theorem 6. *Assume in addition that Δ is not a perfect square. Let d be an integer greater than* 1. *Then the numbers*

$$\sum_{n=1}^{\infty}\frac{n^l\xi^n}{R_{d^n}R_{d^n+k}}\quad (\xi\in\overline{\mathbf{Q}}^{\times}, l\geq 0, k\in\mathbf{N})\quad\text{and}\quad \sum_{n=1}^{\infty}\frac{(-A_2)^n[\log_d n]}{R_nR_{n+1}}\tag{5}$$

are algebraically independent.

2 Lemmas

The following lemma will be used in the proof of Theorems 1 and 4.

Lemma 1 (Tanaka [9]). *Let $\{R_n\}_{n\geq 0}$ be as in Section* 1. *Then the numbers*

$$\sum_{n=d}^{\infty}\frac{(-A_2)^n[\log_d n]}{R_{n+k-1}R_{n+k}}\quad (d\in\mathbf{N}\backslash\{1\}, k\in\mathbf{N})$$

are algebraically independent.

The following lemma plays an essential role in the proof of Theorems 2 and 3.

Lemma 2. *Let $f(x)$ be a real-valued function on $x\geq 0$ such that $f'(x)>0$ for any $x>0$ and $f(\mathbf{N})\subset\mathbf{N}$. Let $f^{-1}(x)$ be the inverse function of $f(x)$. Let K*

be any field of characteristic 0 *endowed with an absolute value* $|\ |_v$. *Let* $\{a_n\}_{n\geq 1}$ *be a sequence in* K *with* $|a_n|_v = o(1/f^{-1}(n))$. *Suppose the sum* $\sum_{n=1}^{\infty} |a_n|_v$ *converges in* $\mathbf{R}$. *Then in the completion* K_v *of* K *we have*

$$\sum_{n=f(1)}^{\infty} [f^{-1}(n)](a_n - a_{n+1}) = \sum_{h=1}^{\infty} a_{f(h)}. \tag{6}$$

Proof. Let $h \in \mathbf{N}$ and $n \in \mathbf{N}$. Since $f'(x) > 0$ for any $x > 0$, $(f^{-1}(x))' > 0$ for any $x \geq f(1)$. Hence, if $f(h) \leq n < f(h+1)$, then $h \leq f^{-1}(n) < h+1$ and so $[f^{-1}(n)] = h$. Therefore, letting

$$\chi(n) = \begin{cases} 1 & (n = f(h)) \\ 0 & (\text{otherwise}) \end{cases} \quad \text{and} \quad s_n = \sum_{k=1}^{n} \chi(k),$$

we see that $s_n = [f^{-1}(n)]$ for $n \geq f(1)$. Then, letting $H \in \mathbf{N}$ and $N = f(H)$, we have

$$\begin{aligned} \sum_{h=1}^{H} a_{f(h)} &= \sum_{n=f(1)}^{N} \chi(n) a_n \\ &= \sum_{n=f(1)}^{N-1} s_n(a_n - a_{n+1}) + s_N a_N \\ &= \sum_{n=f(1)}^{N-1} [f^{-1}(n)](a_n - a_{n+1}) + [f^{-1}(N)] a_N. \end{aligned} \tag{7}$$

Since $|a_n|_v = o(1/f^{-1}(n))$, $[f^{-1}(N)]a_N$ tends to 0 as $N \to \infty$. Since $\sum_{n=1}^{\infty} |a_n|_v$ converges in $\mathbf{R}$, the sum of the subseries $\sum_{h=1}^{\infty} a_{f(h)}$ also converges in K_v. Letting $H \to \infty$ in (7), we have (6). This completes the proof of the lemma.

Remark 1. The condition $|a_n|_v = o(1/f^{-1}(n))$ of Lemma 2 is satisfied if

$$|a_n|_v = o(n^{-1}), \tag{8}$$

since we have $[f^{-1}(n)] = s_n \leq n$. We shall use the condition (8) instead in the proof of Theorems 2 and 3.

The following lemma is a special case of Theorem 3.3.2 in Nishioka [5], since its assumption is satisfied by Masser's vanishing theorem [4].

LEMMA 3. *Let K be an algebraic number field and d an integer greater than* 1. *Suppose that $f_{ij}(z_1, z_2) \in K[[z_1, z_2]]$ $(i = 1, \dots, m, j = 1, \dots, n(i))$ are algebraically independent over $K(z_1, z_2)$ and convergent in a polydisc $U \subset \mathbf{C}^2$ around the origin. Assume that, for every i, $f_{i1}(z_1, z_2), \dots, f_{in(i)}(z_1, z_2)$ satisfy the system of functional equations*

$$\begin{pmatrix} f_{i1}(z_1, z_2) \\ \vdots \\ \vdots \\ f_{in(i)}(z_1, z_2) \end{pmatrix}$$

$$= \begin{pmatrix} a_i & 0 & \cdots & 0 \\ a_{21}^{(i)} & a_i & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ a_{n(i)\,1}^{(i)} & \cdots & a_{n(i)\,n(i)-1}^{(i)} & a_i \end{pmatrix} \begin{pmatrix} f_{i1}(z_1^d, z_2^d) \\ \vdots \\ \vdots \\ f_{in(i)}(z_1^d, z_2^d) \end{pmatrix} + \begin{pmatrix} b_{i1}(z_1, z_2) \\ \vdots \\ \vdots \\ b_{in(i)}(z_1, z_2) \end{pmatrix}, \tag{9}$$

where $a_i, a_{st}^{(i)} \in K$ and $b_{ij}(z_1, z_2) \in K(z_1, z_2)$. If $(\alpha_1, \alpha_2) \in U$ is an algebraic point with $0 < |\alpha_1|, |\alpha_2| < 1$ such that α_1, α_2 are multiplicatively independent, then the values $f_{ij}(\alpha_1, \alpha_2)$ $(i = 1, \dots, m, j = 1, \dots, n(i))$ are algebraically independent.

REMARK 2. It is not necessary in Lemma 3 to assume that $b_{ij}(\alpha_1^{d^k}, \alpha_2^{d^k})$ $(i = 1, \dots, m, j = 1, \dots, n(i))$ are defined for all $k \geq 0$, which is satisfied by (9) and the fact that $f_{ij}(\alpha_1^{d^k}, \alpha_2^{d^k})$ $(i = 1, \dots, m, j = 1, \dots, n(i))$ are defined for all $k \geq 0$ since $(\alpha_1^{d^k}, \alpha_2^{d^k}) \in U$.

LEMMA 4 (Theorem 3.2.1 in Nishioka [5]). *Let C be a field of characteristic* 0. *Suppose that $f_{ij}(z_1, z_2) \in C[[z_1, z_2]]$ $(i = 1, \dots, m, j = 1 \dots, n(i))$ satisfy the functional equations of the form* (9) *with $a_i, a_{st}^{(i)} \in C$, $a_i \neq 0$, $a_{s\,s-1}^{(i)} \neq 0$ $(2 \leq s \leq n(i))$, and $b_{ij}(z_1, z_2) \in C(z_1, z_2)$. If $f_{ij}(z_1, z_2)$ $(i = 1, \dots, m, j = 1, \dots, n(i))$ are algebraically dependent over $C(z_1, z_2)$, then there exists a non-empty subset $\{i_1, \dots, i_r\}$ of $\{1, \dots, m\}$ with $a_{i_1} = \cdots = a_{i_r}$ such that $f_{i_1\,1}, \dots, f_{i_r\,1}$ are linearly dependent over C modulo $C(z_1, z_2)$, that is, there exist $c_1, \dots, c_r \in C$, not all zero, such that*

$$c_1 f_{i_1\,1} + \cdots + c_r f_{i_r\,1} \in C(z_1, z_2).$$

LEMMA 5 (Nishioka [6, Lemmas 2, 3, and 6]). *Let ξ be a nonzero complex number and $a_1, \dots, a_n$ nonzero complex numbers satisfying $|a_i| \neq 1$, $|a_i| \neq |a_j|$ $(i \neq j)$. Let $f_i(z) \in \mathbf{C}[[z]]$ $(0 \leq i \leq n)$ satisfy the functional equations*

$$f_0(z) = \xi f_0(z^d) + \frac{z^r}{1+\varepsilon z^r},$$

$$f_i(z) = \xi f_i(z^d) + \frac{z^r}{1+a_i z^r} \quad (1 \le i \le n),$$

where $r \in \mathbf{N}$ *and* $\varepsilon = \pm 1$. *If* $d = \xi = 2$ *and* $\varepsilon = 1$, *then* $f_i(z)$ $(1 \le i \le n)$ *are linearly independent over* $\mathbf{C}$ *modulo* $\mathbf{C}(z)$, *otherwise so are* $f_i(z)$ $(0 \le i \le n)$.

Remark 3. If $d = \xi = 2$ and $\varepsilon = 1$, then

$$f_0(z) = \sum_{h=0}^{\infty} \frac{2^h z^{r2^h}}{1+z^{r2^h}} = \frac{z^r}{1-z^r} \in \mathbf{C}(z).$$

Lemma 6 (A special case of Theorem 3.3.10 in Nishioka [5]). *Let* C *be a field and* F *a subfield of* C. *If*

$$f(z_1, z_2) \in C[[z_1, z_2]] \cap F(z_1, z_2),$$

then there exist $A(z_1, z_2), B(z_1, z_2) \in F[z_1, z_2]$ *such that*

$$f(z_1, z_2) = \frac{A(z_1, z_2)}{B(z_1, z_2)}, \quad B(0,0) \ne 0.$$

3 Proof of Theorems 1, 2, 3, and 4

Proof of Theorem 1. Let

$$S_{d,k} = \sum_{n=1}^{\infty} \frac{(-A_2)^n [\log_d n]}{R_n R_{n+k}} = \sum_{n=d}^{\infty} \frac{(-A_2)^n [\log_d n]}{R_n R_{n+k}},$$

$$S_{d,k}^* = \sum_{n=1}^{\infty} \frac{A_2^n [\log_d n]}{R_n R_{n+k}} = \sum_{n=d}^{\infty} \frac{A_2^n [\log_d n]}{R_n R_{n+k}},$$

and

$$T_{d,k} = \sum_{n=1}^{\infty} \frac{(-A_2)^n [\log_d n]}{R_{n+k-1} R_{n+k}} = \sum_{n=d}^{\infty} \frac{(-A_2)^n [\log_d n]}{R_{n+k-1} R_{n+k}} \quad (d \in \mathbf{N} \backslash \{1\}, k \in \mathbf{N}).$$

Letting $f(x) = d^x$ in Theorem 2, we see that for any fixed d

$$S_{d,k} = \frac{1}{F_k^*} \sum_{l=1}^{k} (-A_2)^{l-1} T_{d,l} \quad (k \in \mathbf{N}).$$

Hence the sets of the numbers $\{S_{d,l} \mid 2 \le d \le m, 1 \le l \le k\}$ and $\{T_{d,l} \mid 2 \le d \le m, 1 \le l \le k\}$ generate the same vector space over $\mathbf{Q}$ for any fixed $m \in \mathbf{N} \setminus \{1\}$ and for any fixed $k \in \mathbf{N}$. Since the numbers $T_{d,k}$ $(d \in \mathbf{N} \setminus \{1\}, k \in \mathbf{N})$ are algebraically independent by Lemma 1, the numbers $S_{d,k}$ $(d \in \mathbf{N} \setminus \{1\}, k \in \mathbf{N})$ are algebraically independent.

Again letting $f(x) = d^x$ and noting that $f(n) \equiv f(1) \pmod 2$ for any $n \in \mathbf{N}$, we see by Theorem 3 that for any fixed d

$$S^*_{d,2k} = \frac{(-1)^{f(1)}}{F^*_{2k}} \sum_{l=1}^{2k} A_2^{l-1} T_{d,l} \quad (k \in \mathbf{N}).$$

Hence the numbers $\{S^*_{d,2l} \mid 2 \le d \le m, 1 \le l \le k\}$ are expressed as linearly independent linear combinations over $\mathbf{Q}$ of the numbers $\{T_{d,l} \mid 2 \le d \le m, 1 \le l \le 2k\}$ for any $m \in \mathbf{N} \setminus \{1\}$ and for any $k \in \mathbf{N}$. Since the numbers $T_{d,k}$ $(d \in \mathbf{N} \setminus \{1\}, k \in \mathbf{N})$ are algebraically independent by Lemma 1, the numbers $S^*_{d,2k}$ $(d \in \mathbf{N} \setminus \{1\}, k \in \mathbf{N})$ are algebraically independent, which completes the proof of the theorem.

Before stating the proof of Theorems 2 and 3, we recall that $\{R_n\}_{n \ge 0}$ is expressed as

$$R_n = a\alpha^n + b\beta^n \quad (n \ge 0),$$

where α, β are the roots of $\Phi(X) = X^2 - A_1 X - A_2$ such that $|\alpha| > |\beta| > 0$ and $a, b \in \mathbf{Q}(\sqrt{\Delta})$ satisfy $|a| \ge |b| > 0$. Using the same α and β, we can express the sequence $\{F^*_n\}_{n \ge 0}$ defined before Theorem 2 by

$$F^*_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad (n \ge 0).$$

PROOF OF THEOREM 2. Since $R_n = a\alpha^n + b\beta^n$ $(n \ge 0)$ and $-A_2 = \alpha\beta$, we have

$$\begin{aligned} \frac{(-A_2)^n}{R_n R_{n+k}} &= \frac{1}{a(\alpha^k - \beta^k)} \left(\frac{\beta^n}{a\alpha^n + b\beta^n} - \frac{\beta^{n+k}}{a\alpha^{n+k} + b\beta^{n+k}} \right) \\ &= \frac{1}{a(\alpha^k - \beta^k)} \left(\frac{\beta^n}{R_n} - \frac{\beta^{n+k}}{R_{n+k}} \right). \end{aligned} \tag{10}$$

Hence, noting that $n|\beta^n / R_n| \to 0$ as $n \to \infty$, we have by Lemma 2 with Remark 1

$$S_k = \frac{1}{a(\alpha^k - \beta^k)} \sum_{n=f(1)}^{\infty} [f^{-1}(n)] \left(\sum_{l=0}^{k-1} \frac{\beta^{n+l}}{R_{n+l}} - \sum_{l=0}^{k-1} \frac{\beta^{n+l+1}}{R_{n+l+1}} \right)$$

$$= \frac{1}{a(\alpha^k - \beta^k)} \sum_{h=1}^{\infty} \sum_{l=0}^{k-1} \frac{\beta^{f(h)+l}}{R_{f(h)+l}}. \tag{11}$$

Letting $k = 1$ and replacing n by $n + l - 1$ in (10), we have

$$\frac{(-A_2)^{n+l-1}}{R_{n+l-1} R_{n+l}} = \frac{1}{a(\alpha - \beta)} \left(\frac{\beta^{n+l-1}}{R_{n+l-1}} - \frac{\beta^{n+l}}{R_{n+l}} \right).$$

Hence by Lemma 2

$$T_l = \frac{(-A_2)^{1-l}}{a(\alpha - \beta)} \sum_{n=f(1)}^{\infty} [f^{-1}(n)] \left(\frac{\beta^{n+l-1}}{R_{n+l-1}} - \frac{\beta^{n+l}}{R_{n+l}} \right)$$

$$= \frac{(-A_2)^{1-l}}{a(\alpha - \beta)} \sum_{h=1}^{\infty} \frac{\beta^{f(h)+l-1}}{R_{f(h)+l-1}}. \tag{12}$$

Therefore we have

$$S_k = \frac{1}{F_k^*} \sum_{l=1}^{k} (-A_2)^{l-1} T_l.$$

Replacing n by $f(h)$ in (10), we have

$$\frac{(-A_2)^{f(h)}}{R_{f(h)} R_{f(h)+k}} = \frac{1}{a(\alpha^k - \beta^k)} \left(\frac{\beta^{f(h)}}{R_{f(h)}} - \frac{\beta^{f(h)+k}}{R_{f(h)+k}} \right). \tag{13}$$

Hence

$$U_k = \frac{1}{a(\alpha^k - \beta^k)} \sum_{h=1}^{\infty} \left(\frac{\beta^{f(h)}}{R_{f(h)}} - \frac{\beta^{f(h)+k}}{R_{f(h)+k}} \right)$$

and so

$$U_k = \frac{1}{F_k^*} (T_1 - (-A_2)^k T_{k+1}),$$

which completes the proof of the theorem.

Proof of Theorem 3. Replacing k by $2k$ in (10) and multiplying its both sides by $(-1)^n$, we have

$$\frac{A_2^n}{R_n R_{n+2k}} = \frac{1}{a(\alpha^{2k} - \beta^{2k})}\left(\frac{(-\beta)^n}{R_n} - \frac{(-\beta)^{n+2k}}{R_{n+2k}}\right)$$
$$= \frac{1}{a(\alpha^{2k} - \beta^{2k})}\left(\sum_{l=0}^{2k-1}\frac{(-\beta)^{n+l}}{R_{n+l}} - \sum_{l=0}^{2k-1}\frac{(-\beta)^{n+l+1}}{R_{n+l+1}}\right).$$

Hence, noting that $n|\beta^n/R_n| \to 0$ as $n \to \infty$, we have by Lemma 2 with Remark 1

$$S_{2k}^* = \frac{1}{a(\alpha^{2k} - \beta^{2k})}\sum_{n=f(1)}^{\infty}[f^{-1}(n)]\left(\sum_{l=0}^{2k-1}\frac{(-\beta)^{n+l}}{R_{n+l}} - \sum_{l=0}^{2k-1}\frac{(-\beta)^{n+l+1}}{R_{n+l+1}}\right)$$
$$= \frac{1}{a(\alpha^{2k} - \beta^{2k})}\sum_{h=1}^{\infty}\sum_{l=0}^{2k-1}\frac{(-\beta)^{f(h)+l}}{R_{f(h)+l}}$$
$$= \frac{1}{a(\alpha^{2k} - \beta^{2k})}\sum_{l=0}^{2k-1}(-1)^{l+f(1)}\sum_{h=1}^{\infty}\frac{\beta^{f(h)+l}}{R_{f(h)+l}},$$

since $f(h) \equiv f(1) \pmod 2$ for any $h \geq 1$. Therefore we have by (12)

$$S_{2k}^* = \frac{(-1)^{f(1)}}{F_{2k}^*}\sum_{l=1}^{2k}A_2^{l-1}T_l,$$

which completes the proof of the theorem.

Proof of Theorem 4. Let

$$U_{d,k} = \sum_{n=1}^{\infty}\frac{A_2^{d^n}}{R_{d^n}R_{d^n+k}}$$

and

$$T_{d,k} = \sum_{n=1}^{\infty}\frac{(-A_2)^n[\log_d n]}{R_{n+k-1}R_{n+k}} = \sum_{n=d}^{\infty}\frac{(-A_2)^n[\log_d n]}{R_{n+k-1}R_{n+k}} \quad (d \in \mathbf{N}\backslash\{1\}, k \in \mathbf{N}).$$

Letting $f(x) = d^x$ in Theorem 2 and noting that $(-1)^{d^n} = (-1)^d$ $(n \geq 1)$, we see that for any fixed d

$$(-1)^d U_{d,k} = \sum_{n=1}^{\infty}\frac{(-A_2)^{d^n}}{R_{d^n}R_{d^n+k}} = \frac{1}{F_k^*}(T_{d,1} - (-A_2)^k T_{d,k+1}) \quad (k \in \mathbf{N}).$$

Hence the numbers $\{U_{d,l} \mid 2 \le d \le m, 1 \le l \le k\}$ are expressed as linearly independent linear combinations over $\mathbf{Q}$ of the numbers $\{T_{d,l} \mid 2 \le d \le m, 1 \le l \le k+1\}$ for any $m \in \mathbf{N}\backslash\{1\}$ and for any $k \in \mathbf{N}$. Since the numbers $T_{d,k}$ $(d \in \mathbf{N}\backslash\{1\}, k \in \mathbf{N})$ are algebraically independent by Lemma 1, the numbers $U_{d,k}$ $(d \in \mathbf{N}\backslash\{1\}, k \in \mathbf{N})$ are algebraically independent, which completes the proof of the theorem.

4 Proof of Theorems 5 and 6

Remark 4. For $Q(z_1, z_2) \in \mathbf{C}(z_1, z_2)$ with $Q(0,0) = 0$, we define

$$f(x, z_1, z_2) = \sum_{n=1}^{\infty} x^n Q(z_1^{d^n}, z_2^{d^n}),$$

where x is a variable and d is an integer greater than 1. Letting $D = x\partial/\partial x$, we see that

$$f_l(x, z_1, z_2) := D^l f(x, z_1, z_2) = \sum_{n=1}^{\infty} n^l x^n Q(z_1^{d^n}, z_2^{d^n}) \quad (l \ge 0)$$

satisfy

$$f_0(x, z_1, z_2) = x f_0(x, z_1^d, z_2^d) + x Q(z_1^d, z_2^d),$$

$$f_1(x, z_1, z_2) = x f_1(x, z_1^d, z_2^d) + x f_0(x, z_1^d, z_2^d) + x Q(z_1^d, z_2^d),$$

$$\vdots$$

$$f_m(x, z_1, z_2) = \sum_{l=0}^{m} \binom{m}{l} x f_l(x, z_1^d, z_2^d) + x Q(z_1^d, z_2^d).$$

Hence for a complex number x, the functions $f_0(x, z_1, z_2), \dots, f_m(x, z_1, z_2)$ satisfy a system of functional equations of the form (9).

Proof of Theorem 5. Let $c = a^{-1}b$, $\gamma = \alpha^{-1}\beta$, and

$$f_{\xi l k}(z) = \sum_{n=1}^{\infty} n^l \xi^n \left(\frac{z^{d^n}}{1 + cz^{d^n}} - \frac{\gamma^k z^{d^n}}{1 + c\gamma^k z^{d^n}} \right) \quad (\xi \in \overline{\mathbf{Q}}^{\times}, l \ge 0, k \in \mathbf{N}).$$

Then

$$f_{\xi l k}(\gamma) = a^2(\alpha^k - \beta^k) \sum_{n=1}^{\infty} \frac{n^l \xi^n (-A_2)^{d^n}}{R_{d^n} R_{d^n + k}}. \tag{14}$$

Using (11) in the proof of Theorem 2 and letting $k = 1$, $f(x) = d^x$, and $g(z) = \sum_{n=1}^{\infty} z^{d^n}/(1 + cz^{d^n})$, we have

$$\sum_{n=1}^{\infty} \frac{(-A_2)^n [\log_d n]}{R_n R_{n+1}} = \frac{1}{a(\alpha - \beta)} \sum_{n=1}^{\infty} \frac{\beta^{d^n}}{R_{d^n}} = \frac{g(\gamma)}{a^2(\alpha - \beta)}. \tag{15}$$

Therefore it is enough by (14) and (15) to prove the algebraic independence of the values $f_{\xi lk}(\gamma)$ $(\xi \in \overline{\mathbf{Q}}^{\times}, l \geq 0, k \in \mathbf{N})$ and $g(\gamma)$. We see that each $f_{\xi 0k}(z)$ $(\xi \in \overline{\mathbf{Q}}^{\times}, k \in \mathbf{N})$ satisfies the functional equation

$$f_{\xi 0k}(z) = \xi f_{\xi 0k}(z^d) + \xi \left(\frac{z^d}{1 + cz^d} - \frac{\gamma^k z^d}{1 + c\gamma^k z^d} \right)$$

and $f_{\xi lk}(z)$ $(l \geq 0)$ satisfy a system of functional equations of the form (9) for every fixed ξ and k by Remark 4. We see also that $g(z)$ satisfies the functional equation

$$g(z) = g(z^d) + \frac{z^d}{1 + cz^d}.$$

Hence by Lemma 3 the values $f_{\xi lk}(\gamma)$ $(\xi \in \overline{\mathbf{Q}}^{\times}, l \geq 0, k \in \mathbf{N})$ and $g(\gamma)$ are algebraically independent if the functions $f_{\xi lk}(z)$ $(\xi \in \overline{\mathbf{Q}}^{\times}, l \geq 0, k \in \mathbf{N})$ and $g(z)$ are algebraically independent over $\mathbf{C}(z)$.

We assert that for every fixed $\xi \neq 1$ the functions $f_{\xi 0k}(z)$ $(k \in \mathbf{N})$ are linearly independent over $\mathbf{C}$ modulo $\mathbf{C}(z)$ and so are the functions $f_{10k}(z)$ $(k \in \mathbf{N})$ with $g(z)$, which implies by Lemma 4 that the functions $f_{\xi lk}(z)$ $(\xi \in \overline{\mathbf{Q}}^{\times}, l \geq 0, k \in \mathbf{N})$ and $g(z)$ are algebraically independent over $\mathbf{C}(z)$. Let

$$h_{\xi k}(z) = \sum_{n=1}^{\infty} \frac{\gamma^k \xi^n z^{d^n}}{1 + c\gamma^k z^{d^n}} \quad (\xi \in \overline{\mathbf{Q}}^{\times}, k \geq 0).$$

Then

$$f_{\xi 0k}(z) = h_{\xi 0}(z) - h_{\xi k}(z)$$

for every fixed $\xi \in \overline{\mathbf{Q}}^{\times}$ and $k \in \mathbf{N}$ and each $h_{\xi k}(z)$ $(\xi \in \overline{\mathbf{Q}}^{\times}, k \geq 0)$ satisfies the functional equation

$$h_{\xi k}(z) = \xi h_{\xi k}(z^d) + \frac{\xi \gamma^k z^d}{1 + c\gamma^k z^d}.$$

Suppose there exists a $\xi \neq 1$ such that $f_{\xi 01}(z), \ldots, f_{\xi 0k}(z)$ are linearly dependent over $\mathbf{C}$ modulo $\mathbf{C}(z)$ for some k. If $d = \xi = 2$ and $c = 1$, we see by Remark 3 that $h_{20}(z) = 2z^2/(1-z^2) \in \mathbf{C}(z)$ and so $h_{21}(z), \dots, h_{2k}(z)$ are linearly dependent over $\mathbf{C}$ modulo $\mathbf{C}(z)$; otherwise, so are $h_{\xi 0}(z), h_{\xi 1}(z), \dots, h_{\xi k}(z)$, which contradicts Lemma 5, since $H_{\xi k}(z) := \xi^{-1}\gamma^{-k}h_{\xi k}(z)$ satisfies the functional equation

$$H_{\xi k}(z) = \xi H_{\xi k}(z^d) + \frac{z^d}{1 + c\gamma^k z^d}.$$

Therefore, if $f_{\xi lk}(z)$ $(\xi \in \overline{\mathbf{Q}}^\times, l \geq 0, k \in \mathbf{N})$ and $g(z) = h_{10}(z)$ are algebraically dependent over $\mathbf{C}(z)$, then $h_{10}(z), f_{101}(z), \dots, f_{10k}(z)$ are linearly dependent over $\mathbf{C}$ modulo $\mathbf{C}(z)$ for some k, and hence so are $h_{10}(z), h_{11}(z), \dots, h_{1k}(z)$, which contradicts Lemma 5. Therefore the functions $f_{\xi lk}(z)$ $(\xi \in \overline{\mathbf{Q}}^\times, l \geq 0, k \in \mathbf{N})$ and $g(z)$ are algebraically independent over $\mathbf{C}(z)$ and so the values $f_{\xi lk}(\gamma)$ $(\xi \in \overline{\mathbf{Q}}^\times, l \geq 0, k \in \mathbf{N})$ and $g(\gamma)$ are algebraically independent, which completes the proof of the theorem.

Proof of Theorem 6. First we consider the case where α, β are multiplicatively dependent. Then there exist integers m, n, not both zero, with $\alpha^m\beta^n = 1$. Since α and β are field conjugates in the quadratic number field $\mathbf{Q}(\sqrt{\Delta})$, $\beta^m\alpha^n = 1$ must also hold. This implies

$$(\alpha\beta)^{m+n} = (\alpha/\beta)^{m-n} = 1.$$

Since $|\alpha/\beta| > 1$, we have $m = n \neq 0$, and hence $\alpha\beta$ must be a real root of unity, i.e., $-A_2 = \alpha\beta = \pm 1$. Therefore this case is proved by Theorem 5 since $(-A_2)^{d^n} = (-A_2)^d$ $(n \geq 1)$.

Secondly we consider the case where α, β are multiplicatively independent. Define

$$f_{\xi lk}(z_1, z_2) = \sum_{n=1}^{\infty} n^l \xi^n \left(\frac{z_1^{d^n}}{1 + cz_2^{d^n}} - \frac{\gamma^k z_1^{d^n}}{1 + c\gamma^k z_2^{d^n}} \right) \quad (\xi \in \overline{\mathbf{Q}}^\times, l \geq 0, k \in \mathbf{N}),$$

where $c = a^{-1}b$ and $\gamma = \alpha^{-1}\beta$. Then

$$f_{\xi lk}(\alpha^{-2}, \gamma) = a^2(\alpha^k - \beta^k) \sum_{n=1}^{\infty} \frac{n^l \xi^n}{R_{d^n} R_{d^n+k}}.$$

Using (11) in the proof of Theorem 2 and letting $k = 1$, $f(x) = d^x$, and $g(z_1, z_2) = \sum_{n=1}^{\infty} z_2^{d^n}/(1 + cz_2^{d^n})$, we have

$$\sum_{n=1}^{\infty} \frac{(-A_2)^n [\log_d n]}{R_n R_{n+1}} = \frac{1}{a(\alpha - \beta)} \sum_{n=1}^{\infty} \frac{\beta^{d^n}}{R_{d^n}} = \frac{g(\alpha^{-2}, \gamma)}{a^2(\alpha - \beta)}.$$

Therefore it is enough to prove the algebraic independence of the values $f_{\xi lk}(\alpha^{-2}, \gamma)$ $(\xi \in \overline{\mathbf{Q}}^{\times}, l \geq 0, k \in \mathbf{N})$ and $g(\alpha^{-2}, \gamma)$. We see that each $f_{\xi 0k}(z_1, z_2)$ $(\xi \in \overline{\mathbf{Q}}^{\times}, k \in \mathbf{N})$ satisfies the functional equation

$$f_{\xi 0k}(z_1, z_2) = \xi f_{\xi 0k}(z_1^d, z_2^d) + \xi\left(\frac{z_1^d}{1 + cz_2^d} - \frac{\gamma^k z_1^d}{1 + c\gamma^k z_2^d}\right)$$

and $f_{\xi lk}(z_1, z_2)$ $(l \geq 0)$ satisfy a system of functional equations of the form (9) for every fixed ξ and k by Remark 4. We see also that $g(z_1, z_2)$ satisfies the functional equation

$$g(z_1, z_2) = g(z_1^d, z_2^d) + \frac{z_2^d}{1 + cz_2^d}.$$

Hence, noting that α^{-2}, γ are multiplicatively independent, we see by Lemma 3 that the values $f_{\xi lk}(\alpha^{-2}, \gamma)$ $(\xi \in \overline{\mathbf{Q}}^{\times}, l \geq 0, k \in \mathbf{N})$ and $g(\alpha^{-2}, \gamma)$ are algebraically independent if the functions $f_{\xi lk}(z_1, z_2)$ $(\xi \in \overline{\mathbf{Q}}^{\times}, l \geq 0, k \in \mathbf{N})$ and $g(z_1, z_2)$ are algebraically independent over $\mathbf{C}(z_1, z_2)$. We assert that for every fixed $\xi \neq 1$ the functions $f_{\xi 0k}(z_1, z_2)$ $(k \in \mathbf{N})$ are linearly independent over $\mathbf{C}$ modulo $\mathbf{C}(z_1, z_2)$ and so are the functions $f_{10k}(z_1, z_2)$ $(k \in \mathbf{N})$ with $g(z_1, z_2)$, which implies by Lemma 4 that the functions $f_{\xi lk}(z_1, z_2)$ $(\xi \in \overline{\mathbf{Q}}^{\times}, l \geq 0, k \in \mathbf{N})$ and $g(z_1, z_2)$ are algebraically independent over $\mathbf{C}(z_1, z_2)$.

Suppose there exists a $\xi \neq 1$ such that $f_{\xi 01}(z_1, z_2), \dots, f_{\xi 0k}(z_1, z_2)$ are linearly dependent over $\mathbf{C}$ modulo $\mathbf{C}(z_1, z_2)$ for some k. Thus there are complex numbers $c_1, \dots, c_k$, not all zero, such that

$$c_1 f_{\xi 01}(z_1, z_2) + \cdots + c_k f_{\xi 0k}(z_1, z_2) \in \mathbf{C}(z_1, z_2).$$

Since $f_{\xi 01}(z_1, z_2), \dots, f_{\xi 0k}(z_1, z_2) \in \mathbf{C}[[z_1, z_2]]$, by Lemma 6 there exist $A(z_1, z_2)$, $B(z_1, z_2) \in \mathbf{C}[z_1, z_2]$ such that

$$c_1 f_{\xi 01}(z_1, z_2) + \cdots + c_k f_{\xi 0k}(z_1, z_2) = \frac{A(z_1, z_2)}{B(z_1, z_2)}, \quad B(0, 0) \neq 0.$$

Letting $z_1 = z_2 = z$, we have

$$c_1 f_{\xi 01}(z, z) + \cdots + c_k f_{\xi 0k}(z, z) \in \mathbf{C}(z),$$

which contradicts Lemma 5 by the same way as in the proof of Theorem 5. Therefore, if $f_{\xi lk}(z_1, z_2)$ $(\xi \in \overline{\mathbf{Q}}^{\times}, l \geq 0, k \in \mathbf{N})$ and $g(z_1, z_2)$ are algebraically dependent over $\mathbf{C}(z_1, z_2)$, then $g(z_1, z_2), f_{101}(z_1, z_2), \dots, f_{10k}(z_1, z_2)$ are linearly dependent over $\mathbf{C}$ modulo $\mathbf{C}(z_1, z_2)$ for some k. By the same way as above $g(z, z)$, $f_{101}(z, z), \dots, f_{10k}(z, z)$ are linearly dependent over $\mathbf{C}$ modulo $\mathbf{C}(z)$, which again contradicts Lemma 5. Therefore the functions $f_{\xi lk}(z_1, z_2)$ $(\xi \in \overline{\mathbf{Q}}^{\times}, l \geq 0, k \in \mathbf{N})$ and $g(z_1, z_2)$ are algebraically independent over $\mathbf{C}(z_1, z_2)$ and so the values $f_{\xi lk}(\alpha^{-2}, \gamma)$ $(\xi \in \overline{\mathbf{Q}}^{\times}, l \geq 0, k \in \mathbf{N})$ and $g(\alpha^{-2}, \gamma)$ are algebraically independent, which completes the proof of the theorem.

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References

[1] André-Jeannin, R., Irrationalité de la somme des inverses de certaines suites récurrentes, C. R. Acad. Sci. Paris, Sér. I Math. **308** (1989), 539–541.
[2] Brousseau, Bro. A., Summation of infinite Fibonacci series, Fibonacci Quart. **7** (1969), 143–168.
[3] Lucas, E., Théorie des fonctions numériques simplement périodiques, Amer. J. Math. **1** (1878), 184–240.
[4] Masser, D. W., A vanishing theorem for power series, Invent. Math. **67** (1982), 275–296.
[5] Nishioka, K., Mahler functions and transcendence, Lecture Notes in Mathematics No. **1631**, Springer, 1996.
[6] Nishioka, K., Algebraic independence of reciprocal sums of binary recurrences, Monatsh. Math. **123** (1997), 135–148.
[7] Nishioka, K., Tanaka, T., and Toshimitsu, T., Algebraic independence of sums of reciprocals of the Fibonacci numbers, Math. Nachr. **202** (1999), 97–108.
[8] Rabinowitz, S., Algorithmic summation of reciprocals of products of Fibonacci numbers, Fibonacci Quart. **37** (1999), 122–127.
[9] Tanaka, T., Algebraic independence of Fibonacci reciprocal sums associated with Newton's method, Tsukuba J. Math. **27** (2003), 375–387.

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