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NEIGHBORHOODS OF A CERTAIN CLASS OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

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ABSTRACT. A certain subclass of analytic functions in the open unit disc with negative coefficients is introduced. The new class is defined by means of multiplier transformations. By making use of the familiar concept of neighborhoods of analytic function, the author proves coefficient inequalities, distortion theorems and associated inclusion relations for the (n, δ)-neighborhoods of functions belonging to the new class, which satisfy a certain nonhomogeneous Cauchy-Euler differential equation.

1. INTRODUCTION AND DEFINITIONS

Let $\mathcal{H}$ be the class of analytic functions in the open unit disc

$$U = \{z \in \mathbb{C} : \ |z| < 1\}$$

and $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$. Let $\mathcal{A}(n)$ denote the class of functions $f(z)$ normalized by

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, \quad (n \in \mathbb{N} := \{1, 2, 3, \dots\}) \tag{1.1}$$

which are analytic in the open unit disc. In particular, we set

$$\text{and} \quad \mathcal{A}(1) := \mathcal{A}.$$

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For two functions $f(z)$ given by (1.1) and $g(z)$ given by

$$g(z) = z + \sum_{k=n+1}^{\infty} b_k z^k, \quad (n \in \mathbb{N})$$

the Hadamard product (or convolution) $(f * g)(z)$ is defined, as usual, by

$$(f * g)(z) := z + \sum_{k=n+1}^{\infty} a_k b_k z^k := (g * f)(z).$$

Definition 1.1. [8] Let $f \in \mathcal{A}(n)$. For $\delta, \lambda \in \mathbb{R}$, $\lambda \geq 0$, $\delta \geq 0$, $l \geq 0$, we define the multiplier transformations $I(\delta, \lambda, l)$ on $\mathcal{A}(n)$ by the following infinite series

$$I(\delta, \lambda, l) f(z) := z + \sum_{k=n+1}^{\infty} \left[\frac{1 + \lambda(k-1) + l}{1 + l}\right]^{\delta} a_k z^k. \tag{1.2}$$

It follows from (1.2) that

$$I(0, \lambda, l) f(z) = f(z)$$
$$(1 + l) I(2, \lambda, l) f(z) = (1 - \lambda + l) I(1, \lambda, l) f(z) + \lambda z (I(1, \lambda, l) f(z))'$$
$$I(\delta_1, \lambda, l)(I(\delta_2, \lambda, l) f(z)) = I(\delta_2, \lambda, l)(I(\delta_1, \lambda, l) f(z)).$$
$$(1 + l) I(\delta + 1, \lambda, l) f(z) = (1 - \lambda + l) I(\delta, \lambda, l) f(z) + \lambda z (I(\delta, \lambda, l) f(z))'$$

Remark 1.2. For $l = 0$, $\lambda \geq 0$, $\delta = m$, $m \in \mathbb{N}_0$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ the operator $D_{\lambda}^{m} := I(m, \lambda, 0)$ was introduced and studied by Al-Oboudi [1] which reduces to the Sălăgean differential operator [16] for $\lambda = 1$. The operator $I_l^m := I(m, 1, l)$ was studied recently by Cho and Srivastava [10] and Cho and Kim [11]. The operator $I_m := I(m, 1, 1)$ was studied by Uralegaddi and Somanatha [21], the operator $D_{\lambda}^{\delta} := I(\delta, \lambda, 0)$ was introduced by Acu and Owa [7] and the operator $I(m, l) := I(m, 1, l)$ was investigated recently by Kumar, Taneja and Ravichandran [19].

If f is given by (1.1) then we have

$$I(\delta, \lambda, l) f(z) = (f * \varphi_{\lambda, l}^{\delta})(z)$$

where

$$\varphi_{\lambda, l}^{\delta}(z) = z + \sum_{k=n+1}^{\infty} \left[\frac{1 + \lambda(k-1) + l}{1 + l}\right]^{\delta} z^k.$$

Let $\mathcal{T}(n)$ denote the class of functions $f(z)$ of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k, \quad a_k \geq 0, \quad n \in \mathbb{N}, \tag{1.3}$$

which are analytic in the open unit disc.

Following the earlier investigations by Goodman [12], Ruscheweyh [15] and Alintaş et al. [6], we define (n, η)-neighborhood of a function $f(z) \in \mathcal{T}(n)$ by

$$N_{n,\eta}(f) := \left\{ g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k \in \mathcal{T}(n) : \sum_{k=n+1}^{\infty} k |a_k - b_k| \leq \eta \right\} \tag{1.4}$$

or,

$$N_{n,\eta}(h) := \left\{ g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k \in \mathcal{T}(n) : \sum_{k=n+1}^{\infty} k|b_k| \le \eta \right\} \tag{1.5}$$

where

$$h(z) = z.$$

Let $\mathcal{S}_n^*(\alpha)$ denote the subclass of $\mathcal{T}(n)$ consisting of functions which satisfy

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha,\ z \in U,\ 0 \le \alpha < 1.$$

A function $f(z)$, in $\mathcal{S}_n^*(\alpha)$ is said to be starlike of order α in U.

A function $f(z) \in \mathcal{T}(n)$ is said to be convex of order α it it satisfies

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha,\ z \in U,\ 0 \le \alpha < 1.$$

We denote by $\mathcal{C}_n(\alpha)$ the subclass of $\mathcal{T}(n)$ consisting of all such functions [4].

An interesting unification of the function classes $\mathcal{S}_n^*(\alpha)$ and $\mathcal{C}_n(\alpha)$ is provided by the class $\mathcal{T}_n(\alpha,\gamma)$ of functions $f(z) \in \mathcal{T}(n)$, which also satisfy the following inequality

$$\mathrm{Re}\left(\frac{zf'(z) + \gamma z^2 f''(z)}{\gamma z f'(z) + (1-\gamma) f(z)}\right) > \alpha,\ z \in U,\ 0 \le \alpha < 1,\ 0 \le \gamma \le 1.$$

The class $\mathcal{T}_n(\alpha,\gamma)$ was investigated by Alintaş et al. [3].

2. Coefficient Inequalities

In this section we will define a new class of starlike functions by using the multiplier transformations $I(m,\lambda,l)$, $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \lambda \ge 0, l \ge 0$ as in (1.2) and we will establish certain coefficient inequalities relating to the new introduced class.

Definition 2.1. Let $0 \le \alpha < 1$, $0 \le \gamma \le 1$, $m \in \mathbb{N}_0$, $l \ge 0$, $\lambda \ge 0$. A function f belonging to $\mathcal{T}(n)$ is said to be in the class $\mathcal{T}_{\lambda,l}^m(n,\alpha,\gamma)$ if and only if

$$\mathrm{Re}\left\{\frac{(1-\gamma)z(I(m,\lambda,l)f(z))' + \gamma z(I(m+1,\lambda,l)f(z))'}{(1-\gamma)z(I(m,\lambda,l)f(z)) + \gamma z(I(m+1,\lambda,l)f(z))}\right\} > \alpha, \quad z \in U. \tag{2.1}$$

Remark 2.2. The class $\mathcal{T}_{\lambda,l}^m(n,\alpha,\gamma)$ is a generalization of the subclasses

i) $\mathcal{T}_{1,0}^0(1,\alpha,0) \equiv \mathcal{T}^*(\alpha) \equiv \mathcal{S}_1^*(\alpha)$ and $\mathcal{T}_{1,0}^1(1,\alpha,0) \equiv \mathcal{C}(\alpha) \equiv \mathcal{C}_1(\alpha)$ defined and studied by Silverman [18];

ii)$\mathcal{T}_{1,0}^0(n,\alpha,0)$ and $\mathcal{T}_{1,0}^1(n,\alpha,0)$ studied by Chatterjea [9] and Srivastava et al. [20];

iii) $\mathcal{T}_{1,0}^m(1,\alpha,0) \equiv \mathcal{T}(m,\alpha)$ studied by Hur and Oh [13];

iv) $\mathcal{T}_{1,0}^0(n,\alpha,\gamma)$ studied by Kamali [14].

Theorem 2.3. *Let the function f be defined by (1.3). Then f belongs to the class $\mathcal{T}_{\lambda,l}^m(n,\alpha,\gamma)$ if and only if*

$$\sum_{k=n+1}^{\infty} c_k(m,\lambda,l)(k-\alpha)[1+l+\gamma\lambda(k-1)]a_k \leq (1+l)(1-\alpha). \tag{2.2}$$

where

$$c_k(m,\lambda,l) = \left[\frac{1+\lambda(k-1)+l}{1+l}\right]^m. \tag{2.3}$$

The result is sharp and the extremal functions are

$$f_k(z) = z - \frac{(1+l)(1-\alpha)}{c_k(m,\lambda,l)(k-\alpha)[1+l+\gamma\lambda(k-1)]}\cdot z^k, \quad k \geq n+1. \tag{2.4}$$

Proof. Assume that the inequality (2.2) holds and let $|z|=1$. Then we have

$$\left|\frac{(1-\gamma)z(I(m,\lambda,l)f(z))' + \gamma z(I(m+1,\lambda,l)f(z))'}{(1-\gamma)z(I(m,\lambda,l)f(z)) + \gamma z(I(m+1,\lambda,l)f(z))} - 1\right|$$

$$= \left|\frac{\sum_{k=n+1}^{\infty}\left[\frac{1+\lambda(k-1)+l}{1+l}\right]^m\left[\frac{1+l+\gamma\lambda(k-1)}{1+l}\right](k-1)a_k z^{k-1}}{1-\sum_{k=n+1}^{\infty}\left[\frac{1+\lambda(k-1)+l}{1+l}\right]^m\left[\frac{1+l+\gamma\lambda(k-1)}{1+l}\right]a_k z^{k-1}}\right|$$

$$\leq 1 + \frac{\sum_{k=n+1}^{\infty}\left[\frac{1+\lambda(k-1)+l}{1+l}\right]^m\left[\frac{1+l+\gamma\lambda(k-1)}{1+l}\right]ka_k - 1}{1-\sum_{k=n+1}^{\infty}\left[\frac{1+\lambda(k-1)+l}{1+l}\right]^m\left[\frac{1+l+\gamma\lambda(k-1)}{1+l}\right]a_k} \leq 1-\alpha.$$

Consequently, by the maximum modulus theorem one obtains

$$f(z) \in \mathcal{T}_{\lambda,l}^m(n,\alpha,\gamma).$$

Conversely, suppose that $f(z) \in \mathcal{T}_{\lambda,l}^m(n,\alpha,\gamma)$. Then from (2.1) we find that

$$\operatorname{Re}\left\{\frac{z-\sum_{k=n+1}^{\infty}\left[\frac{1+\lambda(k-1)+l}{1+l}\right]^m\left[\frac{1+l+\gamma\lambda(k-1)}{1+l}\right]ka_k z^k}{z-\sum_{k=n+1}^{\infty}\left[\frac{1+\lambda(k-1)+l}{1+l}\right]^m\left[\frac{1+l+\gamma\lambda(k-1)}{1+l}\right]a_k z^k}\right\} > \alpha.$$

Choose values of z on the real axis such that

$$\frac{(1-\gamma)z(I(m,\lambda,l)f(z))' + \gamma z(I(m+1,\lambda,l)f(z))'}{(1-\gamma)z(I(m,\lambda,l)f(z)) + \gamma z(I(m+1,\lambda,l)f(z))}$$

is real. Letting $z \to 1^-$ through real values, we obtain

$$\operatorname{Re}\left\{\frac{1-\sum_{k=n+1}^{\infty}\left[\frac{1+\lambda(k-1)+l}{1+l}\right]^m\left[\frac{1+l+\gamma\lambda(k-1)}{1+l}\right]ka_k}{1-\sum_{k=n+1}^{\infty}\left[\frac{1+\lambda(k-1)+l}{1+l}\right]^m\left[\frac{1+l+\gamma\lambda(k-1)}{1+l}\right]a_k}\right\} \geq \alpha$$

or, equivalently

$$1-\sum_{k=n+1}^{\infty}\left[\frac{1+\lambda(k-1)+l}{1+l}\right]^m\left[\frac{1+l+\gamma\lambda(k-1)}{1+l}\right]ka_k$$
$$\geq \alpha\left\{1-\sum_{k=n+1}^{\infty}\left[\frac{1+\lambda(k-1)+l}{1+l}\right]^m\left[\frac{1+l+\gamma\lambda(k-1)}{1+l}\right]a_k\right\}$$

which gives (2.2). □

Remark 2.4. In the special case $\lambda = 1, l = 0$, Theorem 2.3 yields a result given earlier by Kamali [14].

Theorem 2.5. *Let the function f defined by (1.3) be in the class $\mathcal{T}^m_{\lambda,l}(n,\alpha,\gamma)$. Then*

$$\sum_{k=n+1}^{\infty} a_k \leq \frac{(1+l)(1-\alpha)}{c_{n+1}(m,\lambda,l)(1+l+\gamma\lambda n)(n+1-\alpha)} \tag{2.5}$$

and

$$\sum_{k=n+1}^{\infty} ka_k \leq \frac{(1+l)(1-\alpha)(n+1)}{c_{n+p}(m,\lambda,l)(1+l+\gamma\lambda n)(n+1-\alpha)}. \tag{2.6}$$

The equality in (2.5) and (2.6) is attained for the function f given by (2.4).

Proof. By using Theorem 2.3, we find from (2.1) that

$$(1+l+\gamma\lambda n)(n+1-\alpha)c_{n+p}(m,\lambda,l)\sum_{k=n+1}^{\infty} a_k$$

$$\leq \sum_{k=n+1}^{\infty} c_k(m,\lambda,l)(k-\alpha)[1+l+\gamma\lambda(k-1)]\}a_k \leq (1+l)(1-\alpha),$$

which immediately yields the first assertion (2.5) of Theorem 2.5.

On the other hand, taking into account the inequality (2.1), we also have

$$(1+l+\gamma\lambda n)c_{n+p}(m,\lambda,l)\sum_{k=n+1}^{\infty}(k-\alpha)a_k \leq (1+l)(1-\alpha)$$

that is

$$(1+l+\gamma\lambda n)c_{n+p}(m,\lambda,l)\sum_{k=n+1}^{\infty} ka_k$$

$$\leq (1+l)(1-\alpha)+\alpha(1+l+\gamma\lambda n)c_{n+p}(m,\lambda,l)\sum_{k=n+1}^{\infty} a_k$$

which, in view of the coefficient inequality (2.5), can be put in the form

$$(1+l+\gamma\lambda n)c_{n+p}(m,\lambda,l)\sum_{k=n+1}^{\infty} ka_k$$

$$\leq (1+l)(1-\alpha)+\alpha(1+l+\gamma\lambda n)c_{n+p}(m,\lambda,l)\frac{(1+l)(1-\alpha)}{c_{n+p}(m,\lambda,l)(1+l+\gamma\lambda n)(n+1-\alpha)}$$

and this completes the proof of (2.6). □

3. Distortion Theorems

Theorem 3.1. *Let the function f defined by (1.3) be in the class $\mathcal{T}^m_{\lambda,l}(n,\alpha,\gamma)$. Then we have*

$$|I(i,\lambda,l)f(z)| \geq |z| - \frac{(1+l)(1-\alpha)}{c_k(m-i,\lambda,l)(n+1-\alpha)(1+l+\gamma\lambda n)}\cdot|z|^{n+1} \tag{3.1}$$

and

$$|I(i,\lambda,l)f(z)| \leq |z| + \frac{(1+l)(1-\alpha)}{c_k(m-i,\lambda,l)(n+1-\alpha)(1+l+\gamma\lambda n)}\cdot|z|^{n+1} \tag{3.2}$$

for $z\in U$, where $0\leq i\leq m$ and $c_k(m-i,\lambda,l)$ is given by (2.3).

The equalities in (3.1) and (3.2) are attained for the function f given by

$$f_{n+1}(z) = z - \frac{(1-\alpha)(1+l)^{m+1}}{(1+\lambda n+l)^m(n+1-\alpha)(1+l+\gamma\lambda n)}z^{n+1}. \tag{3.3}$$

Proof. Note that $f\in\mathcal{T}^m_{\lambda,l}(n,\alpha,\gamma)$ if and only if $I(i,\lambda,l)f(z)\in\mathcal{T}^{m-i}_{\lambda,l}(n,\alpha,\gamma)$, where

$$I(i,\lambda,l)f(z) = z - \sum_{k=n+1}^{\infty} c_k(i,\lambda,l)a_kz^k.$$

By Theorem 2.3, we know that

$$c_k(m-i,\lambda,l)(n+1-\alpha)(1+l+\gamma\lambda n)\sum_{k=n+1}^{\infty} c_k(i,\lambda,l)a_k \leq$$

$$\leq \sum_{k=n+1}^{\infty} c_k(m,\lambda,l)(k-\alpha)[1+l+\gamma\lambda(k-1)]a_k \leq (1+l)(1-\alpha)$$

that is

$$\sum_{k=n+1}^{\infty} c_k(i,\lambda,l)a_k \leq \frac{(1+l)(1-\alpha)}{c_k(m-i,\lambda,l)(n+1-\alpha)(1+l+\gamma\lambda n)}.$$

The assertions of (3.1) and (3.2) of Theorem 3.1 follow immediately. Finally, we note that the equalities (3.1) and (3.2) are attained for the function f defined by

$$I(i,\lambda,l)f(z) = z - \frac{(1+l)(1-\alpha)}{c_k(m-i,\lambda,l)(n+1-\alpha)(1+l+\gamma\lambda n)}z^{n+1}$$

This completes the proof of Theorem 3.1. □

Corollary 3.2. *Let the function f defined by (1.3) be in the class $\mathcal{T}_{\lambda,l}^{m}(n,\alpha,\gamma)$. Then we have*

$$|f(z)| \geq |z| - \frac{(1+l)(1-\alpha)}{c_k(m,\lambda,l)(n+1-\alpha)(1+l+\gamma\lambda n)}|z|^{n+1} \tag{3.4}$$

and

$$|f(z)| \leq |z| + \frac{(1+l)(1-\alpha)}{c_k(m,\lambda,l)(n+1-\alpha)(1+l+\gamma\lambda n)}|z|^{n+1} \tag{3.5}$$

for $z \in U$. The equalities in (3.4) and (3.5) are attained for the function f_{n+1} given in (3.3).

Corollary 3.3. *Let the function f defined by (1.3) be in the class $\mathcal{T}_{\lambda,l}^{m}(n,p,\alpha,\gamma)$. Then we have*

$$|f'(z)| \geq 1 - \frac{(1+l)(1-\alpha)(n+1)}{c_k(m,\lambda,l)(n+1-\alpha)(1+l+\gamma\lambda n)}|z|^{n} \tag{3.6}$$

and

$$|f'(z)| \leq 1 + \frac{(1+l)(1-\alpha)(n+1)}{c_k(m,\lambda,l)(n+1-\alpha)(1+l+\gamma\lambda n)}|z|^{n} \tag{3.7}$$

for $z \in U$. The equalities in (3.6) and (3.7) are attained for the function f_{n+1} given in (3.3).

Corollary 3.4. *Let the function f defined by (1.3) be in the class $\mathcal{T}_{\lambda,l}^{m}(n,\alpha,\gamma)$. Then the unit disc is mapped onto a domain that contains the disc*

$$|w| < \frac{c_k(m,\lambda,l)(n+1-\alpha)(1+l+\gamma\lambda n)-(1+l)(1-\alpha)}{c_k(m,\lambda,l)(n+1-\alpha)(1+l+\gamma\lambda n)}.$$

The result is sharp with the extremal function f_{n+1} given in (3.3).

4. Inclusion relations

In this section we determine certain inclusion relations for the class $\mathcal{T}_{\lambda,l}^{m}(n,\alpha,\gamma)$, some of them involving the familiar concept of (n,η)-neighborhoods of analytic functions, defined by (1.4) and (1.5).

Theorem 4.1. *Let $0 \leq \alpha < 1$, $0 \leq \gamma_1 \leq \gamma_2 \leq 1$, $k \geq n+1$, $n \in \mathbb{N}$ and $\lambda \geq 0$. Then*

$$\mathcal{T}_{\lambda,l}^{m}(n,\alpha,\gamma_2) \subseteq \mathcal{T}_{\lambda,l}^{m}(n,\alpha,\gamma_1).$$

Proof. It follows from Theorem 2.3 that

$$\sum_{k=n+1}^{\infty} c_k(m,\lambda,l)(k-\alpha)[1+l+\gamma_1\lambda(k-1)]a_k \leq$$

$$\leq \sum_{k=n+1}^{\infty} c_k(m,\lambda,l)(k-\alpha)[1+l+\gamma_2\lambda(k-1)]a_k \leq (1+l)(1-\alpha)$$

for $f \in \mathcal{T}_{\lambda,l}^{m}(n,\alpha,\gamma_2)$. Hence f belongs to the class $\mathcal{T}_{\lambda,l}^{m}(n,\alpha,\gamma_1)$. □

Theorem 4.2. *Let $0 \leq \alpha < 1$, $0 \leq \gamma \leq 1$, $k \geq n+1$, $n \in \mathbb{N}$ and $\lambda \geq 0$. Then*

$$\mathcal{T}_{\lambda,l}^{m+1}(n,\alpha,\gamma) \subseteq \mathcal{T}_{\lambda,l}^{m}(n,\alpha,\gamma).$$

Proof. It follows from Theorem 2.3 that

$$\sum_{k=n+1}^{\infty} c_k(m,\lambda,l)(k-\alpha)[1+l+\gamma\lambda(k-1)]a_k \leq$$

$$\leq \sum_{k=n+1}^{\infty} c_k(m+1,\lambda,l)(k-\alpha)[1+l+\gamma\lambda(k-1)]a_k \leq (1+l)(1-\alpha)$$

for $f \in \mathcal{T}_{\lambda,l}^{m+1}(n,\alpha,\gamma)$. Hence, f belongs to the class $\mathcal{T}_{\lambda,l}^{m}(n,\alpha,\gamma)$. □

Remark 4.3. $\mathcal{T}_{\lambda,l}^{m}(n,\alpha,\gamma) \subset \mathcal{T}_{\lambda,l}^{0}(n,\alpha,\gamma) \subset \mathcal{T}_{0,0}^{0}(n,\alpha,0) \equiv \mathcal{S}_n^*(\alpha)$. Hence the functions f are starlike of order α, (univalent).

For the following theorems we shall require Definition 4.4 below.

Definition 4.4. A function $f(z) \in \mathcal{T}(n)$ is said to be in the class $\mathsf{K}_{\lambda,l}^{n}(\alpha,\gamma,\mu)$ if it satisfies the following nonhomogeneous Cauchy-Euler differential equation

$$z^2\frac{d^2f(z)}{dz^2} + 2(\mu+1)z\frac{df(z)}{dz} + \mu(\mu+1)f(z) = (1+\mu)(1+\mu+1)g(z) \tag{4.1}$$

where, $g(z) \in \mathcal{T}_{\lambda,l}^{m}(n,\alpha,\gamma)$, $\mu > -1$, $\mu \in \mathbb{R}$.

Theorem 4.5. *If $f(z) \in \mathcal{T}(n)$ is in the class $\mathcal{T}_{\lambda,l}^{m}(n,\alpha,\gamma)$ then*

$$\mathcal{T}_{\lambda,l}^{m}(n,\alpha,\gamma) \subset N_{n,\eta}(h), \tag{4.2}$$

where

$$h(z) = z,$$

$N_{n,\eta}(h)$ is defined in (1.5) and

$$\eta := \frac{(1+l)(1-\alpha)(n+1)}{c_{n+1}(m,\lambda,l)(n+1-\alpha)(1+l+\gamma\lambda n)}.$$

Proof. The assertion (4.2) would follow easily from the definition (1.5) of $N_{n,\eta}(h)$ and from the second assertion (2.6) of Theorem 2.5. □

Theorem 4.6. *If $f(z) \in \mathcal{T}(n)$ is in the class $\mathrm{K}_{\lambda,l}^{n}(\alpha,\gamma,\mu)$ then*

$$\mathrm{K}_{\lambda,l}^{n}(\alpha,\gamma,\mu) \subset N_{n,\eta}(g)\,,$$

where

$$\eta := \frac{(1+l)(1-\alpha)(n+1)}{c_{n+1}(m,\lambda,l)(n+1-\alpha)(1+l+\gamma\lambda n)} \cdot \left\{\frac{n+(1+\mu)(1+\mu+2)}{n+1+\mu}\right\}.$$

Proof. Suppose that $f \in \mathrm{K}_{\lambda,l}^{n}(\alpha,\gamma,\mu)$ and f is given by (1.3). From (4.1) we deduce that

$$a_k = \frac{(1+\mu)(2+\mu)}{(k+\mu)(k+\mu+1)} \cdot b_k, \quad (k = n+1,\ n+2,\dots) \tag{4.3}$$

so that

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k = z - \sum_{k=n+1}^{\infty} \frac{(1+\mu)(2+\mu)}{(k+\mu)(k+\mu+1)} \cdot b_k z^k;$$

$$g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k.$$

One obtains

$$\sum_{k=n+1}^{\infty} k|a_k - b_k| \leq \sum_{k=n+1}^{\infty} k(|a_k| + |b_k|) = \sum_{k=n+1}^{\infty} ka_k + \sum_{k=n+1}^{\infty} kb_k, \ a_k \geq 0, b_k \geq 0.$$

Substituting from (4.3) into the above coefficient inequality, we have

$$\sum_{k=n+1}^{\infty} k|a_k - b_k| \leq \sum_{k=n+1}^{\infty} \frac{(1+\mu)(2+\mu)}{(k+\mu)(k+\mu+1)} \cdot kb_k + \sum_{k=n+1}^{\infty} kb_k. \tag{4.4}$$

Next, since $g(z) \in \mathcal{T}^m_{\lambda,l}(n,\alpha,\gamma)$, the second assertion (2.6) of the Theorem 2.5 yields

$$kb_k \leq \frac{(1+l)(1-\alpha)(n+1)}{c_{n+1}(m,\lambda,l)(n+1-\alpha)(1+l+\gamma\lambda n)}, \ k = n+1, \ n+2, \ldots \tag{4.5}$$

Finally, by making use of (2.6) as well as (4.5) on the right-hand side of (4.4), we find that

$$\sum_{k=n+1}^{\infty} k|a_k - b_k| \leq \frac{(1+l)(1-\alpha)(n+1)}{c_{n+1}(m,\lambda,l)(n+1-\alpha)(1+l+\gamma\lambda n)} \cdot$$
$$\cdot \left(1 + \sum_{k=n+1}^{\infty} \frac{(1+\mu)(2+\mu)}{(k+\mu)(k+\mu+1)}\right).$$

In view of the telescopic sum

$$\sum_{k=n+1}^{\infty} \frac{1}{(k+\mu)(k+\mu+1)} = \sum_{k=n+1}^{\infty} \left(\frac{1}{k+\mu} - \frac{1}{k+\mu+1}\right) =$$
$$= \lim_{s\to\infty} \sum_{k=n+1}^{s} \left(\frac{1}{k+\mu} - \frac{1}{k+\mu+1}\right) =$$
$$= \lim_{s\to\infty} \left(\frac{1}{n+1+\mu} - \frac{1}{s+1+\mu}\right) = \frac{1}{n+1+\mu},$$

$(\mu \in \mathbb{R} - \{-1-n, \ -2-n, \ldots\})$ immediately yields

$$\sum_{k=n+1}^{\infty} k|a_k - b_k| \leq$$
$$\leq \frac{(1+l)(1-\alpha)(n+1)}{c_{n+1}(m,\lambda,l)(n+1-\alpha)(1+l+\gamma\lambda n)} \cdot \left[1 + \frac{(1+\mu)(2+\mu)}{n+1+\mu}\right] =$$

$$= \frac{(1+l)(1-\alpha)(n+1)}{c_{n+1}(m,\lambda,l)(n+1-\alpha)(1+l+\gamma\lambda n)} \cdot \left\{ \frac{n+(1+\mu)(3+\mu)}{n+1+\mu} \right\} = \eta.$$

Thus, by the definition (1.3) $f \in N_{n,\eta}(g)$. This, evidently, completes the proof of Theorem 4.2. □

By setting $m = 0, \gamma = 0, l = 0, \lambda = 1$ in Theorem 4.1, we arrive to the next corollary obtained earlier in [4].

Corollary 4.7. *If $f(z) \in \mathcal{T}(n)$ is in the class $\mathcal{T}_{0,0}^0(n,\alpha,0) \equiv \mathcal{S}_n^*(\alpha)$ then*

$$\mathcal{S}_n^*(\alpha) \subset N_{n,\eta}(h),$$

where

$$h(z) = z,$$

$N_{n,\eta}(h)$ is defined in (1.5) and

$$\eta := \frac{(1-\alpha)(n+1)}{n+1-\alpha}.$$

By setting $m = 0, \gamma = 1, l = 0, \lambda = 1$ in Theorem 4.1, we get the next corollary obtained also in [4].

Corollary 4.8. *If $f(z) \in \mathcal{T}(n)$ is in the class $\mathcal{T}_{0,0}^0(n,\alpha,1) \equiv \mathcal{C}_n(\alpha)$ then*

$$\mathcal{C}_n(\alpha) \subset N_{n,\eta}(h),$$

where

$$h(z) = z,$$

$N_{n,\eta}(h)$ is defined in (1.5) and

$$\eta := \frac{1-\alpha}{n+1-\alpha}.$$

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