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# JORDAN-VON NEUMANN CONSTANT FOR BANAŚ-FRĄCZEK SPACE

CHANGSEN YANG

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Abstract. For any $\lambda \geq 1$, $\mathbb{R}^2_\lambda$ is Banaś-Frączek space, the exact value of the Jordan–von Neumann constant $C_{NJ}(\mathbb{R}^2_\lambda)$ is investigated. By careful calculations, $C_{NJ}(\mathbb{R}^2_\lambda) = 2 - \frac{1}{\lambda^2}$ is given.

## 1. Introduction

In order to study the geometric structure of Banach spaces, many recent studies have focused on the Jordan–von Neumann (NJ)constant. It is proved that the NJ constant is strongly connected with some geometric structures, such as uniform non-squareness and uniform normal structure. Hence many papers have appeared for how to compute the NJ constant.

Throughout this paper, $X$ is a nontrivial Banach space. We will use $B_X$, $S_X$ and $ex(B_X)$ to denote the unit ball, unit sphere of $X$ and the set of extreme points of $B_X$, respectively.

Recall that the Jordan–von Neumann (NJ) constant of a Banach space $X$ was introduced by Clarkson [2] as the smallest constant C for which

$$\frac{1}{C} \leq \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} \leq C$$

holds for all $x, y \in X$ with $(x, y) \neq (0, 0)$.

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An equivalent definition of the Jordan–von Neumann constant is found in [4] as the following form:

$$C_{NJ}(X) = \sup\{\frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} : x \in S_X, y \in B_X\}.$$

We know that (see [9]) the Jordan–von Neumann constant $C_{NJ}(X)$ can be computed by

$$C_{NJ}(X) = \sup\{\frac{\gamma_X(t)}{1+t^2} : 0 \le t \le 1\},$$

where $\gamma_X(t) = \sup\{\frac{\|x+ty\|^2+\|x-ty\|^2}{2} : x, y \in S_X\}$.

We also note that $\gamma_X(t) = \sup\{\frac{\|x+ty\|^2+\|x-ty\|^2}{2} : x, y \in ex(B_X)\}$ for a finite dimensional Banach space $X$. Therefore, for a finite dimensional Banach space $X$,

$$C_{NJ}(X) = \sup\{\frac{\|x+ty\|^2 + \|x-ty\|^2}{2(1+t^2)} : x, y \in ex(B_X), 0 \le t \le 1\}. \tag{1.1}$$

The modulus of convexity of a Banach space $X$ is defined for $\varepsilon \in [0, 2]$ as

$$\delta_X(\epsilon) = \inf\{1 - \frac{\|x+y\|}{2}, x, y \in S_X, \|x-y\| \ge \varepsilon\}.$$

The function $\delta_X(\epsilon)$ is continuous on $[0, 2)$ and strictly increasing on $[\epsilon_0(X), 2]$, where $\epsilon_0(X) = \sup\{\epsilon \in [0, 2], \delta_X(\epsilon) = 0\}$ is the characteristic of convexity of $X$. The non-square or James constant is defined as

$$J(X) = \sup\{\min(\|x+y\|, \|x-y\|), x, y \in S_X\}.$$

In [6, 7, 8], an important relationship between the James constant and the NJ constant is given as follows,

$$C_{NJ}(X) \le J(X).$$

In [1], the authors considered the spaces $\mathbb{R}^2_\lambda := (\mathbb{R}^2, \|\cdot\|_\lambda)$, where $\lambda > 1$ and

$$\|(a, b)\| = \max\{\lambda|a|, \sqrt{a^2 + b^2}\}.$$

They asserted that

$$\delta_{\mathbb{R}^2_\lambda}(\varepsilon) = \begin{cases} 0 & \text{for } \ 0 \le \varepsilon \le 2\sqrt{1 - \frac{1}{\lambda^2}}, \\ 1 - \lambda\sqrt{1 - \frac{\varepsilon^2}{4}} & \text{for } \ 2\sqrt{1 - \frac{1}{\lambda^2}} \le \varepsilon \le \frac{2\lambda}{\sqrt{1+\lambda^2}}, \\ 1 - \sqrt{1 - \frac{\varepsilon^2}{4\lambda^2}} & \text{for } \ \frac{2\lambda}{\sqrt{1+\lambda^2}} \le \varepsilon \le 2. \end{cases}$$

Thus, the James constant $J(X)$ can be easily computed as $J(X) = \frac{2\lambda}{\sqrt{1+\lambda^2}}$ by the modulus of convexity $\delta_{\mathbb{R}^2_\lambda}(\varepsilon)$ and the following formula [3]

$$J(X) = \sup\{\varepsilon \in (0, 2) : \delta_X(\varepsilon) \le 1 - \frac{\varepsilon}{2}\}.$$

Naturally we want to ask "what is its NJ constant?" In this note, we prove that $C_{NJ}(\mathbb{R}^2_\lambda) = 2 - \frac{1}{\lambda^2}$ , for any $\lambda \ge 1$.

## 2. Main results

The following theorem is our main result in this paper.

**Theorem 2.1.** *Let $\lambda \geq 1$ and $\mathbb{R}^2_\lambda$ is the Banaś–Frączek space. Then,*

$$C_{NJ}(\mathbb{R}^2_\lambda) = 2 - \frac{1}{\lambda^2}.$$

In order to prove this theorem, first we give the following lemmas.

**Lemma 2.2.** *If $\lambda \geq \sqrt{2}$ and $|x_1| \leq \frac{1}{\lambda}$, $|y_1| \leq \frac{1}{\lambda}$, then*

$$(\lambda^2 - 1)|x_1 y_1| + \sqrt{1 - x_1^2}\sqrt{1 - y_1^2} \leq 2 - \frac{2}{\lambda^2}. \tag{2.1}$$

*Proof.* By $|x_1| \leq \frac{1}{\lambda}$ and $|y_1| \leq \frac{1}{\lambda}$, we have

$$(\lambda^2 - 2)|x_1 y_1| \leq \frac{\lambda^2 - 2}{\lambda^2},$$

that is

$$1 - |x_1 y_1| \leq (\lambda^2 - 1)(\frac{2}{\lambda^2} - |x_1 y_1|).$$

Therefore

$$\sqrt{1 - x_1^2}\sqrt{1 - y_1^2} \leq 1 - |x_1 y_1| \leq (\lambda^2 - 1)(\frac{2}{\lambda^2} - |x_1 y_1|).$$

Thus, (2.1) is valid. □

**Lemma 2.3.** *Let $0 \leq \tau \leq 1, 1 < \lambda < \sqrt{2}$ and $0 \leq x_1, y_1 \leq \frac{1}{\lambda}$. If $F(x_1, y_1) = \lambda^2(x_1^2 + \tau^2 y_1^2) + 2\tau x_1 y_1(\lambda^2 - 1) + 2\tau\sqrt{1 - x_1^2}\sqrt{1 - y_1^2}$, then*

$$\max\{F(x_1, y_1) : 0 \leq x_1, y_1 \leq \frac{1}{\lambda}\}$$
$$= \max\{F(x_1, y_1) : (x_1, y_1) \in \partial\{[0, \frac{1}{\lambda}] \times [0, \frac{1}{\lambda}]\}\},$$

*where $\partial\{[0, \frac{1}{\lambda}] \times [0, \frac{1}{\lambda}]\}$ denotes the boundary of the square $[0, \frac{1}{\lambda}] \times [0, \frac{1}{\lambda}]$.*

*Proof.* Assume that $\tau \in (0, 1]$. Suppose that $\max\{F(x_1, y_1) : 0 \leq x_1, y_1 \leq \frac{1}{\lambda}\}$ attains at some $(x_1, y_1) \in (0, \frac{1}{\lambda}) \times (0, \frac{1}{\lambda})$, then $F_{x_1}(x_1, y_1) = F_{y_1}(x_1, y_1) = 0$. And this implies

$$\lambda^2 + \tau(\lambda^2 - 1)\frac{y_1}{x_1} = \tau\sqrt{\frac{1 - y_1^2}{1 - x_1^2}}, \tag{2.2}$$

and

$$\tau\lambda^2 + (\lambda^2 - 1)\frac{x_1}{y_1} = \sqrt{\frac{1 - x_1^2}{1 - y_1^2}}. \tag{2.3}$$

Now, through (2.2) multiplied by (2.3), then we have

$$\tau = \tau\lambda^4 + \lambda^2(\lambda^2 - 1)\frac{x_1}{y_1} + \tau^2\lambda^2(\lambda^2 - 1)\frac{y_1}{x_1} + \tau(\lambda^2 - 1)^2. \tag{2.4}$$

But (2.4) is equivalent to

$$0 = 2\tau\lambda^2 + \lambda^2 \frac{x_1}{y_1} + \tau^2\lambda^2 \frac{y_1}{x_1},$$

and which is a contradiction. □

**Lemma 2.4.** *(i) If* $1 \le \lambda \le \sqrt{2}$ *and* $0 \le z \le \frac{1}{\lambda^2}$*, then*

$$9\lambda^4 z^2 + 48z + 25 - 34\lambda^2 z - 8\lambda^2 z^2 - \frac{16}{\lambda^2} z + \frac{16}{\lambda^4} - \frac{40}{\lambda^2} \ge 0.$$

*(ii) If* $0 \le y \le \frac{1}{\lambda}$ *and* $1 \le \lambda \le \sqrt{2}$*, then*

$$2y\sqrt{\lambda^2 - 1}\sqrt{1 - y^2} \le 5 + 2y^2 - 3\lambda^2 y^2 - \frac{4}{\lambda^2}. \tag{2.5}$$

*Proof.* (i) Now, by letting $f(z) = 9\lambda^4 z^2 + 48z + 25 - 34\lambda^2 z - 8\lambda^2 z^2 - \frac{16}{\lambda^2} z + \frac{16}{\lambda^4} - \frac{40}{\lambda^2}$, then we have

$$f'(z) = 18\lambda^4 z + 48 - 34\lambda^2 - 16\lambda^2 z - \frac{16}{\lambda^2}.$$

By $f''(z) = 18\lambda^4 - 16\lambda^2 > 0$, we have

$$f'(z) \le f'(\frac{1}{\lambda^2}) = 32 - 16(\lambda^2 + \frac{1}{\lambda^2}) \le 0.$$

Thus, $f(z) \ge f(\frac{1}{\lambda^2}) = 0$.

(ii) Because (2.5) is equivalent to

$$4y^2(\lambda^2 - 1)(1 - y^2) \le 25 + 4y^4 + 20y^2 - (10 + 4y^2)(3\lambda^2 y^2 + \frac{4}{\lambda^2}) + 9\lambda^4 y^4 + \frac{16}{\lambda^4} + 24y^2,$$

that is

$$34\lambda^2 y^2 + 8\lambda^2 y^4 + \frac{16}{\lambda^2} y^2 + \frac{40}{\lambda^2} \le 9\lambda^4 y^4 + \frac{16}{\lambda^4} + 48y^2 + 25.$$

Now, by letting $z = y^2$, we can obtain (2.5) by (i). □

**Lemma 2.5.** *(i) Let* $A, B > 0$ *and* $\tau \in [0, 1]$*, then*

$$\max\{\frac{A + B\tau}{2(1 + \tau^2)}\} \le \frac{A + \sqrt{A^2 + B^2}}{4}. \tag{2.6}$$

*(ii) Let* $1 < \lambda \le \sqrt{2}, 0 \le y \le \frac{1}{\lambda}$ *and* $\tau \in [0, 1]$*, then*

$$\frac{1}{2} + \frac{1 + \lambda^2\tau^2 y^2 + \frac{2\tau}{\lambda}[(\lambda^2 - 1)y + \sqrt{\lambda^2 - 1}\sqrt{1 - y^2}]}{2(1 + \tau^2)} \le 2 - \frac{1}{\lambda^2}. \tag{2.7}$$

*Proof.* (i) Obviously, the function $g(\tau) \equiv \frac{A + B\tau}{2(1 + \tau^2)}$ attains its maximum at $\tau = \frac{\sqrt{A^2 + B^2} - A}{B}$, hence (2.6) is valid.

(ii) By use of (2.6), we have

$$\begin{aligned}
&\frac{1}{2}+\frac{1+\lambda^2\tau^2y^2+\frac{2\tau}{\lambda}[(\lambda^2-1)y+\sqrt{\lambda^2-1}\sqrt{1-y^2}]}{2(1+\tau^2)}\\
&=\frac{1+\lambda^2y^2}{2}+\frac{1-\lambda^2y^2+\frac{2\tau}{\lambda}[(\lambda^2-1)y+\sqrt{\lambda^2-1}\sqrt{1-y^2}]}{2(1+\tau^2)}\\
&\le\frac{3+\lambda^2y^2}{4}+\frac{\sqrt{(1-\lambda^2y^2)^2+\frac{4}{\lambda^2}[(\lambda^2-1)y+\sqrt{\lambda^2-1}\sqrt{1-y^2}]^2}}{4}.
\end{aligned}$$

So, we only need to prove the following inequality

$$\sqrt{(1-\lambda^2y^2)^2+\frac{4}{\lambda^2}[(\lambda^2-1)y+\sqrt{\lambda^2-1}\sqrt{1-y^2}]^2}\le 5-\frac{4}{\lambda^2}-\lambda^2y^2. \tag{2.8}$$

Now, (2.8) is equivalent to the following inequality

$$\begin{aligned}
&(1-\lambda^2y^2)^2+\frac{4}{\lambda^2}[(\lambda^2-1)y+\sqrt{\lambda^2-1}\sqrt{1-y^2}]^2\\
&\le[(1-\lambda^2y^2)+4(1-\frac{1}{\lambda^2})]^2.
\end{aligned} \tag{2.9}$$

(2.9) can be changed into

$$\frac{4}{\lambda^2}[\sqrt{\lambda^2-1}y+\sqrt{1-y^2}]^2\le\frac{16(\lambda^2-1)}{\lambda^4}+\frac{8}{\lambda^2}(1-\lambda^2y^2). \tag{2.10}$$

By a simple computation, we have that (2.10) is equivalent to (2.5). □

**Lemma 2.6.** *Let* $1<\lambda\le\sqrt{2},\tau\in[0,1]$ *and* $0\le y\le\frac{1}{\lambda}$, *then*

$$\frac{1}{2}+\frac{\lambda^2\tau^2y^2+2\tau\sqrt{1-y^2}}{2(1+\tau^2)}\le 2-\frac{1}{\lambda^2}, \tag{2.11}$$

*and*

$$\frac{1}{2}+\frac{\lambda^2y^2+2\tau\sqrt{1-y^2}}{2(1+\tau^2)}\le 2-\frac{1}{\lambda^2}. \tag{2.12}$$

*Proof.* Obviously, (2.12) implies (2.11).

Now, we prove (2.12). By $\lambda^2(1-y^2)+\frac{\tau^2}{\lambda^2}\ge 2\tau\sqrt{1-y^2}$, we have $\lambda^2y^2+2\tau\sqrt{1-y^2}\le\lambda^2+\frac{\tau^2}{\lambda^2}$. Hence

$$\frac{1}{2}+\frac{\lambda^2y^2+2\tau\sqrt{1-y^2}}{2(1+\tau^2)}\le\frac{1}{2}+\frac{\lambda^2+\frac{\tau^2}{\lambda^2}}{2(1+\tau^2)}\le\frac{1}{2}+\frac{\lambda^2}{2}\le 2-\frac{1}{\lambda^2},$$

where the last inequality is obtained by $(\lambda^2-1)(\lambda^2-2)\le 0$. □

**Proof of Theorem2.1**

Assume that $\lambda>1$. Note that $ex(B_X)=\{(z_1,z_2):z_1^2+z_2^2=1,|z_1|\le\frac{1}{\lambda}\}$.

Now we prove that

$$\frac{\|x+\tau y\|^2+\|x-\tau y\|^2}{2(1+\tau^2)}\le 2-\frac{1}{\lambda^2}, \tag{2.13}$$

holds for any $x,y\in ex(B_X)$ and any $\tau\in[0,1]$.

Case(I). $\lambda\ge\sqrt{2}$. Letting $x=(x_1,x_2), y=(y_1,y_2)$, then we have the following three cases.

Ia). If $\|x+\tau y\|_2\le|\lambda(x_1+\tau y_1)|$ and $\|x-\tau y\|_2\le|\lambda(x_1-\tau y_1)|$, then

$$\begin{aligned}\|x+\tau y\|^2+\|x-\tau y\|^2 &= \lambda^2[(x_1+\tau y_1)^2+(x_1-\tau y_1)^2]\\ &= 2\lambda^2(x_1^2+\tau^2y_1^2)\le 2(1+\tau^2).\end{aligned} \tag{2.14}$$

Ib). If $\|x+\tau y\|_2>|\lambda(x_1+\tau y_1)|$ and $\|x-\tau y\|_2>|\lambda(x_1-\tau y_1)|$, then

$$\|x+\tau y\|^2+\|x-\tau y\|^2=\|x+\tau y\|_2^2+\|x-\tau y\|_2^2=2(1+\tau^2). \tag{2.15}$$

Ic). If $\|x+\tau y\|_2\le|\lambda(x_1+\tau y_1)|$ and $\|x-\tau y\|_2>|\lambda(x_1-\tau y_1)|$, or $\|x+\tau y\|_2>|\lambda(x_1+\tau y_1)|$ and $\|x-\tau y\|_2\le|\lambda(x_1-\tau y_1)|$, then

$$\begin{aligned}\|x+\tau y\|^2+\|x-\tau y\|^2 &= \lambda^2(x_1\pm\tau y_1)^2+(x_1\mp\tau y_1)^2+(x_2\mp\tau y_2)^2\\ &\le 1+\tau^2+\lambda^2(x_1^2+\tau^2y_1^2)+2\tau(\lambda^2-1)|x_1y_1|\\ &+ 2\tau\sqrt{1-x_1^2}\sqrt{1-y_1^2}\\ &\le 2(1+\tau^2)+2\tau(2-\frac{2}{\lambda^2}),\end{aligned} \tag{2.16}$$

holds by Lemma 2.2. Hence, (2.14)-(2.16) imply (2.13).

Case(II) $1<\lambda\le\sqrt{2}$. Letting $x=(x_1,x_2), y=(y_1,y_2)$ again, then (2.14) and (2.15) is also valid. For the third case, first we consider the following function $F(u,v)=\lambda^2(u^2+\tau^2v^2)+2\tau(\lambda^2-1)uv+2\tau\sqrt{1-u^2}\sqrt{1-v^2}$, where $u,v\in[0,\frac{1}{\lambda}]$. By applying (2.11),(2.12) and (2.7), we have

$$\begin{aligned}\frac{1}{2}+\frac{F(0,v)}{2(1+\tau^2)} &= \frac{1}{2}+\frac{\lambda^2\tau^2v^2+2\tau\sqrt{1-v^2}}{2(1+\tau^2)}\le 2-\frac{1}{\lambda^2}; \qquad (2.17)\\ \frac{1}{2}+\frac{F(u,0)}{2(1+\tau^2)} &= \frac{1}{2}+\frac{\lambda^2u^2+2\tau\sqrt{1-u^2}}{2(1+\tau^2)}\le 2-\frac{1}{\lambda^2};\\ \frac{1}{2}+\frac{F(\frac{1}{\lambda},v)}{2(1+\tau^2)} &= \frac{1}{2}+\frac{1+\lambda^2\tau^2v^2+\frac{2\tau}{\lambda}[(\lambda^2-1)v+\sqrt{\lambda^2-1}\sqrt{1-v^2}]}{2(1+\tau^2)}\\ &\le 2-\frac{1}{\lambda^2}\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2}+\frac{F(u,\frac{1}{\lambda})}{2(1+\tau^2)} &= \frac{1}{2}+\frac{\tau^2+\lambda^2u^2+\frac{2\tau}{\lambda}[(\lambda^2-1)u+\sqrt{\lambda^2-1}\sqrt{1-u^2}]}{2(1+\tau^2)}\\ &\le \frac{1}{2}+\frac{1+\lambda^2\tau^2u^2+\frac{2\tau}{\lambda}[(\lambda^2-1)u+\sqrt{\lambda^2-1}\sqrt{1-u^2}]}{2(1+\tau^2)}\\ &\le 2-\frac{1}{\lambda^2}. \end{aligned} \tag{2.18}$$

Hence, by applying (2.17)-(2.18) and Lemma 2.3, we have that

$$\begin{aligned}&\frac{\|x+\tau y\|^2+\|x-\tau y\|^2}{2(1+\tau^2)}\\ &\le \frac{1}{2}+\frac{F(|x_1|,|y_1|)}{2(1+\tau^2)}\\ &\le \frac{1}{2}+\max\{\frac{F(u,v)}{2(1+\tau^2)};(u,v)\in\partial[0,\frac{1}{\lambda}]\times[0,\frac{1}{\lambda}]\}\\ &\le 2-\frac{1}{\lambda^2}.\end{aligned}$$

Therefore, (2.13) is valid. Whence (2.13) and (1.1) imply $C_{NJ}(\mathbb{R}^2_\lambda)\le 2-\frac{1}{\lambda^2}$. On the other hand, if taking $x=(\frac{1}{\lambda},\sqrt{1-\frac{1}{\lambda^2}}), y=(\frac{1}{\lambda},-\sqrt{1-\frac{1}{\lambda^2}})$, we have

$$C_{NJ}(\mathbb{R}^2_\lambda)\ge\frac{\|x+y\|^2+\|x-y\|^2}{4}=2-\frac{1}{\lambda^2},$$

which completes the proof of Theorem 2.1.

**Corollary 2.7.** *If $1\le\lambda<1+\frac{\sqrt{3}}{3}$, then Banaś–Frączek space $\mathbb{R}^2_\lambda$ and its dual space have uniform normal structure.*

*Proof.* Because $1\le\lambda<1+\frac{\sqrt{3}}{3}$ implies

$$C_{NJ}(\mathbb{R}^2_\lambda)\le 2-\frac{1}{\lambda^2}<\frac{1+\sqrt{3}}{2}.$$

Hence, by a Satit Saejung's result (see [5]), we complete the proof of Corollary 2.7. □

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College of Mathematics and Information Science, University of Henan Normal, Xinxiang 453007, Henan, P.R.China.
*E-mail address*: yangchangsen0991@sina.com