



CERTAIN DISTANCE ESTIMATES FOR OPERATORS ON THE BERGMAN SPACE

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ABSTRACT. Let \mathbb{D} be the open unit disk with its boundary $\partial\mathbb{D}$ in the complex plane \mathbb{C} and $dA(z) = \frac{1}{\pi} dx dy$, the normalized area measure on \mathbb{D} . Let $L_a^2(\mathbb{D}, dA)$ be the Bergman space consisting of analytic functions on \mathbb{D} that are also in $L^2(\mathbb{D}, dA)$. In this paper we obtain certain distance estimates for bounded linear operators defined on the Bergman space.

1. INTRODUCTION

Let dA denote Lebesgue area measure on the unit disk \mathbb{D} , normalized so that the measure of \mathbb{D} equals 1. The Bergman space $L_a^2(\mathbb{D})$ is the Hilbert space consisting of the analytic functions on \mathbb{D} that are also in $L^2(\mathbb{D}, dA)$. Since point evaluation at $z \in \mathbb{D}$ is a bounded linear functional on the Hilbert space $L_a^2(\mathbb{D})$, the Riesz representation theorem implies that there exists a unique function K_z in $L_a^2(\mathbb{D})$ such that

$$f(z) = \int_{\mathbb{D}} f(w) \overline{K_z(w)} dA(w)$$

for all $f \in L_a^2(\mathbb{D})$. Let $K(z, w)$ be the function on $\mathbb{D} \times \mathbb{D}$ defined by $K(z, w) = \overline{K_z(w)} = \frac{1}{(1-z\bar{w})^2}$. The function $K(z, w)$ is called the Bergman kernel of \mathbb{D} or the

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reproducing kernel of $L_a^2(\mathbb{D})$ because the formula

$$f(z) = \int_{\mathbb{D}} f(w)K(z, w)dA(w)$$

reproduces each $f \in L_a^2(\mathbb{D})$. The sequence of functions $e_n(z) = \sqrt{n+1}z^n$, $n = 0, 1, 2, \dots$ form an orthonormal basis for $L_a^2(\mathbb{D})$ and $K(z, w) = \sum_{n=0}^{\infty} e_n(z)\overline{e_n(w)}$.

For $a \in \mathbb{D}$, let ϕ_a be the analytic map from \mathbb{D} onto \mathbb{D} defined by

$$\phi_a(z) = \frac{a-z}{1-\bar{a}z}, z \in \mathbb{D}.$$

Let $\mathcal{L}(L_a^2(\mathbb{D}))$ be the set of all bounded linear operators from $L_a^2(\mathbb{D})$ into itself. Let $\mathcal{LC}(L_a^2(\mathbb{D}))$ denote the ideal of compact operators in $\mathcal{L}(L_a^2(\mathbb{D}))$. Let $L^\infty(\mathbb{D})$ denote the usual space of bounded measurable functions f on the unit disk \mathbb{D} in the complex plane with $\|f\|_\infty = \text{ess sup}\{|f(z)| : z \in \mathbb{D}\} < \infty$. Let $H^\infty(\mathbb{D})$ be the space of bounded analytic functions on \mathbb{D} and $h^\infty(\mathbb{D})$ be the space of all bounded harmonic functions on \mathbb{D} .

For $T \in \mathcal{L}(L_a^2(\mathbb{D}))$, the Berezin transform of T is the function \tilde{T} on \mathbb{D} defined by

$$\tilde{T}(z) = \langle Tk_z, k_z \rangle$$

where $k_z(w) = \frac{K(w, z)}{\sqrt{K(z, z)}} = \frac{1-|z|^2}{(1-\bar{z}w)^2}$. These functions are called normalized reproducing kernels of $L_a^2(\mathbb{D})$; it is clear that they are unit vectors in $L_a^2(\mathbb{D})$. For $\phi \in L^\infty(\mathbb{D})$, the Toeplitz operator with symbol ϕ is the operator T_ϕ on $L_a^2(\mathbb{D})$ defined by $T_\phi f = P(\phi f)$, where P is the orthogonal projection of $L^2(\mathbb{D}, dA)$ onto $L_a^2(\mathbb{D})$. The Berezin transform of the function ϕ , denoted by $\tilde{\phi}$, is defined to be the Berezin transform of the Toeplitz operator T_ϕ . This definition easily leads to the formula

$$\tilde{\phi}(z) = (1-|z|^2)^2 \int_{\mathbb{D}} \frac{\phi(w)}{|1-\bar{z}w|^4} dA(w).$$

If $\phi \in h^\infty(\mathbb{D})$, then $\tilde{T}_\phi = \tilde{\phi} = \phi$. Let $\phi : \mathbb{D} \rightarrow \mathbb{D}$ be analytic. Then ϕ induces a composition operator C_ϕ on $L_a^2(\mathbb{D})$ defined by $C_\phi f = f \circ \phi$, for all $f \in L_a^2(\mathbb{D})$. An operator $A \in \mathcal{L}(L_a^2(\mathbb{D}))$ is positive if $\langle Af, f \rangle \geq 0$, for every $f \in L_a^2(\mathbb{D})$. In this case we write $A \geq 0$. Let $\sigma(A)$ denote the spectrum of $A \in \mathcal{L}(L_a^2(\mathbb{D}))$. In section 2, we introduce the operator $U_a \in \mathcal{L}(L_a^2(\mathbb{D}))$, $a \in \mathbb{D}$ and discuss certain algebraic properties of these classes of operators. In section 3, we propose some distance estimates for operators defined on the Bergman space. In section 4, we are concerned only with Toeplitz operators with bounded harmonic symbols and obtained certain distance estimates between Toeplitz operators.

2. PRELIMINARIES

In this section we introduce the operator $U_a \in \mathcal{L}(L_a^2(\mathbb{D}))$, $a \in \mathbb{D}$ and discuss certain algebraic properties of these classes of operators. Given $a \in \mathbb{D}$, we define the operator U_a from $L_a^2(\mathbb{D})$ into itself as $U_a f = (f \circ \phi_a)k_a$. The Proposition 2.1

is well known. For example see [11]. To make the presentation self-contained, we give a proof of the proposition.

Proposition 2.1. *Let $a \in \mathbb{D}$. The following hold:*

- (i) U_a is self-adjoint unitary operator.
- (ii) $U_a K_z = \overline{k_a(z)} K_{\phi_a(z)}$.
- (iii) $U_a k_z = \alpha k_{\phi_a(z)}$ where $\alpha = \alpha_a(z) \in \partial\mathbb{D}$.

Proof. (i). Let $f, g \in L_a^2(\mathbb{D})$ and let ϕ_a be the Möbius transformation for $a \in \mathbb{D}$. Then

$$\langle f \circ \phi_a, g \circ \phi_a \rangle = \langle k_a f, k_a g \rangle.$$

First we will prove U_a is self-adjoint. If $f, g \in L_a^2(\mathbb{D})$ then

$$\langle U_a^* f, g \rangle = \langle f, U_a g \rangle = \langle f, k_a(g \circ \phi_a) \rangle = \langle k_a f \circ \phi_a, k_a(k_a \circ \phi_a) g \rangle = \langle U_a f, (k_a(k_a \circ \phi_a) g) \rangle$$

where we have used the above change of variable and the identity $\phi_a \circ \phi_a(w) = w$. Since $(k_a(k_a \circ \phi_a)) = 1$, so $\langle U_a^* f, g \rangle = \langle U_a f, g \rangle$. That U_a is unitary follows similarly. Simply rewrite the change of variable replacing f by $f \circ \phi_a$, simplify the result with the identity $k_z \circ \phi_z(w) = k_z(w)^{-1}$. Combining these result yields $U_a^{-1} = U_a^* = U_a$. This proves (i). For (ii), let $f \in L_a^2(\mathbb{D})$. Now

$$\begin{aligned} \langle f, U_a K_z \rangle &= \langle U_a f, K_z \rangle = U_a f(z) = k_a(z) f(\phi_a(z)) \\ &= k_a(z) \langle f, K_{\phi_a(z)} \rangle = \langle f, \overline{k_a(z)} K_{\phi_a(z)} \rangle. \end{aligned}$$

This proves (ii). We shall now prove (iii). Notice that

$$U_a k_z = \frac{1}{\|K_z\|} U_a K_z = \frac{1}{\|K_z\|} \overline{k_a(z)} K_{\phi_a(z)} = \left(\frac{\|K_{\phi_a(z)}\|}{\|K_z\|} \overline{k_a(z)} \right) k_{\phi_a(z)}.$$

Setting $\alpha = \alpha_a(z) = \frac{\|K_{\phi_a(z)}\|}{\|K_z\|} \overline{k_a(z)}$ we get the desired identity. Since

$$1 - |\phi_a(z)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \bar{a}z|^2}.$$

To prove $\alpha_a(z) \in \partial\mathbb{D}$ we calculate

$$\alpha_a(z) = \frac{1 - |z|^2}{1 - |\phi_a(z)|^2} \frac{1 - |a|^2}{(1 - \bar{a}z)^2} = \frac{|1 - a\bar{z}|^2}{(1 - \bar{a}z)^2}.$$

□

Now since $k_a(z)k_a(\phi_a(z)) = 1$ for each $a \in \mathbb{D}$, hence $U_a k_a = 1$ and $U_a k_{m_a} = k_{m_a}$ where m_a is the geodesic midpoint between 0 and a (see [6]), i.e, $m_a = \frac{1 - \sqrt{1 - |a|^2}}{|a|^2} a$. Further $U_a k_z = k_{\phi_a(z)}$.

Lemma 2.2. *Let $T \in \mathcal{L}(L_a^2(\mathbb{D}))$. The following hold:*

- (i) If $TU_a = U_a T$ for all $a \in \mathbb{D}$, then $T = \alpha I$ for some $\alpha \in \mathbb{C}$.
- (ii) If $TU_{\phi_a} = U_{\phi_a} T$ for all $a \in \mathbb{D}$, then $T = \beta I$ for some $\beta \in \mathbb{C}$.
- (iii) If $TU_a = U_a T$ for some $a \in \mathbb{D}$, then $M_a = \{k_{m_a}(g \circ \phi_{m_a}) : g \text{ even}\}$ is a reducing subspace of T .

Proof. For proof see [8].

□

Lemma 2.3. *Let $T \in \mathcal{L}(L_a^2(\mathbb{D}))$, $T \geq 0$. Then $T = \sum_{n=1}^{\infty} \sqrt{T}e_n \otimes (\sqrt{T}e_n)^*$, where $\{e_n\}$ is an orthonormal basis for $L_a^2(\mathbb{D})$, and the infinite sum means the limit of partial sums in the strong operator topology.*

Proof. Define $P_N = \sum_{n=1}^N e_n \otimes e_n^*$. Clearly $P_N \uparrow I$ strongly. Thus $\sqrt{T}P_N\sqrt{T} \longrightarrow T$ strongly and $\sqrt{T}P_N\sqrt{T} = \sum_{n=1}^N \sqrt{T}e_n \otimes (\sqrt{T}e_n)^*$. \square

3. THE OPERATOR U_a AND CERTAIN DISTANCE ESTIMATES

In this section we propose some distance estimates for operators defined on the Bergman space. Such inequalities are useful in the theory of best approximation in C^* -algebras [5], complex interpolation [3], the theory of generalized inverses and operator approximation (see, [9], [10] and [1]). We obtain estimates for $\|U_a - T\|$ where $T \in \mathcal{L}(L_a^2(\mathbb{D}))$, $T \geq 0$ and $a \in \mathbb{D}$.

It is well known [7] that if $T \in \mathcal{L}(L_a^2(\mathbb{D}))$ and there exists δ , $0 < \delta < 1$ such that $\|I - T^n\| \leq \delta < 1$ for all $n \in \mathbb{N}$, then $T = I$, the identity operator. Thus it follows from this that $\|I - U_a\| \geq 1$ since $U_a \neq I$ and $U_a^2 = I$. Further, since U_a is not positive, it follows from [7] that there exists no positive integer N such that $\left\| I - \frac{1}{n+1} \sum_{j=k}^{k+n} U_a^j \right\| \leq 1$ for all $n \geq N$ and $k \in \mathbb{N} \cup \{0\}$. But the following holds:

Theorem 3.1. *Let A be a positive operator in $\mathcal{L}(L_a^2(\mathbb{D}))$. Suppose for some $a \in \mathbb{D}$, $\|U_a - A\| < 1$. Then A is invertible. Further, $\|I - A\| \leq \|U_a - A\| \leq \|I + A\|$.*

Proof. If $f \in L_a^2(\mathbb{D})$ and $\|f\| = 1$ then

$$\begin{aligned}
\|(U_a - A)f\|^2 &= \langle (U_a - A)f, (U_a - A)f \rangle \\
&= \|U_af\|^2 + \|Af\|^2 - \langle U_af, Af \rangle - \langle Af, U_af \rangle \\
&= \|f\|^2 + \|Af\|^2 - \langle f, U_a^*Af \rangle - \langle U_a^*Af, f \rangle \\
&= 1 + \|Af\|^2 - \langle f, U_aAf \rangle - \langle U_aAf, f \rangle \\
&= 1 + \|Af\|^2 - \overline{\langle U_aAf, f \rangle} - \langle U_aAf, f \rangle \\
&= 1 + \|Af\|^2 - 2 \operatorname{Re} \langle U_aAf, f \rangle \\
&\geq 1 + \|Af\|^2 - 2|\langle U_aAf, f \rangle| \\
&\geq 1 + \|Af\|^2 - 2\|U_aAf\|\|f\| \\
&= 1 + \|Af\|^2 - 2\|Af\|\|f\| \\
&= 1 + \|Af\|^2 - 2\|Af\| = (1 - \|Af\|)^2.
\end{aligned}$$

Now since A is positive, $\inf_{\|f\|=1} \langle Af, f \rangle = \inf_{\|f\|=1} \|Af\|$ and $\sup_{\|f\|=1} \langle Af, f \rangle = \sup_{\|f\|=1} \|Af\|$ and we have

$$\begin{aligned} \|U_a - A\| &= \sup_{\|f\|=1} \|(U_a - A)f\| \\ &\geq \sup_{\|f\|=1} |1 - \|Af\|| \\ &= \sup_{\|f\|=1} |1 - \langle Af, f \rangle| \\ &= \sup_{\|f\|=1} |\langle (I - A)f, f \rangle| = \|I - A\|. \end{aligned}$$

Thus if $\|U_a - A\| < 1$ then $\|I - A\| < 1$ and this implies A is invertible [4]. Further,

$$\begin{aligned} \|U_a - A\| &= \sup_{\|f\|=1} \|U_a f - Af\| \\ &\leq \sup_{\|f\|=1} (1 + \|Af\|) \\ &= \sup_{\|f\|=1} \langle (I + A)f, f \rangle = \|I + A\|. \end{aligned}$$

□

Theorem 3.2. *Let $T \in \mathcal{L}(L_a^2(\mathbb{D}))$. The following hold:*

- (i): *If for some $a \in \mathbb{D}$, $\|U_a - T\| < 1$, then T is invertible.*
- (ii): *If T is invertible and $\|U_a - rT\| = 1$ for some real number $r > 1$ and $a \in \mathbb{D}$, then $\|U_a - T\| < 1$.*

Proof. (i) The operator U_a is a unitary operator and $U_a^2 = I$, $U_a^* = U_a$. Since

$$\|I - U_a T\| = \|U_a - T\| < 1,$$

the operator $U_a T$ is invertible. Hence T is invertible as $T = U_a(U_a T)$. This completes the proof of (i). To prove (ii), we first show that if $S, R \in \mathcal{L}(L_a^2(\mathbb{D}))$ and $\|S - R\| = \|S\| + \|R\|$ then there exists a sequence $\{f_n\}$ of unit vectors in $\mathcal{L}(L_a^2(\mathbb{D}))$ such that $\lim_{n \rightarrow \infty} \|(\|R\|S + \|S\|R) f_n\| = 0$. The converse holds if one of S and R is an isometric operator. To prove this we use the well known fact that if f and g are two vectors in a normed linear space such that $\|f + g\| = \|f\| + \|g\|$ then $\|af + bg\| = a\|f\| + b\|g\|$ for any nonnegative real numbers a and b . Suppose $S \neq 0$, $R \neq 0$, $\|S - R\| = \|S\| + \|R\|$. Let $K = \frac{S}{\|S\|}$ and $L = \frac{-R}{\|R\|}$. Since $\|S - R\| = \|S\| + \|R\|$ we have $\|K + L\| = 2$. So there exists a sequence $\{f_n\}$ of unit vectors such that $\lim_{n \rightarrow \infty} \|K f_n + L f_n\| = 2$. This implies that $\lim_{n \rightarrow \infty} \|K f_n - L f_n\| = 0$ as $\|K f_n\| \leq 1$, $\|L f_n\| \leq 1$ for all n and by Parallelogram law

$$\|K f_n + L f_n\|^2 + \|K f_n - L f_n\|^2 = 2\|K f_n\|^2 + 2\|L f_n\|^2.$$

It thus follows that $\lim_{n \rightarrow \infty} \|(\|R\|S + \|S\|R) f_n\| = 0$. Conversely, let S be an isometric operator and $\lim_{n \rightarrow \infty} \|(\|R\|S + R) f_n\| = 0$ for some sequence $\{f_n\} \in L_a^2(\mathbb{D})$

of unit vectors. Then we have

$$2\|R\| \geq \|(\|R\|S - R) f_n\| \geq 2\|R\| \|S f_n\| - \|(\|R\|S + R) f_n\| \rightarrow 2\|R\| \text{ as } n \rightarrow \infty.$$

Hence, $\|(\|R\|S - R)\| = 2\|R\|$, equivalently, $\|S - R\| = 1 + \|R\|$. Thus we obtain that if $S, R \in \mathcal{L}(L_a^2(\mathbb{D}))$ and S is an isometric operator then $\|S - R\| = 1 + \|R\|$ if and only if there exists a sequence $\{f_n\}$ of unit vectors in $L_a^2(\mathbb{D})$ such that $\lim_{n \rightarrow \infty} \|(\|R\|S + R) f_n\| = 0$. Notice that

$$\begin{aligned} \|(\|T\|(r-1)S + \|(r-1)S\|T) f_n\| &= \|(\|T\|(r-1)S + (r-1)\|S\|T) f_n\| \\ &= (r-1) \|(\|T\|S + T) f_n\|. \end{aligned}$$

It thus follows that $\|(r-1)S - T\| < (r-1) + \|T\|$ if and only if there exists no sequence $\{f_n\} \in L_a^2(\mathbb{D})$ of unit vectors such that $\lim_{n \rightarrow \infty} \|(\|T\|S + T) f_n\| = 0$. Now we shall establish **(ii)**. Suppose T is invertible and $\|U_a - rT\| = 1$ for some real number $r > 1$. From the above argument it follows that there exists no sequence $\{f_n\} \in L_a^2(\mathbb{D})$ of unit vectors such that

$$\lim_{n \rightarrow \infty} \|(\|U_a - rT\|U_a + (rT - U_a)) f_n\| = 0$$

if and only if

$$\|(r-1)U_a - (rT - U_a)\| < (r-1) + \|rT - U_a\| = r.$$

We thus show that if there exists no sequence $\{f_n\} \in L_a^2(\mathbb{D})$ of unit vectors such that $\lim_{n \rightarrow \infty} \|T f_n\| = 0$ then

$$r\|U_a - T\| = \|rU_a - rT\| = \|(r-1)U_a - (rT - U_a)\| < r.$$

That is, $\|U_a - T\| < 1$. Hence if T is invertible and $\|U_a - rT\| = 1$ for some $a \in \mathbb{D}$ and some real number $r > 1$ then $\|U_a - T\| < 1$. \square

Corollary 3.3. *Let $T \in \mathcal{L}(L_a^2(\mathbb{D}))$. Then $\|U_a - T\| = 1 + \|T\|$ for some $a \in \mathbb{D}$ if and only if the operator $\|T\|U_a + T$ is not invertible.*

Proof. Suppose $r > 1 = \|T\|$ and $V_1 = \frac{U_a + T}{r}$. Then $T = rV_1 - U_a$ and $\|U_a - rV_1\| = \|T\| = 1$. Thus $V_1 = \frac{U_a + T}{r}$ is not invertible if and only if $\|U_a - \frac{U_a + T}{r}\| \geq 1$. But

$$\|(r-1)U_a - T\| \leq (r-1)\|U_a\| + \|T\| = r.$$

Thus V_1 is not invertible if and only if $\|(r-1)U_a - T\| = r$. That is, if and only if, $\|U_a - T\| = 1 + \|T\|$. That is, $rV_1 = U_a + T$ is not invertible if and only if $\|U_a - T\| = 1 + \|T\|$. For the general case, suppose $T \in \mathcal{L}(L_a^2(\mathbb{D}))$. Then $\left\| \frac{T}{\|T\|} \right\| = 1$. From the first part we obtain $U_a + \frac{T}{\|T\|}$ is not invertible if and only if

$$\left\| U_a - \frac{T}{\|T\|} \right\| = 1 + \left\| \frac{T}{\|T\|} \right\| = 2.$$

Thus $\|T\|U_a + T$ is not invertible if and only if $\| \|T\|U_a - T\| = 2\|T\|$. That is, if and only if $\|U_a - T\| = 1 + \|T\|$. This completes the proof. \square

Corollary 3.4. *Let $T \in \mathcal{L}(L_a^2(\mathbb{D}))$. Let $R = U_a T - T U_a$ for some $a \in \mathbb{D}$. Then $\|I - R\| \geq 1$.*

Proof. Notice that $\|U_a\| = 1$ and the spectral radius of $U_a = r(U_a) = 1$. In fact the spectrum of $U_a = \sigma(U_a) = \{-1, 1\}$. Hence there is a sequence of unit vectors $f_n \in L_a^2(\mathbb{D})$, $(n = 1, 2, 3, \dots)$ such that $(U_a - I)f_n \rightarrow 0$. Now

$$\|I - R\| \geq |\langle (I - R)f_n, f_n \rangle| = |1 - \langle Rf_n, f_n \rangle| \geq 1 - |\langle Rf_n, f_n \rangle|.$$

The result follows if we can show that $\langle Rf_n, f_n \rangle \rightarrow 0$. But

$$\begin{aligned} \langle Rf_n, f_n \rangle &= \langle ((U_a - I)T - T(U_a - I))f_n, f_n \rangle \\ &= \langle Tf_n, (U_a - I)f_n \rangle - \langle (U_a - I)f_n, T^*f_n \rangle. \end{aligned}$$

So

$$|\langle Rf_n, f_n \rangle| \leq \|T\| (\|(U_a - I)f_n\| + \|(U_a - I)f_n\|) \rightarrow 0 \text{ as } n \rightarrow \infty. \quad \square$$

The Schatten-Von Neumann class $S_p = S_p(H)$, $0 < p < \infty$, consists of all operators $T \in \mathcal{LC}(H)$ such that

$$\|T\|_{S_p} = \left(\sum_{n=0}^{\infty} (s_n(T))^p \right)^{\frac{1}{p}} < \infty. \quad (3.1)$$

Corollary 3.5. *If A is positive and $A \in \mathcal{L}(L_a^2(\mathbb{D}))$ and $U_a - A \in S_p$, $0 < p < \infty$ for some $a \in \mathbb{D}$, then $I - A \in S_p$.*

Proof. Suppose $U_a - A \in S_p$ for some $a \in \mathbb{D}$. Then

$$-AU_a + U_aA = (U_a - A)U_a - U_a(U_a - A) \in S_p.$$

Hence

$$I - A^2 = (U_a - A)(U_a + A) + (AU_a - U_aA) \in S_p.$$

Since A is positive, hence $I + A$ is invertible and so $I - A = (I - A^2)(I + A)^{-1} \in S_p$. \square

4. TOEPLITZ OPERATORS WITH BOUNDED HARMONIC SYMBOLS

For $T \in \mathcal{L}(L_a^2(\mathbb{D}))$, $T \geq 0$, let $W(T) = \{\langle Tf, f \rangle : \|f\| = 1\}$, the numerical range of T and $w(T) = \sup\{|\langle Tf, f \rangle| : \|f\| = 1\}$, the numerical radius of T . It is well known that $W(T)$ is convex and its closure contains $\sigma(T)$. There are certain standard norm estimates like (see [1]), $w(T) \leq \|T\| \leq 2w(T)$ and $w(T^n) \leq w(T)^n$, $n \in \mathbb{N}$. It is not hard to check that for $T \in \mathcal{L}(L_a^2(\mathbb{D}))$, the range of \tilde{T} is contained in $W(T)$ and $\|\tilde{T}\|_{\infty} \leq w(T)$. Thus if $\phi \in L^{\infty}(\mathbb{D})$, then $\|\tilde{\phi}\|_{\infty} = \|\tilde{T}_{\phi}\|_{\infty} \leq w(T_{\phi}) \leq \|T_{\phi}\| \leq \|\phi\|_{\infty}$. The natural question that arises at this point is the following: Is there a real number $M > 0$ with $\|T_{\phi}\| \leq M\|\tilde{\phi}\|_{\infty}$ for all $\phi \in L^{\infty}(\mathbb{D})$ and if so what are the distance estimates for $\|T_{\phi} - T_{\psi}\|$ and $\|U_a - T_{\phi}\|$ where, $\phi, \psi \in L^{\infty}(\mathbb{D})$? In this section, we are concerned only with Toeplitz operators with bounded harmonic symbols and obtained such distance estimates.

Theorem 4.1. *If $\phi, \psi \in h^{\infty}(\mathbb{D})$ and $T_{\phi}, T_{\psi} \geq 0$, then*

- (i): *For all $a \in \mathbb{D}$, $\|T_{\psi \circ \phi_a} - T_{\phi}\| \leq \max\{\|\phi\|_{\infty}, \|\psi\|_{\infty}\}$.*
- (ii): *$\|T_{\phi}T_{\psi} - T_{\psi}T_{\phi}\| \leq (\max\{\|\phi\|_{\infty}, \|\psi\|_{\infty}\})^2$.*

(iii): If $r > 0$ and $T_{\phi+\psi} \geq rI \geq 0$, then $\|T_\phi U_a + U_a T_\psi\| \geq r$.

Proof. **(i)** Let $\tilde{\phi}(z) = \langle T_\phi k_z, k_z \rangle$, where k_z is the normalized reproducing kernel for the Hilbert space $L_a^2(\mathbb{D})$. Hence $\tilde{\phi}(z) = \int_{\mathbb{D}} \phi(w) |k_z(w)|^2 dA(w)$, $z \in \mathbb{D}$. Since $\phi'_z = -k_z$, making a change of variable we also have $\tilde{\phi}(z) = \int_{\mathbb{D}} \phi(\phi_z(w)) dA(w) = \phi(\phi_z(0)) = \phi(z)$, $z \in \mathbb{D}$. Thus $|\phi(z)| = |\tilde{\phi}(z)| = |\langle T_\phi k_z, k_z \rangle| \leq \|T_\phi\|$ for all $z \in \mathbb{D}$. Hence $\|\phi\|_\infty \leq \|T_\phi\|$. Now since the Bergman projection has norm 1, we have $\|T_\phi\| = \|\phi\|_\infty$.

Suppose $a \in \mathbb{D}$. Now on $L_a^2(\mathbb{D}) \oplus L_a^2(\mathbb{D})$, let

$$M = \begin{pmatrix} T_\phi & 0 \\ 0 & T_\psi \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 0 & U_a \\ 0 & 0 \end{pmatrix}.$$

Then $M \geq 0$ and $MN - NM = \begin{pmatrix} 0 & T_\phi U_a - U_a T_\psi \\ 0 & 0 \end{pmatrix}$. From [2] we have

$$\begin{aligned} \|T_{\psi \circ \phi_a} - T_\phi\| &= \|T_\phi U_a - U_a T_\psi\| = \|MN - NM\| \leq \|M\| \|N\| \\ &= \max\{\|T_\phi\|, \|T_\psi\|\} \|U_a\| = \max\{\|\phi\|_\infty, \|\psi\|_\infty\}. \end{aligned}$$

This completes the proof of **(i)**. To prove **(ii)**, observe that

$$\begin{aligned} 2(T_\psi T_\phi - T_\phi T_\psi) &= (T_\phi + T_\psi)(T_\phi - T_\psi) - (T_\phi - T_\psi)(T_\phi + T_\psi) \\ &= T_{\phi+\psi} T_{\phi-\psi} - T_{\phi-\psi} T_{\phi+\psi}. \end{aligned}$$

Since $T_{\phi+\psi}$ is positive and $\|T_{\phi+\psi}\| = \|\phi + \psi\|_\infty$, we have from [2] that

$$\begin{aligned} 2\|T_\psi T_\phi - T_\phi T_\psi\| &\leq \|T_{\phi+\psi}\| \|T_{\phi-\psi}\| = \|\phi + \psi\|_\infty \|T_\phi - T_\psi\| \\ &= \|\phi + \psi\|_\infty \max\{\|\phi\|_\infty, \|\psi\|_\infty\}. \end{aligned}$$

Thus

$$\|T_\psi T_\phi - T_\phi T_\psi\| \leq \frac{1}{2} \{\|\phi\|_\infty + \|\psi\|_\infty\} \max\{\|\phi\|_\infty, \|\psi\|_\infty\} \leq (\max\{\|\phi\|_\infty, \|\psi\|_\infty\})^2.$$

This completes the proof of **(ii)**. To prove **(iii)**, notice that the spectral radius of $U_a = r(U_a) = 1$ and $\|U_a\| = 1$ and $\sigma(U_a) = \{-1, 1\}$. Hence there is a sequence of unit vectors $\{f_n\} \in L_a^2(\mathbb{D})$ such that $(U_a - I)f_n \rightarrow 0$. Now

$$\begin{aligned} \|T_\phi U_a + U_a T_\psi\| &\geq |\langle (T_\phi U_a + U_a T_\psi) f_n, f_n \rangle| \\ &= |\langle T_\phi (U_a - I) f_n, f_n \rangle + \langle (U_a - I) T_\psi f_n, f_n \rangle + \langle (T_\phi + T_\psi) f_n, f_n \rangle| \\ &= |\langle (U_a - I) f_n, T_\phi f_n \rangle + \langle T_\psi f_n, (U_a - I) f_n \rangle + \langle (T_\phi + T_\psi) f_n, f_n \rangle| \\ &\geq |\langle (T_\phi + T_\psi) f_n, f_n \rangle| - [\|(U_a - I) f_n\| \|T_\phi f_n\| + \|T_\psi f_n\| \|(U_a - I) f_n\|] \\ &\geq r - r_n \end{aligned}$$

where r_n is a term depending on n and $r_n \rightarrow 0$ as $n \rightarrow \infty$. Hence

$$\|T_\phi U_a + U_a T_\psi\| \geq r.$$

□

Theorem 4.2. (i): If $\phi \in L^\infty(\mathbb{D})$ and $T_\phi^2 + T_{\phi \circ \phi_a}^2 = 2T_{\phi \circ \phi_a} T_\phi$ for some $a \in \mathbb{D}$, then $\lim_{n \rightarrow \infty} \frac{\|T_\phi^{n-1}(T_{\phi \circ \phi_a - \phi})\|}{\|T_\phi^{n-1}\|} = 0$, $T_{\phi \circ \phi_a - \phi}$ is not invertible and $\|U_a - T_{\phi \circ \phi_a - \phi}\| \geq 1$.

(ii): If $\phi \in L^\infty(\mathbb{D})$ and T_ϕ is positive, then also $\|U_a - T_{\phi \circ \phi_a - \phi}\| \geq 1$.

Proof. (i) Suppose $T_\phi^2 + T_{\phi \circ \phi_a}^2 = 2T_{\phi \circ \phi_a} T_\phi$ for some $a \in \mathbb{D}$. Then

$$T_{\phi \circ \phi_a} T_\phi - T_\phi^2 = T_{\phi \circ \phi_a}^2 - T_{\phi \circ \phi_a} T_\phi.$$

That is,

$$\begin{aligned} U_a T_\phi U_a T_\phi - T_\phi^2 &= (U_a T_\phi U_a)(U_a T_\phi U_a) - (U_a T_\phi U_a) T_\phi \\ &= U_a T_\phi^2 U_a - (U_a T_\phi U_a) T_\phi. \end{aligned}$$

Hence $T_\phi U_a T_\phi - U_a T_\phi^2 = T_\phi^2 U_a - T_\phi U_a T_\phi$. That is, $(T_\phi U_a - U_a T_\phi) T_\phi = T_\phi (T_\phi U_a - U_a T_\phi)$. Thus T_ϕ commutes with $T_\phi U_a - U_a T_\phi$. Suppose there exists a $m \in \mathbb{Z}_+$ such that $T_\phi^m = 0$. Now since

$$\begin{aligned} T_\phi^n U_a - U_a T_\phi^n &= \sum_{i=0}^{n-1} T_\phi^{n-1-i} (T_\phi U_a - U_a T_\phi) T_\phi^i \\ &= n T_\phi^{n-1} - \sum_{i=0}^{n-1} T_\phi^{n-1-i} (I - (T_\phi U_a - U_a T_\phi)) T_\phi^i, \end{aligned}$$

we have $0 = T_\phi^m U_a - U_a T_\phi^m = m T_\phi^{m-1} (T_\phi U_a - U_a T_\phi)$ and hence

$$\lim_{n \rightarrow \infty} \frac{\|T_\phi^{n-1} (T_\phi U_a - U_a T_\phi)\|}{\|T_\phi^{n-1}\|} = \lim_{n \rightarrow \infty} \frac{\|T_\phi^{n-1} (T_{\phi \circ \phi_a - \phi})\|}{\|T_\phi^{n-1}\|} = 0.$$

If there does not exist $m \in \mathbb{Z}_+$ such that $T_\phi^m = 0$ then the inequality

$$\begin{aligned} n \|T_\phi^{n-1} T_{\phi \circ \phi_a - \phi}\| &= n \|T_\phi^{n-1} (T_\phi U_a - U_a T_\phi)\| = \|T_\phi^n U_a - U_a T_\phi^n\| \\ &\leq 2 \|T_\phi\| \|U_a\| \|T_\phi^{n-1}\| = 2 \|T_\phi\| \|T_\phi^{n-1}\| \end{aligned}$$

implies that $\frac{\|T_\phi^{n-1} (T_{\phi \circ \phi_a - \phi})\|}{\|T_\phi^{n-1}\|} \rightarrow 0$ as $n \rightarrow \infty$. Suppose now that $T_{\phi \circ \phi_a - \phi}$ is invertible and $\|T_{\phi \circ \phi_a - \phi}^{-1}\| \leq C$ for some constant $C > 0$. Then

$$\begin{aligned} 1 &= \frac{\|T_\phi^{n-1} T_{\phi \circ \phi_a - \phi} T_{\phi \circ \phi_a - \phi}^{-1}\|}{\|T_\phi^{n-1}\|} \\ &\leq \frac{\|T_\phi^{n-1} T_{\phi \circ \phi_a - \phi}\| \|T_{\phi \circ \phi_a - \phi}^{-1}\|}{\|T_\phi^{n-1}\|} \\ &\leq \frac{C \|T_\phi^{n-1} T_{\phi \circ \phi_a - \phi}\|}{\|T_\phi^{n-1}\|} \leq \frac{2C \|T_\phi\|}{n}. \end{aligned}$$

Taking limit as $n \rightarrow \infty$ both the sides we obtain a contradiction. Hence $T_{\phi \circ \phi_a - \phi}$ is not invertible. That is, $U_a T_\phi U_a - T_\phi = U_a (T_\phi U_a - U_a T_\phi)$ is not invertible.

Hence $T_\phi U_a - U_a T_\phi$ is not invertible and $\|I - (T_\phi U_a - U_a T_\phi)\| \geq 1$. That is,

$$\|U_a - T_{\phi \circ \phi_a - \phi}\| = \|U_a - T_{\phi \circ \phi_a} + T_\phi\| = \|I - (T_\phi U_a - U_a T_\phi)\| \geq 1.$$

This completes the proof of **(i)**. To prove **(ii)**, assume T_ϕ is positive. Then $\|T_\phi^m\| = \|T_\phi\|^m$ for all $m \in \mathbb{Z}_+$. Thus it follows that

$$\begin{aligned} n\|T_\phi\|^{n-1} &= n\|T_\phi^{n-1}\| \\ &\leq \|T_\phi^n U_a - U_a T_\phi^n\| + \left\| \sum_{i=0}^{n-1} T_\phi^{n-1-i} (I - (T_\phi U_a - U_a T_\phi)) T_\phi^i \right\| \\ &\leq 2\|T_\phi\| \|U_a\| \|T_\phi^{n-1}\| + \|I - (T_\phi U_a - U_a T_\phi)\| \sum_{i=0}^{n-1} \|T_\phi^{n-1-i}\| \|T_\phi^i\| \\ &= 2\|T_\phi\| \|T_\phi\|^{n-1} + \|I - (T_\phi U_a - U_a T_\phi)\| \sum_{i=0}^{n-1} \|T_\phi\|^{n-1-i} \|T_\phi\|^i \\ &= 2\|T_\phi\|^n + n\|I - (T_\phi U_a - U_a T_\phi)\| \|T_\phi\|^{n-1}. \end{aligned}$$

Thus

$$1 \leq \frac{2}{n} \|T_\phi\| + \|I - (T_\phi U_a - U_a T_\phi)\| \text{ for all } n \in \mathbb{Z}_+.$$

If now $\|I - (T_\phi U_a - U_a T_\phi)\| < 1$ then we get a contradiction. Hence

$$\|U_a - T_{\phi \circ \phi_a - \phi}\| = \|I - (T_\phi U_a - U_a T_\phi)\| \geq 1.$$

□

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