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ONE CLASS OF REPRESENTATIONS OVER TRIVIAL EXTENSIONS OF ITERATED TILTED ALGEBRAS

By

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Abstract. Let $T(A)=A\ltimes D(A)$ be the trivial extension of iterated tilted algebra A of type $\vec{\Delta}$. In this paper, we study the indecomposable $T(A)$-modules belonging to the components of form $Z\vec{\Delta}$, which are called the modules on platform. Our main results are as follows: (1) The number of the modules on platform which have the same dimension vector is equal to or less than the number of simple A-modules. (2) The module on platform is uniquely determined by its top and socle. (3) The module on platform is uniquely determined by its Loewy factor and by its socle factor.

§1. Introduction.

Throughout this paper, we denoted by k an algebraically closed field, by A a basic, connected and finite-dimensional k-algebra, and by A-*mod* (*mod*-A, respectively) the category of all finitely generated left (right, respectively) modules over A. We write $D=Hom_k(, k)$ for the usual dual functor between A-*mod* and *mod*-A, then $D(A)$ has a cononical $A-A$-bimodule structure. The trivial extension $T(A)=A\ltimes D(A)$ of A is defind as the k-algebra whose additive structure is that of $A\oplus D(A)$ and whose multiplication is given by $(a, \varphi)\cdot(b, \phi)=(ab, a\phi+\varphi b)$ for $a, b\in A$ and $\varphi, \phi\in D(A)$. Note that $T(A)$ is a self-injective algebra, see [1].

Tilted and iterated tilted algebra are important in representation theory of algebra and are extesively studied. It is well known that the AR quiver of a tilted algebra must have a connecting component as well as preprojective and preinjective ones, see [2] and [3]. All of these components consist of directing modules, which enjoy very pleasant properties, for example, being uniquely determined by their composition factors and by their tops and socles.

On the other hand, as a special class of self-injective algebras, the trivial

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extensions of iterated tilted algebra of type $\vec{\Delta}$ also enjoy some good properties, such as their stable module categories must have components of form $\boldsymbol{Z}\vec{\Delta}$([4]), but unfortunately, no indecompoable $T(A)$-module is directing; the indecomposable $T(A)$-module is directing; the indecomposable $T(A)$-modules belonging to the components of form $\boldsymbol{Z}\vec{\Delta}$ are no longer determined by their composition factors. However, our results show that these modules still have some interesting properties.

For stating our results, we recall some notations. Let A be an iterated tilted algebra of type $\vec{\Delta}$, the repetitive algebra $\hat{A}$ has the additive structure of $(\bigoplus_{i\in\boldsymbol{Z}} A_i)\oplus(\bigoplus_{i\in\boldsymbol{Z}} Q_i)$ with $A_i=A$ and $Q_i=D(A)$ for $i\in\boldsymbol{Z}$, whose multiplication is defined as follows

$$(a_i,\ \varphi_i)_i\cdot(b_i,\ \phi_i)_i=(a_ib_i,\ a_{i+1}\phi_i+\varphi_ib_i)_i\,,$$

where $(a_i,\ \varphi_i)_i,\ (b_i,\ \phi_i)_i,\ \in\hat{A}$ with $a_i,\ b_i\in A$, and $\varphi_i,\ \phi_i\in D(A)$ for $i\in\boldsymbol{Z}$. Note that $\hat{A}$ is an infinite-dimensional and locally bounded self-injective algebra. Defining Nakayama automorphism $v\colon \hat{A}\to\hat{A}$ as in [5], we know that $T(A)=\hat{A}/v$ and that the functor v induce Galois covering functor $\pi\colon \hat{A}\to T(A)$ and an automorphim of $\hat{A}$-*mod*. By Happel's result in [4] we know that $\hat{A}$-$\underline{mod}\simeq D^b(A)$ and $\Gamma_s(T(A))\simeq\Gamma(D^b(k\vec{\Delta}))/\langle T^2\tau\rangle$, where $\hat{A}$-$\underline{mod}$ is the stable module category of $\hat{A}$-*mod*; $D^b(A)$ is the derived category of A and $T^2\tau$ is just the automorphism of $\hat{A}$ induced by Nakayama functor v. In the following we still denote by π the covering functor from $\hat{A}$-*mod* to $T(A)$-*mod* induced by $\pi\colon \hat{A}\to T(A)$.

DEFINITION. Let A be an iterated tilted algebra of type $\vec{\Delta}$, the indecompoable $T(A)$-module M is said to be a module on platform, if there is $X\in\hat{A}$-*mod* such that $\pi(X)=M$ and that X as an object of $\hat{A}$-$\underline{mod}$ belongs to a component of form $\boldsymbol{Z}\vec{\Delta}$ of $\Gamma(\hat{A}\text{-}\underline{mod})\simeq\Gamma(D^b(k\vec{\Delta}))$.

REMARK. (1) If $\vec{\Delta}$ is of Dynkin type, then any indecompoable $T(A)$-module is on platform.

(2) The module on platform is non-projective.

For a finite dimensional k-algebra Λ, we denote by Q the Gabriel quiver of Λ ([6]), by $P(x)(I(x),\ S(x)$ respectively) the indecomposable projective (injective, simple, respectively) module corresponding to the vertex $x\in Q$, i.e.. $top\,P(x)\cong soc\,I(x)$. For $M\in\Lambda$-*mod*, we define its dimension vector as

$$\begin{aligned}\underline{dim}\,M&=(dim_k Hom_A(P(x),\ M))_{x\in Q_0}\\&=(dim_k Hom_A(M,\ I(x)))_{x\in Q_0}\end{aligned}$$

is just the number of composion factors of form $S(a)$ in any fixed composition series. The Loewy factor of M is defined as the matrix

$$\underline{L dim} M=\begin{pmatrix} \underline{dim} M/radM \\ \underline{dim} radM/rad^2M \\ \vdots \\ \underline{dim} rad^i M/rad^{i+1} M \\ \vdots \end{pmatrix}$$

and the socle factor of M is the matrix

$$\underline{S dim} M=\begin{pmatrix} \vdots \\ \underline{dim} soc^{i+1} M/soc^i M \\ \vdots \\ \underline{dim} soc^2 M/socM \\ \underline{dim} socM \end{pmatrix}$$

Now we can state our main results as follows:

THEOREM 1. *Let $T(A)$ be the trivial extension of an iterated tilted algebra A of type $\vec{\Delta}$, X a $T(A)$-module on platform, then the number of isoclass of the $T(A)$-modules on platform which have the same dimension vector with X is at most n, where n is the number of vertices of $\vec{\Delta}$.*

THEOREM 2. *If $T(A)$ is as above, X, Y are two $T(A)$-modules on platform, then $X\simeq Y$ if and only if $topX\simeq topY$ and $socX\simeq socY$.*

THEOREM 3. *If the assumptions are as in Theorem 2, then the following are equivalent*

(1) $X\simeq Y$

(2) $\underline{L dim} X=\underline{L dim} Y$

(3) $\underline{S dim} X=\underline{S dim} Y$

§2. Proof of Theorem 1.

LEMMA 1 ([7] p. 15) *Let A be locally bounded self-injective algebra.*

(1) *If M is indecomposable non-projective, $f: M\to N$ is epic, then $\underline{f}$ is nonzero in A-$\underline{mod}$.*

(2) *If N is indecomposable non-projective, $g: M\to N$ is mono, then $\underline{g}$ is nonzero in A-$\underline{mod}$.*

LEMMA 2 ([7] p. 15). *Assume that A is as above, M, N are indecomposable non-projective with $Hom(M, N)\neq 0$, then there exists a A-module L such that*

$\underline{Hom}(M, L) \neq 0 \neq \underline{Hom}(L, N)$.

LEMMA 3. *Let A be as above, then M is directing as A-module iff M is directing as object in A-$\underline{mod}$.*

PROOF. Suppose that X is directing in A-$\underline{mod}$. If X is not directing as A-module, then we get a chain of nonzero nonisomorphisms $X \to X_1 \to X_2 \cdots \to X_r = X$ with $r \geqq 1$, If no X_i is projective, then X is not directing in A-$\underline{mod}$ by Lemma 2, so we may assume that $X_i = P(a)$ is projective, considering the AR sequence

$$0 \longrightarrow radP(a) \longrightarrow (P(a) \oplus Y \longrightarrow P(a)/socP(a) \longrightarrow 0$$

then we have

$$X \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_{i-1} \longrightarrow radP(a) \longrightarrow Y \longrightarrow$$
$$P(a)/socP(a) \longrightarrow X_{i+1} \longrightarrow \cdots \longrightarrow X,$$

which doesn't contain the projective module X_i. Repeating this process if necessary, we finally get a chain which doesn't contain any projective module, a contradiction by Lemma 2.

PROOF OF THEOREM 1. Assume that $\pi(M) = X$ with M lying on the component of form $\boldsymbol{Z}\vec{\Delta}$ of $\hat{A}$-$\underline{mod}$. Choose a complete slice $\boldsymbol{S}$ of this component such that $M \in \boldsymbol{S}$, from the structure of $D^b(k\vec{\Delta})$ we know that $\boldsymbol{S}$ is path-closed in $\hat{A}$-$\underline{mod}$. Let B be the support algebra of ${}_{\hat{A}}S$ in $\hat{A}$, where $\boldsymbol{S} = add_{\hat{A}}S$.

(1) First we claim that ${}_BM$ is directing. Since B-mod is full subcategory of $\hat{A}$-*mod*, it is enough to prove that M is directing in $\hat{A}$-$\underline{mod}$. In the following we always identify $\hat{A}$-$\underline{mod}$ with $D^b(k\vec{\Delta})$. If there is a chain of nonzero nonisomorphims in $\hat{A}$-$\underline{mod}$ $M = X_0 \to X_1 \to \cdots \to X_r = M$ with $r \geqq 1$, then by the structcre of $D^b(k\vec{\Delta})$ we have a chain in $D^b(k\vec{\Delta})$

$$T^{i_0}Y_0 \longrightarrow T^{i_1}Y_1 \longrightarrow \cdots \longrightarrow T^{i_r}Y_r = T^{i_0}Y_0$$

with $Y_i \in k\vec{\Delta}$-*mod* for $0 \leqq i \leqq r$, so $i_0 \leqq i_l \cdots \leqq i_r = i_0$, therefore we have a chain in $k\vec{\Delta}$-*mod* $Y_0 \to Y_1 \to \cdots \to Y_r = Y_0$ which implies that Y_0 is not directing. But since $M = T^{i_0}Y_0 \in \boldsymbol{S}$, Y_0 must be preprojective or preinjective $k\vec{\Delta}$-module, which is a contradiction with above.

(2) Denoting by $Q_{\hat{A}}$ and Q_B the Gabriel quiver of $\hat{A}$ and B respectively, we wish to prove that Q_B is path-closed in $Q_{\hat{A}}$. For this let $x \to \cdots \to y \to \cdots \to z$ be a path in Q_B with $x, z \in Q_B$, so we have

$$P_{\hat{A}}(x) \longrightarrow \cdots \longrightarrow P_{\hat{A}}(y) \longrightarrow \cdots \longrightarrow P_{\hat{A}}(z)$$

and

$$I_{\hat{A}}(x) \longrightarrow \cdots \longrightarrow I_{\hat{A}}(y) \longrightarrow \cdots \longrightarrow I_{\hat{A}}(z).$$

Considering the chain

$$P_{\hat{A}}(y)/socP_{\hat{A}}(y) \xrightarrow{f} S(y) \xrightarrow{g} radI_{\hat{A}}(y),$$

where $topP_{\hat{A}}(y) \simeq S(y) \simeq socI_{\hat{A}}(y)$. It follows from Lemma 1 that $\underline{f} \neq 0 \neq \underline{g}$ in $D^b(k\vec{\Delta})$. Since $x, z \in Q_B$, we have $P_{\hat{A}}(y)/socP_{\hat{A}}(y) \prec \boldsymbol{S} \prec radI_{\hat{A}}(y)$.

By the structure of $D^b(k\vec{\Delta})$ we know that $S(y) \prec S$ or $S \prec S(y)$. Assume that $S(y) \leqq S$ and that S correspond to the all indecomposable projective $k\vec{\Delta}$-modules. Let $radI_{\hat{A}}(y) = T^iY'$ with $Y' \in k\vec{\Delta}\text{-}mod$, since $\boldsymbol{S} \prec I_{\hat{A}}(y)$, we have $i \geqq 0$. If $i>0$, then from the isomorphism

$$Hom_{D^b(k\vec{\Delta})}(S(y), radI_{\hat{A}}(y)) \cong DHom_{D^b(k\vec{\Delta})}(T^{-1}\tau^{-1}radI_{\hat{A}}(y), S(y))$$

we get

$$\boldsymbol{S} \prec T^{-1}radI_{\hat{A}}(y) = T^{i-1}Y' \prec T^{-1}\tau^{-1}radI_{\hat{A}}(y) \prec S(y) \leqq \boldsymbol{S},$$

hence $T^{-1}radI_{\hat{A}}(y), \tau^{-1}T^{-1}radI_{\hat{A}}(y) \in \boldsymbol{S}$, which is a contradiction with $\boldsymbol{S}$ being a complete-slice of the component. So $i=0$ and we have a chain in $\hat{A}\text{-}\underline{mod}$ $S \rightarrow radI_{\hat{A}}(y)$ which implies $Hom_{\hat{A}}(S, I_{\hat{A}}(y)) \neq 0$, i.e., $y \in Q_B$.

If $S \prec S(y)$, we may use $\underline{f} \neq 0$ and get dually the chain $P_{\hat{A}}(y)/socP_{\hat{A}}(y) \rightarrow S$.

(3) We now prove that Q_B is a complete v-slice of $Q_{\hat{A}}$ in the sense of [5]. For this it is enough to prove that for any $a \in Q_{\hat{A}}$ the v-orbit of a contains only one vertex in O_B. If it is not the case, we assume that $a, v^m a \in Q_B$, i.e., there are $S_1, S_2 \in \boldsymbol{S}$ such that

$$Hom_{\hat{A}}(P_{\hat{A}}(a)/socP_{\hat{A}}(a), S_1) \neq 0 \neq Hom_{\hat{A}}(P_{\hat{A}}(v^m a)/socP_{\hat{A}}(v^m a), S_2)$$

then $P_{\hat{A}}(a)/socP_{\hat{A}}(a) = T^iX$ with $i=0$ or -1. On the other hand,

$$P_{\hat{A}}(v^m a)/socP_{\hat{A}}(v^m a) = v^m(P_{\hat{A}}(a)/socP_{\hat{A}}(a)) = \tau^m T^{2m+i}X.$$

Let $\tau^m T^{2m+i}X = T^jY$, then $j=0$ or -1, this force $m=-1$, so we have

$$S_2 \prec I_{\hat{A}}(v^{-1}a) = P_{\hat{A}}(a) \prec S_1$$

$$S_2 \prec radP_{\hat{A}}(a) \prec P_{\hat{A}}(a)/socP_{\hat{A}}(a) \prec S_1$$

and then $radP_{\hat{A}}(a), P_{\hat{A}}(a)/socP_{\hat{A}}(a) = \tau^{-1}radP_{\hat{A}}(a) \in S$ since $\boldsymbol{S}$ being path-closed, this is a contradiction with $\boldsymbol{S}$ being a complete slice of the component of form $\boldsymbol{Z}\vec{\Delta}$. This shows that for any $a \in Q_{\hat{A}}$, the τ-orbit of v contains at most one vertex in Q_B, so it remains to prove that the number of vertices of Q_B is not less than n, where n is the number of vertices of $\vec{\Delta}$. For this purpose it is enough to prove that $_BS$ is partial tilting module. First we claim that $p.d.$ $_BS\leqq 1$, or equivalently that $Hom_B(I, \tau_B I)=0$ for any indecomposable injective B-module I. Otherwise, there are $S_1, S_2\in \boldsymbol{S}$ with $S_1\to I\to\tau_B S_2\prec S_2$, by Lemma 2 we know this chain can occur in $\hat{A}$-$\underline{mod}$, so we have $\tau_B S_2, I\in\boldsymbol{S}$ and then the three terms of the AR sequence of B-mod $0\to\tau_B S_2\to *\to S_2\to 0$ are in $\boldsymbol{S}$, this contradicts with the fact that B is the support algebra of $_{\hat{A}}S$ and $\boldsymbol{S}$ is a complete slice. And then we may use Auslander-Reiten formula to show $Ext_B^1(S, S)$ $=DHom(S, \tau_B S)=0$, hence $_BS$ is partial tilting and it follows that Q_B is a complete v-slice of $Q_{\hat{A}}$.

(4) Now suppose that Y is an arbitrary $T(A)$-*module* on platform with $\underline{dim}Y=\underline{dim}X$, then $Y=\pi(N)$ for some N lying on the component of form $\boldsymbol{Z}\vec{\Delta}$. We may assume that N and M lie in the same v-period. By the above analysis we know that N is a directing module over some finite-dimenional k-algebra D and Q_D is a complete v-reflections. By [5] (Lemma 2.10) we know that D can be obtained from B by a series of v-reflections. On the other hand, the indecomposable D-module which has the same dimension vector with N must be $_DN$ itself, so the number of $T(A)$-modules on platform which the same dimension vector with X is at most m, where m is the number of all v-reflections from B within one v-period. Since within one v-period there are just n algebras which are obtained from B by a series of v-reflections, we have $m=n$, which finishes the proof of Theorem 1.

REMARK, We have an example showing that the number of $T(A)$-modules on platform which have the same dimension vector is n, where n is the vertices of A.

§2. Proof of Theorems 2 and 3.

Let Λ be a locally bounded k-algebra and X, Y two Λ-modules. Define

$$R_P^1(X, Y)=Hom_\Lambda(X, Y),$$

$$R_P^1(X, Y)=\{f\in Hom_\Lambda(X, Y)/f=\sum_i f_{i1}g_i \text{ (for finite } i),$$

$$\text{where } f_{i1}\in R(X, P_{i1}),\ P_{i1} \text{ is a projective } \Lambda\text{-module}\}.$$

In general, for $m>1$, we define

$$R_P^m(X, Y)=\{f\in Hom_\Lambda(X, Y)/f=\sum_i f_{i1}\cdots f_{im}g_i \text{ (for finite } i),$$

$$\text{where } f_{i1}\in R(X\text{ ‘}P_{i1}),\ \cdots,\ f_{im}\in R(P_{im-1}, P_{im}),\ P_{i1}, \cdots,$$

$$P_{im} \text{ are projective modules}\}.$$

LEMMA 4 ([8]). *For arbitrary non-negative integer m, there holds*

$$rad^m X/rad^{m+1}X \simeq \bigoplus_{x\in Q_0} k_x\cdot S(x),$$

where $k_x = dim_k R_P^m(P(x), M)/R_P^{m+1}(P(x), M)$.

LEMMA 5. *Let Λ be a locally bounded selfjective k-algebra.*

(1) *If M is an indecomposable non-projective Λ-module and $\varepsilon: P\to M$ is the projective cover of M, then ker ε is indecomposable.*

(2) *If N is an indecomposable non-projective Λ-module and $i: N\to I$ is the injective envelope of N, then coker i is indecomposable.*

PROOF. (2) is the dual of (1), so we consider (1). Assume ker $\varepsilon = \bigoplus_{i=1}^{m} N_i$, N_i indecomposable for all i. We see that every N_i is non-injective since $\varepsilon: P\to M$ is the projective cover. In fact, the natural embedding ker $\varepsilon\to P$ is the injective envelope, otherwise there is a proper direct summand of P isomorphic to the injective envelope $I(ker\ \varepsilon)$ of ker ε, and hence M has a projective direct summand, a contradiction. However, the injective envelope of ker ε is isomorphic to the direct sum of thase of all N_i, so $M = \bigoplus_{i=1}^{m} I(N_j)/N_i$. It follows from the indecomposability of M that $m=1$, which implies that ker ε is indecomposable.

THE PROOF OF THEOREM 3. Let X and Y be $T(A)$-module on platform, then there are indecomposable non-projective $\hat{A}$-modules M, N such that $\pi(M) = X$, $\pi(N) = Y$ with M, N belonging to the $\boldsymbol{Z}\vec{\Delta}$-components of $\hat{A}$-*mod* (it is possible that M, N lie on distinct components). Suppose $\boldsymbol{S}$ is a complete slice of the $\boldsymbol{Z}\vec{\Delta}$-componemt of $\hat{A}$-$\underline{mod}$ such that $M\in\boldsymbol{S}$, without loss of generality, we would assume that $\boldsymbol{S}\leqq N\prec T^2\tau\boldsymbol{S}$. Now $SuppN$ is divided into two parts, namely,

$$\Delta_1 = \{x\in SuppN/P_{\hat{A}}(x)\leqq \boldsymbol{S}\}$$

and

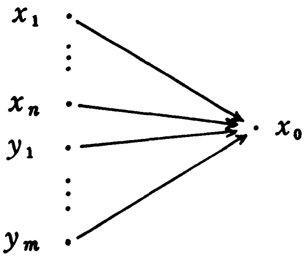
$$\Delta_2 = \{x\in SuppN/P_{\hat{A}}(x)\succ \boldsymbol{S}\}\ .$$

Let B be the full subcategory of $\hat{A}$ whose object is

$$\{x\in\hat{A}/T^{-2}\tau^{-1}\boldsymbol{S}\leqq P(x)\leqq \boldsymbol{S}\}\ ,$$

then B is the support algebra of modules located in $\boldsymbol{S}$. It follows from the proof of Theorem 1 that B is a tilted algebra with $\hat{B}=\hat{A}$ and $T(A)=T(B)$, moreover, we might assert that B is obtained from A by a series of reflections. Clearly $SuppN\subseteqq B$, if $\Delta_2=\emptyset$, then $SuppN\subseteqq B$. Since the covering functor π is induced by $T^2\tau$, M and N as B-modules have the same Loewy factors, hence, the same composition factors. Because B is a tilted algebra and M is directing as B-module, we see $M\simeq N$ by [2], therefore $X\simeq Y$. If $\Delta_1=\emptyset$, we would use $T^{-2}\tau^{-1}N$ to replace N, this amounts to the situation above.

If $\Delta_1\neq\emptyset$, $\Delta_2\neq\emptyset$, we try to get a contradiction. On account of *SuppN* being connected subcategory of $\hat{A}$, we can find $x_0\in\Delta_1$, $Sy_1\in\Delta_2$ and an arrow $y_1\rightarrow x_0$ in the Gabriel quiver of *SuppN*. Assume that all arrows in the Gabriel quiver of $\hat{A}$ ending at x_0 are as follows:



where $P(x_i)\leqq S$, $i=1,\cdots n$, $P(y_i)\succ S$, $i=i,\cdots m$. Therefore we have the following natural exact sequence

$$P(x_0)\longrightarrow\Big(\bigoplus_{i=1}^{m}P(x_i)\Big)\oplus\Big(\bigoplus_{i=1}^{m}P(y_i)\Big)\longrightarrow coker\varepsilon\longrightarrow 0\,.$$

Noticing that $Im\varepsilon$ is indecomposable for $P(x_0)$ is the projective cover of $Im\varepsilon$; and that the natural embedding

$$Im\varepsilon\longrightarrow\Big(\bigoplus_{i=1}^{m}P(x_i)\Big)\oplus\Big(\bigoplus_{i=1}^{m}P(y_i)\Big)$$

is the injective envelope, we see that $coker\varepsilon$ is indecomposable by Lemma 5, it follows that the sequence above is the minimal projective presentation of $coker\varepsilon$. For M being directing, by [9] the morphism

$$\Big(\bigoplus_{i=1}^{n}Hom_{\hat{A}}(P(x_i),M)\Big)\oplus\Big(\bigoplus_{i=1}^{m}Hom_{\hat{A}}(P(y_i),M)\Big)\xrightarrow{\varepsilon^*}Hom_{\hat{A}}(P(x_0),M)$$

is epic or mono, however $Hom_{\hat{A}}(P(y_i),M)=0$ for $i=1,\cdots,m$, then

$$\bigoplus_{i=1}^{n}Hom(P(x_i),M)\longrightarrow Hom_{\hat{A}}(P(y_0),M)$$

is either epic or mono.

For the same reason, the morphism

$$(*)\qquad\Big(\bigoplus_{i=1}^{n}Hom_{\hat{A}}(P(x_i),N)\Big)\oplus\Big(\bigoplus_{i=1}^{m}Hom_{\hat{A}}(P(y_i),N)\Big)\longrightarrow Hom_{\hat{A}}(P(x_0),N)$$

is either epic or mono.

1^0 If $\bigoplus_{i=1}^{n}Hom_{\hat{A}}(P(x_i),M)\rightarrow Hom_{\hat{A}}(P(x_0),M)$ is non-isomorphic and mono, we know by Lemma 4 that $S(x_0)$ is a direct summand of *topM* with multiplicity

$$t=dim_k\, Hom_{\hat{A}}(P(x_0),\, M)-\sum_{i=1}^{n} dim_k\, Hom_{\hat{A}}(P(x_0),\, M)$$
$$>0\,.$$

Since there are not $T^2\tau$-conjugated vertices in $SuppN$ and in $SuppM$, we see that $dim_k\, Hom_{\hat{A}}(P(x_0),\, M)=dim_k\, Hom_{\hat{A}}(P(x_i),\, N)$, $\forall i=1, \cdots, n$. If the morphism $(*)$ is epic, then $S(x_0)$ is not a direct summand of $topN$, which contradicts the fact that X and Y have the same Loewy factors. If $(*)$ is mono, then $S(x_0)$ is a direct summand of $topN$ with multiplicity

$$r=dim_k\, Hom_{\hat{A}}(P(x_0),\, M)-\sum_{i=1}^{m} dim_k\, Hom_{\hat{A}}(P(x_i),\, N)$$
$$-\sum_{i=1}^{m} dim_k\, Hom_{\hat{A}}(P(y_i),\, N)\,.$$

However,

$$r<dim_k\, Hom_{\hat{A}}(P(x_0),\, N)-\sum_{i=1}^{n} dim_k\, Hom_{\hat{A}}(P(x_i),\, N)$$
$$=dim_k\, Hom_{\hat{A}}(P(x_0),\, M)-\sum_{i=1}^{n} dim_k\, Hom_{\hat{A}}(P(x_i),\, M)$$
$$=t\,,$$

a contradiction.

2^0 If $\bigoplus_{i=1}^{n} Hom_{\hat{A}}(P(x_i),\, M)\to Hom_{\hat{A}}(P(x_0),\, M)$ is epic, considering the longest path in $SuppN$ ending at x_0 which is not a zero-relation

$$y'_1 \longrightarrow \cdots \longrightarrow y_1 \longrightarrow x_0\,.$$

It follows from [9] that the natural morphism l:

$$Hom_{\hat{A}}(P(y'_1),\, N) \longrightarrow \cdots \longrightarrow Hom_{\hat{A}}(P(y_1),\, N) \longrightarrow Hom_{\hat{A}}(P(x_0),\, N)$$
$$\longrightarrow Hom_{\hat{A}}(P(x_0),\, N)$$

is non-zero. Hence there exists $f\in Hom_A(P(y'_1),\, N)$ satisfying $l(f)\neq 0$. Since this non-zero path is the longest one, f can be no longer factor through any projective $\hat{A}$-module. By Lemma 4, $S(y'_1)$ is a direct summand of $topN$, hence we can conclude that $S(vy'_1)$ is a direct summand of $topM$. We know by [9] that the natural morphism $Hom_A(P(x_0),\, M)\to Hom_A(P(vy'_1),\, M)$ is mono or epic, therefore it must be non-isomorphic and mono by Lemma 4. Assume that the arrows in $SuppN$ ending at y'_1 are as follows:

$$\begin{tikzcd} z_1 \arrow[dr] & \\ \vdots & y'_1 \\ z_q \arrow[ur] & \end{tikzcd}$$

then $S(y_1')$ is a direct summand of $topN$ with multiplicity $dim_k Hom_A(P(y_1'), N) - \sum_{i=1}^q dim_k\ Hom_A(P(z_i), N) > 0$. Owing to $x_0 \in \Delta_? \subseteq SuppN$, it bears $x_0 \notin \{vz_i\}_{i=1}^q$. Similarly we can show that

$$\left(\bigoplus_{i=1}^{q} Hom_A(P(vz_i), M)\right) \oplus Hom_A(P(x_0), M) \longrightarrow Hom_A(P(vy_1'), M)$$

is non-isomorphic and mono and $S(vy_1')$ is a direct summand of $topN$ with multiplicity s:

$$s < dim_k\ Hom_A(P(vy_1'), M) - \sum_{i=1}^{q} dim_k\ Hom_A(P(vz_i), M)$$

$$= dim_k\ Hom_A(P(y_1'), N) - \sum_{i=1}^{q} dim_k\ Hom_A(P(z_i), N),$$

which contradicts the hypothesis that X and Y have the same Loewy factors. Up to now we finish the proof of (2)⇒(1). The proof of (3)⇒(1) is similar.

PROOF OF THEORDM 2. Let X and Y be two $T(A)$-modules on platform with $topX \simeq topY$ and $socX \simeq socY$. Suppose that M, N, B are same as above, from the proof of Theorem 3 we know that M and N are both B-modules, and as B-modules they have the same top and socle. Since both M and N are directing B-modules, we have $M \simeq N$ by [2], it follows that $X \simeq Y$.

COROLLARY. *Let A be an iterated tilted algeba, X and Y $T(A)$-modules on platform, then the following are equivalent:*

(1) $$X \simeq Y$$

(2) $$\underline{dim}X = \underline{dim}Y, \quad topX \simeq topY$$

(3) $$\underline{dim}X = \underline{dim}Y, \quad socsX \simeq ocY$$

REMARK. (1) We know that every non-projective indecomposable module over a representation-finite trivial extension algebra is a module on platform. So the conclusions of Theorems 2 and 3 in [10] are contained in the results of this article.

(2) At last we leave a space to explain the fact that no directing module exists over a finite-dimensional selfinjective algebra Λ. In fact, let P_1 be a direct summand of the projective cover of an indecompossable module M and P_2 be a direct summand of an injective envelope of M. It is not difficult to see that arbitrary two vertices in the Gabriel quiver Q_Λ of Λ belong to a cycle path of Q_Λ, therefore $P_2 \prec P_1 \prec M \prec P_2$, i.e., M is not directing.

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