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GENERALIZATION OF AN INTEGRAL FORMULA OF GUESSAB AND SCHMEISSER

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Abstract. Weighted version of two-point integral quadrature formula is obtained using $w-$harmonic sequences of functions. Improved version of Guessab and Schmeisser's result is given with new integral inequalities under various regular conditions. As special cases, the generalizations of quadrature formulae of Gauss type are established.

1. Introduction and preliminaries

In this paper we consider the two-point quadrature formulae of the following type:

$$\int_a^b f(t)w(t)\mathrm{d}t = A_w(x)f(x) + B_w(x)f(a+b-x) + E(f,x). \tag{1.1}$$

Here, $x \in [a, \frac{a+b}{2}]$, $w : [a,b] \to \mathbf{R}$ is an integrable function called weight, f is an integrable function defined on $[a,b]$, $E(f,x)$ is a remainder and $A_w(x), B_w(x)$ are coefficients such that $A_w(x) + B_w(x) = \int_a^b w(t)dt$.
Recently, A. Guessab and G. Schmeisser ([1]) studied a class of two-point formulae for $w \equiv \frac{1}{b-a}$:

$$\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t = \frac{1}{2}\left(f(x) + f(a+b-x)\right) + E(f,x). \tag{1.2}$$

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They established sharp estimates for the remainder under various regularity conditions. They proved the following theorem

Theorem 1.1. *Let f be a function defined on $[a, b]$ and having there a piecewise continuous $n-th$ derivative. Let Q_n be any monic polynomial of degree n such that $Q_n(t) \equiv (-1)^n Q_n(a + b - t)$. Define*

$$K_n(t) = \begin{cases} (t-a)^n, & \text{for } a \le t \le x \\ Q_n(t), & \text{for } x < t \le a+b-x \\ (t-b)^n, & \text{for } a+b-x < t \le b. \end{cases} \tag{1.3}$$

Then, for the remainder in (1.2), we have

$$\begin{aligned} E(f;x) &= \sum_{\nu=1}^{n-1} \Big[\frac{(x-a)^{\nu+1}}{(\nu+1)!} - \frac{Q_n^{(n-\nu-1)}(x)}{n!} \Big] \frac{f^{(\nu)}(a+b-x) + (-1)^\nu f^{(\nu)}(x)}{b-a} \\ &+ \frac{(-1)^n}{n!(b-a)} \int_a^b K_n(t) f^{(n)}(t) \mathrm{d}t. \end{aligned} \tag{1.4}$$

A number of error estimates for the identity (1.4) are obtained, and various examples of the general two-point quadrature formula are given in [3].
The goal of this paper is to establish analogous formula for the weighted case, i.e. such case where the integrand can be written as product of two function $f(t)$ and $w(t)$. The main tool used are the $w-$harmonic sequences of functions and related weighted integral identity obtained in [2].

Definition 1.2. Let $w : [a, b] \to \mathbf{R}$ be an integrable weight function and $w_k : [a, b] \to \mathbf{R}$ be differentiable functions for $k \in \mathbf{N}$. We say that $\{w_k\}_{k \in \mathbf{N}}$ is a $w-$**harmonic sequence of functions** if for $k \ge 2$, $w_k'(t) = w_{k-1}(t)$ and $w_1'(t) = w(t)$, for $t \in [a, b]$.

Given a subdivision $\sigma = \{a = x_0 < x_1 < \ldots < x_m = b\}$ of the interval $[a, b]$, let us consider different $w-$harmonic sequences of functions $\{w_{jk}\}_{k \in \mathbf{N}}$ on each interval $[x_{j-1}, x_j]$, $j \in \{1, 2, \ldots, m\}$. Define

$$W_{n,w}(t, \sigma) = \begin{cases} w_{1n}(t), & t \in [a, x_1] \\ w_{2n}(t), & t \in (x_1, x_2] \\ \vdots & \\ w_{mn}(t), & t \in (x_{m-1}, b], \end{cases} \tag{1.5}$$

Then for every function $f : [a,b] \to \mathbf{R}$ such that $f^{(n)}$ is piecewise continuous on $[a,b]$ it is proved in [2] that

$$\begin{aligned}\int_a^b w(t)f(t)\mathrm{d}t &= \sum_{k=1}^{n}(-1)^{k-1}\Big[w_{mk}(b)f^{(k-1)}(b) \\ &+ \sum_{j=1}^{m-1}\left[w_{jk}(x_j)-w_{j+1,k}(x_j)\right]f^{(k-1)}(x_j)-w_{1k}(a)f^{(k-1)}(a)\Big] \\ &+ (-1)^n\int_a^b W_{n,w}(t,\sigma)f^{(n)}(t)\mathrm{d}t.\end{aligned} \tag{1.6}$$

The identity (1.6) is called weighted integral identity. It will be our starting point in deriving weighted two-point quadrature formulas. We shall observe functions f whose higher order derivatives belong to L_p spaces, and establish sharp and best possible constants for such inequalities. For special choices of weights w and nodes x and $a+b-x$ we shall get the generalization of the well-known two-point quadrature formulas of Gauss type. Even more, some new sharp and best possible constants will be established. We shall use a hypergeometric function

$$F(\alpha,\beta,\gamma;x)=\frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-xt)^{-\alpha}\mathrm{d}t,$$

for $\gamma-\beta>0$ and $|x|<1$, where B stands for the famous Beta function

$$B(u,v)=\int_0^1 x^{u-1}(1-x)^{v-1}\mathrm{d}x.$$

2. Main results

Let $w : [a,b] \to \mathbf{R}$ be some integrable function and $x \in [a, \frac{a+b}{2}]$. Consider a subdivision

$$\sigma := \{x_0 = a, x_1 = x, x_2 = a+b-x < x_3 = b\}$$

of $[a,b]$. Let $\{Q_{k,x}\}_{k\in\mathbf{N}}$ be sequence of polynomials such that $\deg Q_{k,x} \le k-1$, $Q'_{k,x}(t) = Q_{k-1,x}(t)$, $k \in \mathbf{N}$ and $Q_{0,x} \equiv 0$. Define functions $w_{jk}(t)$ on $[x_{j-1}, x_j]$, for $j = 1,2,3$ and $k \in \mathbf{N}$:

$$\begin{aligned}w_{1k}(t) &= \frac{1}{(k-1)!}\int_a^t (t-s)^{k-1}w(s)\mathrm{d}s \\ w_{2k}(t) &= \frac{1}{(k-1)!}\int_x^t (t-s)^{k-1}w(s)\mathrm{d}s + Q_{k,x}(t) \\ w_{3k}(t) &= -\frac{1}{(k-1)!}\int_t^b (t-s)^{k-1}w(s)\mathrm{d}s.\end{aligned} \tag{2.1}$$

Obviously, $\{w_{jk}\}_{k\in\mathbf{N}}$ are sequences of $w-$harmonic functions on $[x_{j-1}, x_j]$, for every $j = 1,2,3$. Let us define coefficients $A_k(x)$ and $B_k(x)$ by following:

$$A_k^2(x) = (-1)^{k-1}\left[\frac{1}{(k-1)!}\int_a^x (x-s)^{k-1}w(s)ds - Q_{k,x}(x)\right], \tag{2.2}$$

and

$$B_k^2(x) = (-1)^{k-1}\left[\frac{1}{(k-1)!}\int_x^b (a+b-x-s)^{k-1}w(s)ds + Q_{k,x}(a+b-x)\right]. \tag{2.3}$$

Let $f:[a,b]\to\mathbf{R}$ be such that $f^{(n-1)}$ exists on $[a,b]$ for some $n\in\mathbf{N}$. We introduce the following notation:

$$\begin{aligned} T_{n,w}(x) &= 0, \quad \text{for } n=1 \\ T_{n,w}(x) &:= \sum_{k=2}^{n}\left[A_k(x)f^{(k-1)}(x) + B_k(x)f^{(k-1)}(a+b-x)\right], \quad \text{for } n\geq 2. \end{aligned}$$

Theorem 2.1. *Let $f:[a,b]\to\mathbf{R}$ be such that $f^{(n)}$ is piecewise continuous on $[a,b]$, for some $n\in\mathbf{N}$. Then*

$$\begin{aligned} \int_a^b w(t)f(t)dt = A_1(x)f(x) + B_1(x)f(a+b-x) + T_{n,w}(x) \\ +(-1)^n\int_a^b W_{n,w}(t,x)f^{(n)}(t)dt, \end{aligned} \tag{2.4}$$

where

$$W_{n,w}(t,x) = \begin{cases} w_{1n}(t) & \text{for } t\in[a,x], \\ w_{2n}(t) & \text{for } t\in(x,a+b-x], \\ w_{3n}(t) & \text{for } t\in(a+b-x,b]. \end{cases}$$

Proof. Put (2.1) in identity (1.6). It follows

$$\begin{aligned} \int_a^b w(t)f(t)dt &= \sum_{k=1}^{n}(-1)^{k-1}\Big[\left[w_{1k}(x)-w_{2k}(x)\right]f^{(k-1)}(x) \\ &+ \left[w_{2k}(a+b-x)-w_{3k}(a+b-x)\right]f^{(k-1)}(a+b-x)\Big] \\ &+ (-1)^n\int_a^b W_{n,w}(t,x)f^{(n)}(t)dt, \end{aligned}$$

since $w_{1k}(a)=0$ and $w_{3k}(b)=0$. Further,

$$w_{1k}(x)-w_{2k}(x) = (-1)^{k-1}A_k(x)$$

and

$$w_{2k}(a+b-x)-w_{3k}(a+b-x) = (-1)^{k-1}B_k(x),$$

so the proof is finished. □

Remark 2.2. Let R_n be monic polynomial of degree n such that $R_n(t)=(-1)^nR_n(a+b-t)$. For $w(t)\equiv\frac{1}{b-a}$ and polynomials

$$Q_{k,x}(t) := \frac{R_n^{(n-k)}(t)}{n!(b-a)} - \frac{(t-x)^k}{k!(b-a)}, \quad k=0,1,\dots,n,$$

Guessab–Schmeisser's identity (1.4) is recovered from (2.4). Therefore, we can say that (2.4) is generalization of Guessab–Schmeisser's integral identity.

Remark 2.3. The polynomials $Q_{k,x}$ satisfy

$$Q_{k,x}(t) = \sum_{j=0}^{k-1} Q_{k-j,x}(x)\frac{(t-x)^j}{j!},$$

so the polynomial $Q_{k,x}$ is uniquely determined by values $Q_{j,x}(x)$, for $j = 0, 1, \dots, k$.

Theorem 2.4. *Let $w : [a,b] \to [0,\infty)$ be continuous function on (a,b) and let*

$$Q_{2n,x}(t) \geq -\frac{1}{(2n-1)!}\int_x^t (t-s)^{2n-1} w(s)ds, \quad \forall t \in [x, a+b-x],$$

for some $n \in \mathbf{N}$. If $f : [a,b] \to \mathbf{R}$ is function such that $f^{(2n)}$ is continuous on $[a,b]$, then there exists $\eta \in (a,b)$ such that

$$\begin{aligned}\int_a^b w(t)f(t)dt &= A_1(x)f(x) + B_1(x)f(a+b-x) + T_{2n,w}(x) \\ &\quad + (A_{2n+1}(x) + B_{2n+1}(x)) \cdot f^{(2n)}(\eta). \end{aligned} \tag{2.5}$$

Proof. According to the relation (2.4), we have to prove the identity

$$\int_a^b W_{2n,w}(t,x)f^{(2n)}(t)dt = (A_{2n+1}(x) + B_{2n+1}(x)) \cdot f^{(2n)}(\eta).$$

Observe that $W_{2n,w}(\cdot,x)$ is an even function. Since $W_{2n,w}(\cdot,x)$ does not change the sign, then by the mean value theorem there exists $\eta \in (a,b)$ such that

$$\begin{aligned} & \int_a^b W_{2n,w}(t,x)f^{(2n)}(t)dt \\ = \; & f^{(2n)}(\eta) \cdot \left(\int_a^x w_{1,2n}(t)dt + \int_x^{a+b-x} w_{2,2n}(t)dt + \int_{a+b-x}^b w_{3,2n}(t)dt \right) \\ = \; & f^{(2n)}(\eta) \cdot (w_{1,2n+1}(x) - w_{2,2n+1}(x) + w_{2,2n+1}(a+b-x) - w_{3,2n+1}(a+b-x)) \\ = \; & f^{(2n)}(\eta)\left(A_{2n+1}(x) + B_{2n+1}(x)\right). \end{aligned}$$

□

Theorem 2.5. *Assume (p,q) is a pair of conjugate exponents, that is $1 \leq p, q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n)} \in L_p[a,b]$. Then we have*

$$\left| \int_a^b w(t)f(t)dt - A_1(x)f(x) - B_1(x)f(a+b-x) - T_{n,w}(x) \right| \leq C_2(n,q,x,w) \cdot \|f^{(n)}\|_p, \tag{2.6}$$

where for $1 \leq q < \infty$

$$\begin{aligned} & C_2(n,q,x,w) \\ & = \frac{1}{(n-1)!}\left[\int_a^x |w_{1n}(t)|^q\,dt + \int_x^{a+b-x} |w_{2n}(t)|^q\,dt + \int_{a+b-x}^b |w_{3n}(t)|^q\,dt \right]^{\frac{1}{q}}, \end{aligned}$$

and for $q = \infty$

$$C_2(n,\infty,x,w) = \frac{1}{(n-1)!}\max\left\{\sup_{t\in[a,x]}|w_{1n}(t)|,\ \sup_{t\in[x,a+b-x]}|w_{2n}(t)|,\ \sup_{t\in[a+b-x,b]}|w_{3n}(t)|\right\}.$$

The inequality is the best possible for $p=1$*, and sharp for* $1<p\leq\infty$*. The equality is attained for every function* $f(t)=Mf_*(t)+p_{n-1}(t), t\in[a,b]$ *where* $M\in\mathbf{R}$*,* p_{n-1} *is an arbitrary polynomial of degree at most* $n-1$*, and* f_* *is function on* $[a,b]$ *defined by*

$$f_*(t)=\int_a^t \frac{(t-\xi)^{n-1}}{(n-1)!}\cdot \mathrm{sgn}W_{n,w}(\xi,x)\cdot |W_{n,w}(\xi,x)|^{\frac{1}{p-1}}d\xi, \tag{2.7}$$

for $1<p<\infty$*, and*

$$f_*(t)=\int_a^t \frac{(t-\xi)^{n-1}}{(n-1)!}\cdot \mathrm{sgn}W_{n,w}(\xi,x)d\xi, \tag{2.8}$$

for $p=\infty$*.*

Proof. Applying Hölder inequality to the integral $(-1)^n\int_a^b W_{n,w}(t,x)f^{(n)}(t)dt$ we get

$$\left|(-1)^n\int_a^b W_{n,w}(t,x)f^{(n)}(t)dt\right|\leq \|W_{n,w}(\cdot,x)\|_q\|f^{(n)}\|_p=C_2(n,q,x,w)\cdot\|f^{(n)}\|_p,$$

so the inequality (2.6) holds. In order to prove the sharpness, we need to find function f such that

$$\left|\int_a^b W_{n,w}(t,x)f^{(n)}(t)\mathrm{d}t\right|=C_2(n,q,x,w)\cdot\|f^{(n)}\|_p,$$

for $1<p\leq\infty$ and $1\leq q<\infty$ such that $\frac{1}{p}+\frac{1}{q}=1$. The function f_* defined by (2.7) and (2.8) is n times differentiable, and its $n-$th derivative is piecewise continuous function. Further, f_* is a solution of the differential equation

$$W_{n,w}(t,x)f^{(n)}(t)=|W_{n,w}(t,x)|^q,$$

so the above identity holds.
For $p=1$ we shall prove that

$$\left|\int_a^b W_{n,w}(t,x)f^{(n)}(t)\mathrm{d}t\right|\leq\sup_{t\in[a,b]}|W_{n,w}(t,x)|\cdot\int_a^b|f^{(n)}(t)|\mathrm{d}t \tag{2.9}$$

is the best possible inequality. Suppose that $|W_{n,w}(t,x)|$ attains its supremum at point $t_0\in[a,b]$ and let $\sup_{t\in[a,b]}|W_{n,w}(t,x)|=|w_{kn}(t_0)|$, for some $k=1,2,3$. First, let us assume that $w_{kn}(t_0)>0$. For ϵ small enough define $f_\epsilon^{(n-1)}(t)$ by

$$f_\epsilon^{(n-1)}(t)=\begin{cases}0, & t\leq t_0-\epsilon\\ \frac{t-t_0+\epsilon}{\epsilon}, & t\in[t_0-\epsilon,t_0]\\ 1, & t\geq t_0,\end{cases}$$

if $t_0 \in (x_{k-1}, x_k]$. Then, for ϵ small enough,

$$\Big| \int_a^b W_{n,w}(t,x) f_\epsilon^{(n)}(t)\mathrm{d}t \Big| = \Big| \int_{t_0-\epsilon}^{t_0} w_{kn}(t)\frac{1}{\epsilon}\mathrm{d}t \Big| = \frac{1}{\epsilon}\int_{t_0-\epsilon}^{t_0} w_{kn}(t)\mathrm{d}t. \tag{2.10}$$

Now, relation (2.9) implies

$$\frac{1}{\epsilon}\int_{t_0-\epsilon}^{t_0} w_{kn}(t)\mathrm{d}t \le w_{kn}(t_0)\int_{t_0-\epsilon}^{t_0} \frac{1}{\epsilon}\mathrm{d}t = w_{kn}(t_0). \tag{2.11}$$

Since

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{t_0-\epsilon}^{t_0} w_{kn}(t)\mathrm{d}t = w_{kn}(t_0),$$

the statement follows.
If $t_o = x_{k-1}$, then we define, for $\epsilon > 0$ small enough, function $f_\epsilon^{(n-1)}(t)$ by

$$f_\epsilon^{(n-1)}(t) = \begin{cases} 0, & t \le t_0 \\ \frac{t-t_0}{\epsilon}, & t \in [t_0, t_0+\epsilon] \\ 1, & t \ge t_0+\epsilon, \end{cases}$$

and we argue as above.
For the case $w_{kn}(t_0) < 0$ the proof is similar. □

Guessab and Schmeisser's identity (1.4) has symmetric coefficients, while coefficients $A_k(x)$ and $B_k(x)$ in (2.4) are not symmetric. The next result describes conditions which lead to symmetry.

Theorem 2.6. *If*

$$w(t) = w(a+b-t), \quad t \in [a,b] \tag{2.12}$$

and

$$(-1)^k Q_{k,x}(x) - Q_{k,x}(a+b-x) = \frac{1}{(k-1)!}\int_x^{a+b-x} (s-x)^{k-1} w(s)ds, \tag{2.13}$$

then $A_k(x) = (-1)^{k-1} B_k(x)$.

Proof. Assume (2.12) and (2.13) for some k. Then we have

$$\begin{aligned} B_k(x) &= (-1)^{k-1}\left[\frac{1}{(k-1)!}\int_x^b (a+b-x-s)^{k-1} w(s)ds + Q_{k,x}(a+b-x)\right] \\ &= (-1)^{k-1}\left[\frac{(-1)^{k-1}}{(k-1)!}\int_x^b (a+b-x-s)^{k-1} w(s)ds + Q_{k,x}(a+b-x)\right] \\ &= (-1)^{k-1}\left[\frac{(-1)^{k-1}}{(k-1)!}\int_a^{a+b-x} (s-x)^{k-1} w(a+b-s)ds + (-1)^k Q_{k,x}(x)\right. \\ &- \left.\frac{1}{(k-1)!}\int_x^{a+b-x} (s-x)^{k-1} w(s)ds\right] \\ &= (-1)^{k-1}\left[\frac{(-1)^{k-1}}{(k-1)!}\int_a^x (s-x)^{k-1} w(s)ds + (-1)^k Q_{k,x}(x)\right] \\ &= (-1)^{k-1} A_k(x). \end{aligned}$$

□

What about the degree of exactness of quadrature formula (2.4)? We would like to have as great degree of exactness as possible. For fixed x we choose polynomials $Q_{k,x}(t)$ which are uniquely determined by the following (according to the remark 2.3):

$$Q_{1,x}(x) = \frac{1}{2x-a-b}\left(\int_a^x (x-s)w(s)ds + \int_x^b (a+b-x-s)w(s)ds\right),$$

$$Q_{k,x}(x) = \frac{1}{(k-1)}\int_a^x (x-s)^{k-1}w(s)ds, \quad k=2,3,4$$

$$Q_{k,x}(x) = 0, \quad k \geq 5.$$

Now we have

$$A_1(x) = \frac{1}{a+b-2x}\int_a^b (a+b-x-s)w(s)ds$$

and

$$B_1(x) = \frac{1}{a+b-2x}\int_a^b (s-x)w(s)ds.$$

Further,

$$A_k^2(x) = B_k^2(x) = 0, \quad k=2,3,4. \tag{2.14}$$

Now, assume (2.12) holds. So we have

$$A_1(x) = B_1(x) = \frac{1}{2}\int_a^b w(t)dt.$$

From the condition

$$\int_a^b t^l(t)dt = A_1^2(x)g(x) + B_1^2(x)g(a+b-x), \quad l=2,3 \tag{2.15}$$

we get the equation

$$\int_a^b (t+x-a-b)(t-x)w(t)dt = 0, \tag{2.16}$$

which has exactly one solution $x \in [a, \frac{a+b}{2}]$. For that x we get the generalization of the well-known quadrature formulas of Gauss type. Now identity (2.4) becomes

$$\int_a^b f(t)w(t)dt = A_1(x)\left[f(x)+f(a+b-x)\right] + T_{n,w}(x)$$
$$+(-1)^n\int_a^b W_{n,w}(t,x)f^{(n)}(t)dt,$$

where

$$T_{n,w}(x) = \sum_{k=5}^{n}\left[A_k(x)f^{(k-1)}(x) + B_k(x)f^{(k-1)}(a+b-x)\right].$$

In particular, for $n=2$ from the identity (2.5) we get

$$\int_a^b f(t)w(t)dt = A_1(x)\left[f(x)+f(a+b-x)\right] + \left[A_5(x)+B_5(x)\right]f^{(4)}(\eta). \tag{2.17}$$

3. Special cases of the two-point quadrature formulae

In this section we shall apply results from the previous section to the special cases of weights w. First we shall give the identities, then L_p inequalities and, finally, error estimates for every weight w. Specially, we shall get Gauss quadrature formulas and related inequalities, for appropriate choice of x. In all examples we assume $\{Q_{k,x}\}_{k\in\mathbf{N}}$ is sequence of harmonic polynomials such that $\deg Q_{k,x} \leq k-1$ and $Q_{0,x} \equiv 0$, for fixed $x \in \left[a, \frac{a+b}{2}\right]$.

3.1. **Legendre–Gauss two-point quadrature formula.** .

Let $w(t) = 1, t \in [a,b]$ and $x \in [a, \frac{a+b}{2}]$ an arbitrary and fixed node. Define $\{w_{jk}^{2,LG}\}_{k\in\mathbf{N}}$

$$w_{1k}^{2,LG}(t) = \frac{(t-a)^k}{k!}, \quad t \in [a,x],$$

$$w_{2k}^{2,LG}(t) = \frac{(t-x)^k}{k!} + Q_{k,x}(t), \quad t \in (x, a+b-x]$$

and

$$w_{3k}^{2,LG}(t) = \frac{(t-b)^k}{k!}, \quad t \in (a+b-x, b].$$

Define kernel

$$W_{n,w}^{2,LG}(t,x) = \begin{cases} w_{1n}^{2,LG}(t), & \text{for } t \in [a,x], \\ w_{2n}^{2,LG}(t), & \text{for } t \in (x, a+b-x], \\ w_{3n}^{2,LG}(t), & \text{for } t \in (a+b-x, b]. \end{cases} \tag{3.1}$$

For $k \geq 1$ define

$$A_k^{2,LG}(x) = (-1)^{k-1}\left[\frac{(x-a)^k}{k!} - Q_{k,x}(x)\right],$$

$$B_k^{2,LG}(x) = (-1)^{k-1}\left[\frac{(a+b-2x)^k}{k!} - \frac{(a-x)^k}{k!} + Q_{k,x}(a+b-x)\right].$$

In particular, $A_1^{2,LG}(x) = B_1^{2,LG}(x) = \frac{b-a}{2}$. Let $f : [a,b] \to \mathbf{R}$ be such that for $n \in \mathbf{N}$, $f^{(n-1)}$ exists on $[a,b]$. Define $T_{n,w}^{2,LG}(x)$ by

$$T_{n,w}^{2,LG}(x) = \sum_{k=2}^{n}\left[A_k^{2,LG}(x) f^{(k-1)}(x) + B_k^{2,LG}(x) f^{(k-1)}(a+b-x)\right].$$

Corollary 3.1. *Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n)}$ is piecewise continuous on $[a,b]$. Then we have*

$$\int_a^b f(t)dt = \frac{b-a}{2}\left[f(x) + f(a+b-x)\right] + T_{n,w}^{2,LG}(x) + (-1)^n \int_a^b W_{n,w}^{2,LG}(t,x) f^{(n)}(t)dt. \tag{3.2}$$

Proof. Apply Theorem 2.1 for the case $w(t) = 1$. □

The solution of the equation (2.16) is $x = \frac{a+b}{2} - \frac{b-a}{2\sqrt{3}}$, so we get the generalization of Legendre–Gauss two-point quadrature formula. Further, for the polynomials $Q_{k,x}(t)$ such that $Q_{k,x}(x) = \frac{(x-a)^k}{k!}$ for $k = 2, 3, 4$, we have

$$\begin{aligned}\int_a^b f(t)dt &= \frac{b-a}{2}\left[f\left(\frac{a+b}{2} - \frac{b-a}{2\sqrt{3}}\right) + f\left(\frac{a+b}{2} + \frac{b-a}{2\sqrt{3}}\right)\right] \\ &\quad + T_{n,w}^{2,LG}\left(\frac{a+b}{2} - \frac{b-a}{2\sqrt{3}}\right) \\ &\quad + (-1)^n \int_a^b W_{n,w}^{2,LG}\left(t, \frac{a+b}{2} - \frac{b-a}{2\sqrt{3}}\right) f^{(n)}(t)dt. \end{aligned} \tag{3.3}$$

Corollary 3.2. *If $f : [a, b] \to \mathbf{R}$ is such that $f^{(2n)}$ is continuous on $[a, b]$, and if $w_{2,2n}^{2,LG}(t) \geq 0$, for $t \in [x, a + b - x]$ then there exists $\eta \in [a, b]$ such that*

$$\int_a^b f(t)dt = \frac{b-a}{2}\left[f(x) + f(a + b - x)\right] + T_{2n,w}^{2,LG}(x) + \left[A_{2n+1}^{2,LG}(x) + B_{2n+1}^{2,LG}(x)\right] \cdot f^{(2n)}(\eta). \tag{3.4}$$

For $x = \frac{a+b}{2} - \frac{b-a}{2\sqrt{3}}$, the identity (2.17) becomes Legendre–Gauss quadrature

$$\int_a^b f(t)dt = \frac{b-a}{2}\left[f\left(\frac{a+b}{2} - \frac{b-a}{2\sqrt{3}}\right) + f\left(\frac{a+b}{2} + \frac{b-a}{2\sqrt{3}}\right)\right] + \frac{(b-a)^5}{4320} f^{(4)}(\eta).$$

In particular, from the inequality (2.6) it follows

$$\begin{aligned} &\left|\int_a^b f(t)dt - \frac{b-a}{2}\left[f\left(\frac{a+b}{2} - \frac{b-a}{2\sqrt{3}}\right) + f\left(\frac{a+b}{2} + \frac{b-a}{2\sqrt{3}}\right)\right]\right| \\ &\leq C_2^{LG}\left(n, q, \frac{a+b}{2} - \frac{b-a}{2\sqrt{3}}, w\right) \|f^{(n)}\|_p, \quad n = 1, 2, 3, 4, \end{aligned}$$

where

$$C_2^{LG}\left(1,\infty,\frac{a+b}{2}-\frac{b-a}{2\sqrt{3}},w\right)=\frac{b-a}{2\sqrt{3}},$$
$$C_2^{LG}\left(1,1,\frac{a+b}{2}-\frac{b-a}{2\sqrt{3}},w\right)=\frac{(5-2\sqrt{3})(b-a)^2}{12},$$
$$C_2^{LG}\left(2,\infty,\frac{a+b}{2}-\frac{b-a}{2\sqrt{3}},w\right)=\frac{(2-\sqrt{3})(b-a)^2}{12},$$
$$C_2^{LG}\left(2,1,\frac{a+b}{2}-\frac{b-a}{2\sqrt{3}},w\right)=\frac{\sqrt{26\sqrt{3}-45}(b-a)^3}{18},$$
$$C_2^{LG}\left(3,\infty,\frac{a+b}{2}-\frac{b-a}{2\sqrt{3}},w\right)=\frac{(2-\sqrt{3})\sqrt{2\sqrt{3}-3}(b-a)^3}{72},$$
$$C_2^{LG}\left(3,1,\frac{a+b}{2}-\frac{b-a}{2\sqrt{3}},w\right)=\frac{(9-4\sqrt{3})(b-a)^4}{1728},$$
$$C_2^{LG}\left(4,\infty,\frac{a+b}{2}-\frac{b-a}{2\sqrt{3}},1\right)=\frac{(9-4\sqrt{3})(b-a)^4}{3456},$$
$$C_2^{LG}\left(4,1,\frac{a+b}{2}-\frac{b-a}{2\sqrt{3}},w\right)=\frac{(b-a)^5}{4320}.$$

3.2. **Chebyshev–Gauss two-point quadrature formula.** .

Let $w(t)=\frac{1}{\sqrt{1-t^2}}$, $t\in[-1,1]$ and let $x\in[-1,0]$ be fixed node. Define $\{w_{jk}^{2,C1}\}_{k\in\mathbf{N}}$

$$w_{1k}^{2,C1}(t)=\frac{1}{(k-1)!}\int_{-1}^{t}\frac{(t-s)^{k-1}}{\sqrt{1-s^2}}ds,\quad t\in[-1,x],$$
$$w_{2k}^{2,C1}(t)=\frac{1}{(k-1)!}\int_{x}^{t}\frac{(t-s)^{k-1}}{\sqrt{1-s^2}}ds+Q_{k,x}(t),\quad t\in(x,-x]$$

and

$$w_{3k}^{2,C1}(t)=-\frac{1}{(k-1)!}\int_{t}^{1}\frac{(t-s)^{k-1}}{\sqrt{1-s^2}}ds,\quad t\in(-x,1].$$

Define kernel

$$W_{n,w}^{2,C1}(t,x)=\begin{cases} w_{1n}^{2,C1}(t), & \text{for } t\in[-1,x],\\ w_{2n}^{2,C1}(t), & \text{for } t\in(x,-x],\\ w_{3n}^{2,C1}(t), & \text{for } t\in(-x,1].\end{cases}\tag{3.5}$$

For $k\geq 1$ define

$$A_k^{2,C1}(x)=(-1)^{k-1}\left[\frac{2^{k-1/2}(x+1)^{k-1/2}}{(2k-1)!!}F\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}+k,\frac{x+1}{2}\right)-Q_{k,x}(x)\right],$$
$$B_k^{2,C1}(x)=(-1)^{k-1}\left[\frac{1}{(k-1)!}\int_{x}^{1}\frac{(-x-s)^{k-1}}{\sqrt{1-s^2}}ds+Q_{k,x}(-x)\right].$$

Specially, $A_1^{2,C1}(x)=B_1^{2,C1}(x)=\frac{\pi}{2}$.

Let $f : [-1,1] \to \mathbf{R}$ be such that for $n \in \mathbf{N}$, $f^{(n-1)}$ exists on $[-1,1]$. Define

$$T_{n,w}^{2,C1}(x) = \sum_{k=2}^{n} \left[A_k^{2,C1}(x) f^{(k-1)}(x) + B_k^{2,C1}(x) f^{(k-1)}(-x) \right].$$

Corollary 3.3. *Let $f : [-1,1] \to \mathbf{R}$ be such that $f^{(n)}$ is piecewise continuous. Then we have*

$$\int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}} dt = \frac{\pi}{2} [f(x) + f(-x)]$$
$$+ T_{n,w}^{2,C1}(x) + (-1)^n \int_{-1}^{1} W_{n,w}^{2,C1}(t,x) f^{(n)}(t) dt. \tag{3.6}$$

Proof. Apply Theorem 2.1 for the case $w(t) = \frac{1}{\sqrt{1-t^2}}$. □

The solution of the equation (2.16) is $x = -\frac{\sqrt{2}}{2}$, so we get the generalization of the Chebysev–Gauss two-point quadrature formula. Further, for the polynomials $Q_{k,x}(t)$ such that $Q_{k,x}(x) = \frac{1}{(k-1)!} \int_{-1}^{x} \frac{(x-s)^{k-1}}{\sqrt{1-s^2}} ds$, for $k = 2,3,4$, we have

$$\begin{aligned} \int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}} dt &= \frac{\pi}{2} \left[f\left(-\frac{\sqrt{2}}{2}\right) + f\left(\frac{\sqrt{2}}{2}\right) \right] \\ &+ T_{n,w}^{2,C1}\left(-\frac{\sqrt{2}}{2}\right) + (-1)^n \int_{-1}^{1} W_{n,w}^{2,C1}\left(t, -\frac{\sqrt{2}}{2}\right) f^{(n)}(t) dt. \end{aligned} \tag{3.7}$$

Corollary 3.4. *If $f : [-1,1] \to \mathbf{R}$ is such that $f^{(2n)}$ is continuous on $[-1,1]$, and if $w_{2,2n}^{2,C1}(t) \geq 0$, for $t \in [x,-x]$, then there exists $\eta \in [-1,1]$ such that*

$$\int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}} dt = \frac{\pi}{2} [f(x) + f(-x)] + T_{2n,w}^{2,C1}(x) + \left[A_{2n+1}^{2,C1}(x) + B_{2n+1}^{2,C1}(x) \right] \cdot f^{(2n)}(\eta). \tag{3.8}$$

For $x = -\frac{\sqrt{2}}{2}$, from the identity (2.17) we get Chebyshev–Gauss quadrature formula

$$\int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}} dt = \frac{\pi}{2} \left[f\left(-\frac{\sqrt{2}}{2}\right) + f\left(-\frac{\sqrt{2}}{2}\right) \right] + \frac{\pi}{192} f^{(4)}(\eta).$$

Specially, the inequality (2.6) implies

$$\left| \int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}} dt - \frac{\pi}{2} \left[f\left(-\frac{\sqrt{2}}{2}\right) + f\left(\frac{\sqrt{2}}{2}\right) \right] \right|$$
$$\leq C_2^{C1}\left(n, q, -\frac{\sqrt{2}}{2}, w\right) \|f^{(n)}\|_p, \quad n = 1,2,3,4,$$

where

$$C_2^{C1}\left(1,\infty,-\frac{\sqrt{2}}{2},w\right)=\frac{\pi}{4}\approx 0,785398,$$

$$C_2^{C1}\left(1,1,-\frac{\sqrt{2}}{2},w\right)=2\sqrt{2}-2\approx 0.828427,$$

$$C_2^{C1}\left(2,\infty,-\frac{\sqrt{2}}{2},w\right)\approx 0,151746,$$

$$C_2^{C1}\left(2,1,-\frac{\sqrt{2}}{2},w\right)\approx 0,138151$$

$$C_2^{C1}\left(3,\infty,-\frac{\sqrt{2}}{2},w\right)\approx 0,034537,$$

$$C_2^{C1}\left(3,1,-\frac{\sqrt{2}}{2},w\right)\approx 0,037102,$$

$$C_2^{C1}\left(4,1,-\frac{\sqrt{2}}{2},w\right)=\frac{\pi}{192}\approx 0,0163624.$$

3.3. **Chebyshev–Gauss formula of the second kind.** .

Let $w(t)=\sqrt{1-t^2}, t\in[-1,1]$ and let $x\in[-1,0]$ be fixed node. Define $\{w_{jk}^{2,C2}\}_{k\in\mathbf{N}}$

$$w_{1k}^{2,C2}(t)=\frac{1}{(k-1)!}\int_{-1}^{t}(t-s)^{k-1}\sqrt{1-s^2}ds,\quad t\in[-1,x],$$

$$w_{2k}^{2,C2}(t)=\frac{1}{(k-1)!}\int_{x}^{t}(t-s)^{k-1}\sqrt{1-s^2}ds+Q_{k,x}(t),\quad t\in(x,-x]$$

and

$$w_{3k}^{2,C2}(t)=-\frac{1}{(k-1)!}\int_{t}^{1}(t-s)^{k-1}\sqrt{1-s^2}ds,\quad t\in(-x,1].$$

Define kernel

$$W_{n,w}^{2,C2}(t,x)=\begin{cases} w_{1n}^{2,C2}(t), & \text{for } t\in[-1,x],\\ w_{2n}^{2,C2}(t), & \text{for } t\in(x,-x],\\ w_{3n}^{2,C2}(t), & \text{for } t\in(-x,1].\end{cases}\tag{3.9}$$

For $k\geq 1$ define

$$A_k^{2,C2}(x)=(-1)^{k-1}\left[\frac{2^{j+1/2}(x+1)^{j+1/2}}{(2k+1)!!}F\left(-\frac{1}{2},\frac{3}{2},\frac{3}{2}+j,\frac{x+1}{2}\right)-Q_{k,x}(x)\right],$$

$$B_k^{2,C2}(x)=(-1)^{k-1}\left[\frac{1}{(k-1)!}\int_x^1(-x-s)^{j-1}\sqrt{1-s^2}ds+Q_{k,x}(-x)\right].$$

Specially, $A_1^{2,C2}(x)=B_1^{2,C2}(x)=\frac{\pi}{4}$.

Let $f:[-1,1]\to\mathbf{R}$ be such that for $n\in\mathbf{N}$, $f^{(n-1)}$ exists on $[-1,1]$. Define

$$T_{n,w}^{2,C2}(x)=\sum_{k=2}^{n}\left[A_k^{2,C2}(x)f^{(k-1)}(x)+B_k^{2,C2}(x)f^{(k-1)}(-x)\right].$$

Corollary 3.5. *Let $f:[-1,1]\to\mathbf{R}$ be such that $f^{(n)}$ is piecewise continuous. Then we have*

$$\int_{-1}^{1}f(t)\sqrt{1-t^2}dt=\frac{\pi}{4}\left[f(x)+f(-x)\right]$$
$$+T_{n,w}^{2,C2}(x)+(-1)^n\int_{-1}^{1}W_{n,w}^{2,C2}(t,x)f^{(n)}(t)dt. \tag{3.10}$$

Proof. Apply Theorem 2.1 for the case $w(t)=\sqrt{1-t^2}$. □

The solution of the equation (2.16) is $x=-\frac{1}{2}$, so we get the generalization of the Chebysev–Gauss two-point quadrature formula of the second kind. Further, for the polynomials $Q_{k,x}(t)$ such that $Q_{k,x}(x)=\frac{1}{(k-1)!}\int_{-1}^{x}(x-s)^{k-1}\sqrt{1-s^2}ds$, for $k=2,3,4$, we have

$$\begin{aligned}\int_{-1}^{1}f(t)\sqrt{1-t^2}dt &= \frac{\pi}{4}\left[f\left(-\frac{1}{2}\right)+f\left(\frac{1}{2}\right)\right] \\ &+ T_{n,w}^{2,C2}\left(-\frac{1}{2}\right)+(-1)^n\int_{-1}^{1}W_{n,w}^{2,C2}\left(t,-\frac{1}{2}\right)f^{(n)}(t)dt.\end{aligned} \tag{3.11}$$

Corollary 3.6. *If $f:[-1,1]\to\mathbf{R}$ is such that $g^{(2n)}$ is continuous on $[-1,1]$, and if $w_{2,2n}^{2,C1}(t)\geq 0$, for $t\in[x,-x]$, then there exists $\eta\in[-1,1]$ such that*

$$\int_{-1}^{1}f(t)\sqrt{1-t^2}dt=\frac{\pi}{4}\left[f(x)+f(-x)\right]+T_{2n,w}^{2,C2}(x)+\left[A_{2n+1}^{2,C2}(x)+B_{2n+1}^{2,C2}(x)\right]\cdot f^{(2n)}(\eta). \tag{3.12}$$

For $x=-\frac{1}{2}$, from the identity (2.17) we get Chebyshev–Gauss quadrature formula of the second kind

$$\int_{-1}^{1}f(t)\sqrt{1-t^2}dt=\frac{\pi}{4}\left[f\left(-\frac{1}{2}\right)+f\left(-\frac{1}{2}\right)\right]+\frac{\pi}{768}f^{(4)}(\eta).$$

Specially, the inequality (2.6) for this case looks like

$$\left|\int_{-1}^{1}f(t)\sqrt{1-t^2}dt-\frac{\pi}{4}\left[f\left(-\frac{1}{2}\right)+f\left(\frac{1}{2}\right)\right]\right|$$
$$\leq C_2^{C2}\left(n,q,-\frac{1}{2},w\right)\|f^{(n)}\|_p,\quad n=1,2,3,4,$$

where

$$C_2^{C2}\left(1,\infty,-\frac{1}{2},w\right)\approx 0,478305,$$
$$C_2^{C2}\left(1,1,-\frac{1}{2},w\right)\approx 0.370572,$$
$$C_2^{C2}\left(2,\infty,-\frac{1}{2},w\right)\approx 0,062960,$$
$$C_2^{C2}\left(2,1,-\frac{1}{2},w\right)\approx 0.0547145,$$
$$C_2^{C2}\left(3,\infty,-\frac{1}{2},w\right)\approx 0,012251,$$
$$C_2^{C2}\left(3,1,-\frac{1}{2},w\right)\approx 0,0117195,$$
$$C_2^{C2}\left(4,1,-\frac{1}{2},w\right)=\frac{\pi}{768}.$$

3.4. **Hermite–Gauss two-point formula.** .
Let us consider $w(t)=e^{-t^2}, t\in\mathbf{R}$ and let $x\le 0$. Since this weight function is defined on infinity interval, at first we shall consider it on some finite interval $[-L,L]$, for some $L\in\mathbf{R}_+$ such that $|x|\le L$.

Define

$$w_{1k}^{2,HG,L}(t)=\frac{1}{(k-1)!}\int_{-L}^{t}(t-s)^{k-1}e^{-s^2}ds,\quad t\in[-L,x],$$
$$w_{2k}^{2,HG,L}(t)=\frac{1}{(k-1)!}\int_{x}^{t}(t-s)^{k-1}e^{-s^2}ds+Q_{k,x}(t),\quad t\in(x,-x]$$

and

$$w_{3k}^{2,HG,L}(t)=-\frac{1}{(k-1)!}\int_{t}^{L}(t-s)^{k-1}e^{-s^2}ds,\quad t\in(-x,L].$$

Define kernel

$$W_{n,w}^{2,HG,L}(t,x)=\begin{cases} w_{1n}^{2,HG,L}(t), & \text{for } t\in[-L,x],\\ w_{2n}^{2,HG,L}(t), & \text{for } t\in(x,-x],\\ w_{3n}^{2,HG,L}(t), & \text{for } t\in(-x,L].\end{cases}\tag{3.13}$$

For $k\ge 1$ define

$$A_k^{2,HG,L}(x)=(-1)^{k-1}\left[\frac{1}{(k-1)!}\int_{-L}^{x}(x-s)^{k-1}e^{-s^2}ds-Q_{k,x}(x)\right],$$
$$B_k^{2,HG,L}(x)=(-1)^{k-1}\left[\frac{1}{(k-1)!}\int_{x}^{L}(-x-s)^{k-1}e^{-s^2}ds+Q_{k,x}(-x)\right].$$

Specially, $A_1^{2,HG,L}(x) = B_1^{2,HG,L}(x) = \frac{1}{2}\int_{-L}^{L} e^{-t^2}dt$. Let $f : [-1,1] \to \mathbf{R}$ be such that for $n \in \mathbf{N}$, $f^{(n-1)}$ exists on $[-1,1]$. Define

$$T_{n,w}^{2,HG,L}(x) = \sum_{k=2}^{n}\left[A_k^{2,HG,L}(x)f^{(k-1)}(x) + B_k^{2,HG,L}(x)f^{(k-1)}(-x)\right].$$

Corollary 3.7. *Let $f : [-L,L] \to \mathbf{R}$ be such that $g^{(n)}$ is piecewise continuous. Then we have*

$$\begin{aligned}\int_{-L}^{L} f(t)e^{-t^2}dt &= A_1^{2,HG,L}(x)\left[f(x)+f(-x)\right]\\ &+T_{n,w}^{2,HG,L}(x) + (-1)^n\int_{-L}^{L} W_{n,w}^{2,HG,L}(t,x)f^{(n)}(t)dt. \qquad (3.14)\end{aligned}$$

Proof. Apply Theorem 2.1 forthe case $w(t) = e^{-t^2}$.

□

Now, assume f has all the necessary higher ordered derivatives on $\mathbf{R}$. Let us define

$$\begin{aligned} w_{1k}^{2,HG}(t) &= \frac{1}{(k-1)!}\int_{-\infty}^{t}(t-s)^{k-1}e^{-s^2}ds, \quad t\in(-\infty,x],\\ w_{2k}^{2,HG}(t) &= \frac{1}{(k-1)!}\int_{x}^{t}(t-s)^{k-1}e^{-s^2}ds + Q_{k,x}(t), \quad t\in(x,-x]\\ w_{3k}^{2,HG}(t) &= -\frac{1}{(k-1)!}\int_{t}^{\infty}(t-s)^{k-1}e^{-s^2}ds, \quad t\in(-x,\infty),\end{aligned}$$

$$W_{n,w}^{2,HG}(t,x) = \begin{cases} w_{1n}^{2,HG}(t), & \text{for } t\in(-\infty,x],\\ w_{2n}^{2,HG}(t), & \text{for } t\in(x,-x],\\ w_{3n}^{2,HG}(t), & \text{for } t\in(-x,\infty),\end{cases}$$

$$\begin{aligned} A_k^{2,HG}(x) &= (-1)^{k-1}\left[\frac{1}{(k-1)!}\int_{-\infty}^{x}(x-s)^{k-1}e^{-s^2}ds - Q_{k,x}(x)\right],\\ B_k^{2,HG}(x) &= (-1)^{k-1}\left[\frac{1}{(k-1)!}\int_{x}^{\infty}(-x-s)^{k-1}e^{-s^2}ds + Q_{k,x}(-x)\right],\\ T_{n,w}^{2,HG}(x) &= \sum_{k=2}^{n}\left[A_k^{2,HG}(x)f^{(k-1)}(x) + B_k^{2,HG}(x)f^{(k-1)}(-x)\right].\end{aligned}$$

Specially, $A_1^{2,HG}(x) = B_1^{2,HG}(x) = \frac{\sqrt{\pi}}{2}$. Obviously,

$$\lim_{L\to\infty} w_{jk}^{2,HG,L}(t) = w_{jk}^{2,HG}(t)$$
$$\lim_{L\to\infty} A_k^{2,HG,L}(x) = A_k^{2,HG}(x)$$
$$\lim_{L\to\infty} B_k^{2,HG,L}(x) = B_k^{2,HG}(x)$$
$$\lim_{L\to\infty} T_{n,w}^{2,HG,L}(x) = A_k^{2,HG}(x),$$

so in (3.14) put $L \to \infty$, and we obtain

$$\int_{-\infty}^{\infty} f(t)e^{-t^2}dt = \frac{\sqrt{\pi}}{2}\left[f(x)+f(-x)\right]+T_{n,w}^{2,HG}(x)+(-1)^n\int_{-\infty}^{\infty} W_{n,w}^{2,HG}(t,x)f^{(n)}(t)dt. \tag{3.15}$$

The solution of the equation (2.16) is $x = -\frac{\sqrt{2}}{2}$, so we get the generalization of the Gauss-Hermite two-point quadrature formula. Further, for the polynomials $Q_{k,x}(t)$ such that $Q_{k,x}(x) = \frac{1}{(k-1)!}\int_{-\infty}^{x}(x-s)^{k-1}e^{-s^2}ds$, for $k = 2,3,4$, we have

$$\begin{aligned}\int_{-\infty}^{\infty} f(t)e^{-t^2}dt &= \frac{\sqrt{\pi}}{2}\left[f\left(-\frac{\sqrt{2}}{2}\right)+f\left(\frac{\sqrt{2}}{2}\right)\right] \\ &+ T_{n,w}^{2,HG}\left(-\frac{\sqrt{2}}{2}\right)+(-1)^n\int_{-\infty}^{\infty} W_{n,w}^{2,HG}\left(t,-\frac{\sqrt{2}}{2}\right)f^{(n)}(t)dt.\end{aligned} \tag{3.16}$$

Corollary 3.8. *If $f : [-1,1] \to \mathbf{R}$ is such that $f^{(2n)}$ is continuous on $\mathbf{R}$,and if $w_{2,2n}^{2,HG}(t) \geq 0$, for $t \in [x,-x]$, then there exists $\eta \in \mathbf{R}$ such that*

$$\int_{-\infty}^{\infty} f(t)e^{-t^2}dt = \frac{\sqrt{\pi}}{2}\left(f(x)+f(-x)\right)+T_{2n,w}^{2,HG}(x)+\left[A_{2n+1}^{2,HG}(x)+B_{2n+1}^{2,HG}(x)\right]\cdot f^{(2n)}(\eta). \tag{3.17}$$

For $x = -\frac{\sqrt{2}}{2}$ from identity (2.17) we get Hermite–Gauss quadrature formula

$$\int_{-\infty}^{\infty} f(t)e^{-t^2}dt = \frac{\sqrt{\pi}}{2}\left[f\left(-\frac{\sqrt{2}}{2}\right)+f\left(-\frac{\sqrt{2}}{2}\right)\right]+\frac{\sqrt{\pi}}{48}f^{(4)}(\eta),$$

Specially, the inequality (2.6) implies

$$\begin{aligned}&\left|\int_{-\infty}^{\infty} f(t)e^{-t^2}dt - \frac{\sqrt{\pi}}{2}\left[f\left(-\frac{\sqrt{2}}{2}\right)+f\left(\frac{\sqrt{2}}{2}\right)\right]\right| \\ \leq\ & C_2^{HG}\left(n,q,-\frac{\sqrt{2}}{2},w\right)\|f^{(n)}\|_p, \quad n = 1,2,3,4,\end{aligned}$$

where

$$C_2^{2,HG}\left(1,\infty,-\frac{\sqrt{2}}{2},w\right)\approx 0,605018,$$

$$C_2^{2,HG}\left(1,1,-\frac{\sqrt{2}}{2},w\right)\approx 0,670996,$$

$$C_2^{2,HG}\left(2,\infty,-\frac{\sqrt{2}}{2},w\right)\approx 0,16266,$$

$$C_2^{2,HG}\left(2,1,-\frac{\sqrt{2}}{2},w\right)\approx 0,10442,$$

$$C_2^{2,HG}\left(3,\infty,-\frac{\sqrt{2}}{2},w\right)\approx 0,061041,$$

$$C_2^{2,HG}\left(4,1,-\frac{\sqrt{2}}{2},w\right)=\frac{\sqrt{\pi}}{48}.$$

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