Banach J. Math. Anal. 5 (2011), no. 1, 83–87

Banach Journal of Mathematical Analysis
ISSN: 1735-8787 (electronic)
www.emis.de/journals/BJMA/

CHARACTERIZATIONS OF INNER PRODUCT SPACES BY STRONGLY CONVEX FUNCTIONS

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Communicated by M. S. Moslehian

Abstract. New characterizations of inner product spaces among normed spaces involving the notion of strong convexity are given. In particular, it is shown that the following conditions are equivalent: (1) $(X, \|\cdot\|)$ is an inner product space; (2) $f : X \to \mathbb{R}$ is strongly convex with modulus $c > 0$ if and only if $f - c\|\cdot\|^2$ is convex; (3) $\|\cdot\|^2$ is strongly convex with modulus 1.

1. Introduction

It is well known that in a normed space $(X, \|\cdot\|)$ the following Jordan–von Neumann parallelogram law

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2, \quad x, y \in X,$$

holds if and only if the norm $\|\cdot\|$ is derivable from an inner product (cf.[8], [5]). In the literature one can find many other conditions characterizing inner product spaces among normed spaces. A rich collection of such characterizations is contained in the celebrated book of Amir [5] (cf. also [1, Chpt. 11], [2], [3], [4], [6], [11]). The aim of this note is to present some new results of this type involving strongly convex and strongly midconvex functions.

Date: Received: 19 April 2010; Accepted: 25 June 2010.

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2010 *Mathematics Subject Classification.* Primary 46C15. Secondary 26B25, 39B62.

Key words and phrases. Inner product space, strongly convex function, strongly midconvex function.

This research of the second author has been supported by the Hungarian Scientific Research Fund (OTKA) Grants NK-68040, NK81402.

In what follows $(X, \|\cdot\|)$ is a real normed space, D stands for a convex subset of X and c is a positive constant. A function $f : D \to \mathbb{R}$ is called *strongly convex with modulus* c if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) - ct(1-t)\|x-y\|^2, \tag{1.1}$$

for all $x, y \in D$ and $t \in (0,1)$. We say that f is *strongly midconvex with modulus* c if (1.1) is assumed only for $t = 1/2$, that is

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2} - \frac{c}{4}\|x-y\|^2, \qquad x, y \in D. \tag{1.2}$$

Recall also that f is convex (midconvex) if it satisfies (1.1) ((1.2), respectively) with $c = 0$. Strongly convex functions have been introduced by Polyak [10] and they play an important role in optimization theory. Many properties of them can be found, among other, in [7], [9], [12], [13]. The following result gives relationships between strongly convex (strongly midconvex) and convex (midconvex) functions. In the case where $X = \mathbb{R}^n$ the first part of this result can be found in [7, Prop. 1.1.2].

2. Main result

We start this section with a useful lemma.

Lemma 2.1. *Let $(X, \|\cdot\|)$ be a real inner product space, D be a convex subset of X and c be a positive constant.*

1. *A function $f : D \to \mathbb{R}$ is strongly convex with modulus c if and only if the function $g = f - c\|\cdot\|^2$ is convex.*
2. *A function $f : D \to \mathbb{R}$ is strongly midconvex with modulus c if and only if the function $g = f - c\|\cdot\|^2$ is midconvex.*

Proof. 1. Assume that f is strongly convex with modulus c. Using elementary properties of the inner product and the fact that $\|x\|^2 = \langle x|x\rangle$, we get

$$\begin{aligned}
g(tx + (1-t)y) &= f(tx + (1-t)y) - c\|tx + (1-t)y\|^2 \\
&\le tf(x) + (1-t)f(y) - ct(1-t)\|x-y\|^2 - c\|tx + (1-t)y\|^2 \\
&\le tf(x) + (1-t)f(y) - c\Big(t(1-t)\big(\|x\|^2 - 2\langle x|y\rangle + \|y\|^2\big) \\
&\qquad + t^2\|x\|^2 + 2t(1-t)\langle x|y\rangle + (1-t)^2\|y\|^2\Big) \\
&= tf(x) + (1-t)f(y) - ct\|x\|^2 - c(1-t)\|y\|^2 \\
&= tg(x) + (1-t)g(y),
\end{aligned}$$

which proves that g is convex.
Conversely, if g is convex, then

$$\begin{aligned}f(tx+(1-t)y) &= g(tx+(1-t)y)+c\|tx+(1-t)y\|^2\\ &\le tg(x)+(1-t)g(y)+c\big(t^2\|x\|^2+2t(1-t)\langle x|y\rangle+(1-t)^2\|y\|^2\big)\\ &= t\left(g(x)+c\|x\|^2\right)+(1-t)\left(g(y)+c\|y\|^2\right)\\ &\quad -ct(1-t)\left(\|x\|^2-2\langle x|y\rangle+\|y\|^2\right)\\ &= f(x)+(1-t)f(y)-ct(1-t)\|x-y\|^2,\end{aligned}$$

which shows that f is strongly convex with modulus c.

2. Assume now that f is strongly midconvex with modulus c. Using the parallelogram law we get

$$\begin{aligned}g\Big(\frac{x+y}{2}\Big) &= f\Big(\frac{x+y}{2}\Big)-c\Big\|\frac{x+y}{2}\Big\|^2\\ &\le \frac{f(x)+f(y)}{2}-\frac{c}{4}\|x-y\|^2-\frac{c}{4}\|x+y\|^2\\ &= \frac{f(x)+f(y)}{2}-\frac{c}{4}\big(2\|x\|^2+2\|y\|^2\big)=\frac{g(x)+g(y)}{2}\ .\end{aligned}$$

Similarly, if g is midconvex, then

$$\begin{aligned}f\Big(\frac{x+y}{2}\Big) &= g\Big(\frac{x+y}{2}\Big)+c\Big\|\frac{x+y}{2}\Big\|^2\le\frac{g(x)+g(y)}{2}+\frac{c}{4}\|x+y\|^2\\ &= \frac{g(x)+\|x\|^2}{2}+\frac{g(y)+\|y\|^2}{2}+\frac{c}{4}\big(\|x+y\|^2-2\|x\|^2-2\|y\|^2\big)\\ &= \frac{f(x)+f(y)}{2}-\frac{c}{4}\|x-y\|^2.\end{aligned}$$

□

The following example shows that the assumption that X is an inner product space is essential in the above lemma.

Example 2.2. Let $X=\mathbb{R}^2$ and $\|x\|=|x_1|+|x_2|$, for $x=(x_1,x_2)$. Take $f=\|\cdot\|^2$. Then $g=f-\|\cdot\|^2$ is convex being the zero function. However, f is neither strongly convex nor strongly midconvex with modulus 1. Indeed, for $x=(1,0)$ and $y=(0,1)$ we have

$$f\Big(\frac{x+y}{2}\Big)=1>0=\frac{f(x)+f(y)}{2}-\frac{1}{4}\|x-y\|^2,$$

which contradicts (1.2).

It appears that something stronger can be proved: the assumption that X is an inner product space is necessary in Lemma 2.1. Namely, the following characterizations of inner product spaces hold.

Theorem 2.3. *Let $(X,\|\cdot\|)$ be a real normed space. The following conditions are equivalent to each other:*

1. *For all $c>0$ and for all functions $f:D\to\mathbb{R}$, f is strongly convex with modulus c if and only if $g=f-c\|\cdot\|^2$ is convex;*

2. *For all* $c > 0$ *and for all functions* $f : D \to \mathbb{R}$, f *is strongly midconvex with modulus* c *if and only if* $g = f - c\|\cdot\|^2$ *is midconvex;*
3. *There exists* $c > 0$ *such that, for all functions* $f : D \to \mathbb{R}$, g *is convex if and only if* $f = g + c\|\cdot\|^2$ *is strongly convex with modulus* c*;*
4. *There exists* $c > 0$ *such that, for all functions* $f : D \to \mathbb{R}$, g *is midconvex if and only if* $f = g + c\|\cdot\|^2$ *is strongly midconvex with modulus* c*;*
5. $\|\cdot\|^2 : X \to \mathbb{R}$ *is strongly convex with modulus* 1*;*
6. $\|\cdot\|^2 : X \to \mathbb{R}$ *is strongly midconvex with modulus* 1*;*
7. $(X, \|\cdot\|)$ *is an inner product space.*

Proof. We will show the following chains of implications: *1* $\Rightarrow$ *3* $\Rightarrow$ *5* $\Rightarrow$ *7* $\Rightarrow$ *1* and *2* $\Rightarrow$ *4* $\Rightarrow$ *6* $\Rightarrow$ *7* $\Rightarrow$ *2*.

Implications *1* $\Rightarrow$ *3* and *2* $\Rightarrow$ *4* are obvious. To show *3* $\Rightarrow$ *5* and *4* $\Rightarrow$ *6* take $g = 0$. Then $f = c\|\cdot\|^2$ is strongly convex (resp. strongly midconvex) with modulus c. Consequently, $\frac{1}{c}f = \|\cdot\|^2$ is strongly convex (resp. strongly midconvex) with modulus 1.

To see that *5* $\Rightarrow$ *7* and *6* $\Rightarrow$ *7* also hold, observe that, by the strong convexity or strong midconvexity with modulus 1 of $\|\cdot\|^2$ we have

$$\left\|\frac{x+y}{2}\right\|^2 \leq \frac{\|x\|^2 + \|y\|^2}{2} - \frac{1}{4}\|x-y\|^2 \ ,$$

and hence

$$\|x+y\|^2 + \|x-y\|^2 \leq 2\|x\|^2 + 2\|y\|^2 \tag{2.1}$$

for all $x, y \in X$. Now, putting $u = x + y$ and $v = x - y$ in (2.1), we get

$$2\|u\|^2 + 2\|v\|^2 \leq \|u+v\|^2 + \|u-v\|^2, \qquad u, v \in X. \tag{2.2}$$

Conditions (2.1) and (2.2) mean that the norm $\|\cdot\|$ satisfies the parallelogram law, which implies that $(X, \|\cdot\|)$ is an inner product space.

Implications *7* $\Rightarrow$ *1* and *7* $\Rightarrow$ *2* follow by Lemma 2.1. $\square$

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