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CONFORMAL FLATNESS OF CIRCLE BUNDLE METRIC

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§.1. Introduction and Main Theorem

The aim of this paper is to investigate the conformal flatness of bundle metric on a circle bundle.

A riemannian n-manifold is conformally flat if it is locally conformal to the euclidean space $\boldsymbol{R}^n$ ([1]). Riemann surfaces and space forms are conformally flat. It is further known ([5]) that a riemannian product manifold $M \times N$ is conformally flat if and only if either (1) M is a space form and N is one dimensional, or (2) M and N are space forms of same dimension $n \geq 2$ and they have opposite curvatures.

So (1) means that a trivial circle bundle $M \times S^1$ with the product metric is conformally flat if and only if the base space M is of constant curvature. From this fact we consider the conformal flatness of a bundle metric $g = \gamma^2 + \pi^* h$ on a non-trivial circle bundle $\pi : P \rightarrow M$ where (M, h) is an oriented riemannian manifold and γ is a non-flat Yang-Mills connection.

A typical example is the Hopf bundle $\pi : S^{2n+1} \rightarrow \boldsymbol{CP}^n$. The total space S^{2n+1} is equipped with the standard metric g which is conformally flat and it is easily shown that the metric g can be written as a bundle metric $g = \gamma^2 + \pi^* h$ with respect to the Fubini-Study metric h and a canonical connection γ whose curvature form is proportional to the Kähler form of the Fubini-Study metric.

In this paper we restrict ourselves to a circle bundle $\pi : P \rightarrow M$ such that $\dim M = 4$ and a connection γ has self-dual curvature form.

THEOREM 1.1. *Let $\pi : P \rightarrow M$ be a circle bundle over a connected oriented riemannian 4-manifold (M, h), and γ a non-flat connection on P. Define the bundle metric $g = \gamma^2 + \pi^* h$ on P. If the curvature form Γ of γ is self-dual and g is conformally flat, then*

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(1) $(M,(1/24)\sigma h)$ *is locally isometric and biholomorphic to a domain* D *of* CP^2 *with the Fubini-Study metric, and*

(2) (P,g) *is of positive constant curvature* $(1/24)\sigma$, *where* σ *is the scalar curvature of* (M,h).

This theorem says that if Γ is self-dual and (P,g) is conformally flat, then $\pi: P \to M$ is a part of the Hopf bundle $\pi: S^5 \to CP^2$. In particular, if both M and P are complete and simply connected, then this circle bundle is the Hopf bundle and the bundle metric g is the standard metric on S^5.

§.2. Weyl Conformal Curvature of (P,g)

When $n \geq 4$, the conformal flatness of M^n is equivalent to the vanishing of the Weyl conformal curvature W.

Let $\pi: P \to M$ be a circle bundle over an oriented riemannian 4-manifold (M,h), and γ a non-flat Yang-Mills connection on P, that is,the curvature form Γ of γ satisfies $*^{-1}d*\Gamma = 0$.

We define the bundle metric g on P by $g = \gamma^2 + \pi^* h$. Let $\{e_1,\dots,e_4\}$ be a local orthonormal frame field of (M,h) which is compatible with the orientation of M. Denote by $\{\theta^1,\dots,\theta^4\}$ the dual coframe field of $\{e_1,\dots,e_4\}$. If we put $\theta^0 = \gamma$, then $\{\theta^0,\pi^*\theta^1,\dots,\pi^*\theta^4\}$ is a local orthonormal coframe field of (P,g).

From now on, we determine the range of the Roman indices i, j, k, l, s, t between 1 and 4, the Greek indices α, β, γ, δ between 0 and 4. In addition, we write the pull back π^*T of a tensor T simply by the same letter T. In this manner, $\{\theta^0,\pi^*\theta^1,\dots,\pi^*\theta^4\}$ is represented as $\{\theta^0,\theta^1,\dots,\theta^4\}$.

Let ∇ be the Levi-Civita connection of (M,h). We write the 2-form Γ as

$$\Gamma = \frac{1}{2}\sum_{s,t}\Gamma_{st}\theta^s\wedge\theta^t, \quad \Gamma_{ts} = -\Gamma_{st}. \tag{1}$$

The covariant derivative $\nabla_i\Gamma_{jk}$ of Γ with respect to ∇ is defined by

$$\sum_s \nabla_s\Gamma_{ij}\theta^s = d\Gamma_{ij} - \sum_s \omega_j^s\Gamma_{is} - \sum_s \omega_i^s\Gamma_{sj}, \tag{2}$$

where ω_j^i is the connection form of ∇. Since γ is a Yang-Mills connection and $\Gamma = d\gamma$, the Γ satisfies

$$\sum_s \nabla_s\Gamma_{si} = 0. \tag{3}$$

We denote the trace-free Ricci tensor T of (M,h) by

$$T_{ij} = R_{ij} - \frac{\sigma}{4}\delta_{ij}, \tag{4}$$

where R_{ij} and σ are respectively the Ricci tensor and the scalar curvature of (M,h).

Let $\tilde{\omega}^\alpha_\beta$ be the connection form of the Levi-Civita connection of (P,g). It follows from [3] that $\tilde{\omega}^\alpha_\beta$ is

$$\tilde{\omega}^0_0 = 0, \tag{5}$$

$$\tilde{\omega}^0_i = \frac{1}{2}\sum_s \Gamma_{is}\theta^s \tag{6}$$

$$\tilde{\omega}^i_j = \omega^i_j - \frac{1}{2}\Gamma_{ij}\theta^0. \tag{7}$$

Hence, the curvature form $\tilde{\Omega}^\alpha_\beta$ of $\tilde{\omega}^\alpha_\beta$ is

$$\tilde{\Omega}^0_0 = 0, \tag{8}$$

$$\tilde{\Omega}^0_i = \frac{1}{4}\sum_{s,t}\Gamma_{si}\Gamma_{st}\theta^0 \wedge \theta^t + \frac{1}{2}\sum_{s,t}\nabla_s\Gamma_{it}\theta^s \wedge \theta^t, \tag{9}$$

$$\tilde{\Omega}^i_j = \Omega^i_j - \frac{1}{4}\sum_{s,t}(\Gamma_{ij}\Gamma_{st} + \Gamma_{is}\Gamma_{jt})\theta^s \wedge \theta^t + \frac{1}{2}\sum_s \nabla_s\Gamma_{ij}\theta^0 \wedge \theta^s. \tag{10}$$

Applying the Bianchi identity for Γ, we have the riemannian curvature $K_{\alpha\beta\gamma\delta}$ of (P,g) as

$$K_{ijkl} = R_{ijkl} - \frac{1}{4}(2\Gamma_{ij}\Gamma_{kl} + \Gamma_{ik}\Gamma_{jl} - \Gamma_{il}\Gamma_{jk}), \tag{11}$$

$$K_{0ijk} = \frac{1}{2}\nabla_i\Gamma_{jk}, \tag{12}$$

$$K_{0i0j} = \frac{1}{4}\sum_s \Gamma_{si}\Gamma_{sj}, \tag{13}$$

where R_{ijkl} is the riemannian curvature of (M,h), and $|\Gamma|$ is the norm of Γ with respect to h:

$$|\Gamma|^2 = \sum_{s<t}\Gamma^2_{st}. \tag{14}$$

The Ricci tensor $K_{\alpha\beta}$ of (P,g) is

$$K_{ij} = R_{ij} - \frac{1}{2}\sum_s \Gamma_{si}\Gamma_{sj}, \tag{15}$$

$$K_{0i} = 0, \tag{16}$$

$$K_{00} = \frac{1}{2}|\Gamma|^2, \tag{17}$$

where R_{ij} is the Ricci tensor of (M, h). The scalar curvature κ of (P, g) is

$$\kappa = \sigma - \frac{1}{2}|\Gamma|^2, \tag{18}$$

where σ is the scalar curvature of (M, h). Let $\mathscr{W}_{\alpha\beta\gamma\delta}$ and W_{ijkl} be the Weyl conformal curvatures of (P, g) and of (M, h) respectively. By (3), we have the following:

PROPOSITION 2.1. *If γ is a Yang-Mills connection, then the Weyl conformal curvature $\mathscr{W}_{\alpha\beta\gamma\delta}$ of (P, g) is*

$$\begin{aligned}\mathscr{W}_{ijkl} = {} & W_{ijkl} - \frac{1}{4}(2\Gamma_{ij}\Gamma_{kl} + \Gamma_{ik}\Gamma_{jl} - \Gamma_{il}\Gamma_{jk}) \\ & - \frac{1}{8}|\Gamma|^2(\delta_{jk}\delta_{il} - \delta_{jl}\delta_{ik}) \\ & - \frac{1}{6}(T_{jk}\delta_{il} - T_{jl}\delta_{ik} - T_{ik}\delta_{jl} + T_{il}\delta_{jk}) \\ & - \frac{1}{6}\left\{\left(\sum_s \Gamma_{sj}\Gamma_{sk} - \frac{|\Gamma|^2}{2}\delta_{jk}\right)\delta_{il} - \left(\sum_s \Gamma_{sj}\Gamma_{sl} - \frac{|\Gamma|^2}{2}\delta_{jl}\right)\delta_{ik}\right. \\ & \left. - \left(\sum_s \Gamma_{si}\Gamma_{sk} - \frac{|\Gamma|^2}{2}\delta_{ik}\right)\delta_{jl} + \left(\sum_s \Gamma_{si}\Gamma_{sl} - \frac{|\Gamma|^2}{2}\delta_{il}\right)\delta_{jk}\right\},\end{aligned} \tag{19}$$

$$\mathscr{W}_{0ijk} = \frac{1}{2}\nabla_i\Gamma_{jk}, \tag{20}$$

$$\mathscr{W}_{0i0j} = -\frac{1}{3}T_{ij} + \frac{5}{12}\left(\sum_s \Gamma_{si}\Gamma_{sj} - \frac{|\Gamma|^2}{2}\delta_{ij}\right). \tag{21}$$

§.3. Complex Structure and Curvature of (M, h)

We use the same notation as that in §.2. Suppose that (P, g) is conformally flat. It then follows from (21) that (M, h) is Einstein if and only if Γ satisfies the following equation:

$$\sum_s \Gamma_{si}\Gamma_{sj} - \frac{|\Gamma|^2}{2}\delta_{ij} = 0. \tag{22}$$

In general, a 2-form ω on M satisfies $\sum \omega_{si}\omega_{sj} - (|\omega|^2/2) \cdot \delta_{ij} = 0$ if and only if ω is either self-dual or anti-self-dual. Therefore, if Γ is self-dual, then (M, h) is Einstein. We can define an almost complex structure J on M by

$$\Gamma(X, Y) = \frac{|\Gamma|}{\sqrt{2}} h(JX, Y), \quad X, Y \in T_pM, p \in M. \tag{23}$$

From (20), both Γ and h are parallel with respect to ∇, and so is J. Then, (M, h, J) is a Kähler manifold.

PROPOSITION 3.1. *Let γ be a non-flat connection on P with self-dual curvature Γ. If (P, g) is conformally flat, then (M, h, J) is self-dual, Einstein and Kähler.*

PROOF. It suffices to show that (M, h) is self-dual. By Proposition 2.1, the following equation holds:

$$W_{ijkl} = \frac{1}{4}(2\Gamma_{ij}\Gamma_{kl} + \Gamma_{ik}\Gamma_{jl} - \Gamma_{il}\Gamma_{jk}) + \frac{1}{8}|\Gamma|^2(\delta_{jk}\delta_{il} - \delta_{jl}\delta_{ik}). \tag{24}$$

In order to calculate the anti-self-dual part W^- of the Weyl conformal curvature of (M, h), we take the following basis on $\wedge^2 T^*M$:

$$\theta^1 \wedge \theta^2 \pm \theta^3 \wedge \theta^4, \quad \theta^1 \wedge \theta^3 \pm \theta^4 \wedge \theta^2, \quad \theta^1 \wedge \theta^4 \pm \theta^2 \wedge \theta^3. \tag{25}$$

Then W^- is expressed as

$$W^- = \begin{pmatrix} W_{1212} - W_{1234} & W_{1213} - W_{1242} & W_{1214} - W_{1223} \\ W_{1312} - W_{1334} & W_{1313} - W_{1342} & W_{1314} - W_{1323} \\ W_{1412} - W_{1434} & W_{1413} - W_{1442} & W_{1414} - W_{1423} \end{pmatrix}. \tag{26}$$

From (24) and the self-duality of Γ, we have

$$\begin{aligned} W_{1212} - W_{1234} &= \frac{3}{4}\Gamma_{12}^2 - \frac{1}{8}|\Gamma|^2 - \frac{1}{4}(2\Gamma_{12}\Gamma_{34} + \Gamma_{13}\Gamma_{24} - \Gamma_{14}\Gamma_{23}) \\ &= \frac{3}{4}\Gamma_{12}^2 - \frac{1}{8}|\Gamma|^2 - \frac{3}{4}\Gamma_{12}^2 + \frac{1}{4}(\Gamma_{12}^2 + \Gamma_{13}^2 + \Gamma_{14}^2) \\ &= 0, \end{aligned}$$

$$W_{1213} - W_{1242} = \frac{3}{4}\Gamma_{12}\Gamma_{13} - \frac{3}{4}\Gamma_{12}\Gamma_{42}$$
$$= 0,$$

and so on. Consequently, (M,h) is self-dual. Q.E.D.

§.4. Proof of Main Theorem

Let γ be a non-flat connection on P with self-dual curvature form Γ. Assume that (P,g) is conformally flat. By Proposition 3.1, the J defined by (23) is a complex structure on M, and the base space (M,h,J) is self-dual, Einstein and Kähler.

First, we assert that (M,h,J) is of constant holomorphic sectional curvature. Take arbitrary unit vectors e_1, $e_3 \in T_pM$, $p \in M$ such that e_3 is perpendicular to e_1 and Je_1. Put $e_2 = Je_1$ and $e_4 = Je_3$. From (23), Γ_{12} and Γ_{34} are $|\Gamma|/\sqrt{2}$, and the others are zero. From (24), we have

$$W_{1212} = \frac{3}{4}\Gamma_{12}^2 - \frac{1}{8}|\Gamma|^2 = \frac{1}{4}|\Gamma|^2, \tag{27}$$

$$W_{1313} = -\frac{1}{8}|\Gamma|^2. \tag{28}$$

On the other hand, by the definition of the Weyl conformal curvature, we have

$$W_{1212} = R_{1212} - \frac{\sigma}{12}, \tag{29}$$

$$W_{1313} = R_{1313} - \frac{\sigma}{12}, \tag{30}$$

because (M,h) is Einstein. From (27), (28), (29) and (30), we have

$$R_{1212} = \frac{1}{4}|\Gamma|^2 + \frac{\sigma}{12}, \tag{31}$$

$$R_{1313} = -\frac{1}{8}|\Gamma|^2 + \frac{\sigma}{12}. \tag{32}$$

Since Γ is parallel and (M,h) is Einstein, the right hand side of (31) is constant. Hence, (M,h,J) is of constant holomorphic sectional curvature.

Moreover, the holomorphic sectional curvature of (M,h) is positive. Indeed, since the ratio of the holomorphic sectional curvature to the anti-holomorphic sectional curvature is four ([4]), we have

$$\sigma = 3|\Gamma|^2 > 0 \quad \text{if } \gamma \text{ is non-flat}, \tag{33}$$

by (31) and (32). It then follows that the holomorphic sectional curvature of (M, h) is positive.

The above implies that the base space (M, h, J) is locally biholomorphic to some domain D of CP^2. It is easy to see that $(M, (1/24)\sigma h)$ is isometric to D with the Fubini-Study metric h_{FS}. Note that the sectional curvature $K_{\alpha\beta\alpha\beta}$ of (P, g) is

$$K_{1212} = R_{1212} - \frac{3}{8}|\Gamma|^2 = \frac{\sigma}{24}, \tag{34}$$

$$K_{1313} = R_{1313} = \frac{\sigma}{24}, \tag{35}$$

$$K_{0101} = \frac{|\Gamma|^2}{8} = \frac{\sigma}{24}, \tag{36}$$

and so on. Therefore, we conclude that (P, g) is a space of positive constant curvature $(1/24)\sigma$.

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