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CHARACTERIZING PROJECTIONS AMONG POSITIVE OPERATORS IN THE UNIT SPHERE

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ABSTRACT. Let E and P be subsets of a Banach space X, and let us define the unit sphere around E in P as the set

$$Sph(E;P) := \{x \in P : \|x-b\| = 1 \text{ for all } b \in E\}.$$

Given a C^*-algebra A and a subset $E \subset A$, we shall write $Sph^+(E)$ or $Sph^+_A(E)$ for the set $Sph(E;S(A^+))$, where $S(A^+)$ denotes the unit sphere of A^+. We prove that, for every complex Hilbert space H, the following statements are equivalent for every positive element a in the unit sphere of $B(H)$:

(a) a is a projection;

(b) $Sph^+_{B(H)}\left(Sph^+_{B(H)}(\{a\})\right) = \{a\}$.

We also prove that the equivalence remains true when $B(H)$ is replaced with an atomic von Neumann algebra or with $K(H_2)$, where H_2 is an infinite-dimensional and separable complex Hilbert space.

1. Introduction

In a recent attempt to solve a variant of Tingley's problem for surjective isometries of the set formed by all positive operators in the unit sphere of $M_n(\mathbb{C})$, the space of all $n \times n$ complex matrices endowed with the spectral norm; G. Nagy has established an interesting characterization of those positive norm-one elements in $M_n(\mathbb{C})$ which are projections (see the final paragraph in the proof of [10, Claim 1]). Motivated by the terminology employed by Nagy in the just quoted paper,

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we introduce here the notion of *unit sphere around a subset* in a Banach space. Let E and P be subsets of a Banach space X. We define the *unit sphere around* E *in* P as the set

$$Sph(E;P) := \{x \in P : \|x-b\| = 1 \text{ for all } b \in E\}.$$

If x is an element in X, we write $Sph(x;P)$ for $Sph(\{x\};P)$. Henceforth, given a Banach space X, let $S(X)$ denote the unit sphere of X. The cone of positive elements in a C^*-algebra A will be denoted by A^+. If M is a subset of X, we shall write $S(M)$ for $M \cap S(X)$. To simplify the notation, given a C^*-algebra A and a subset $E \subset A$, we shall write $Sph^+(E)$ or $Sph^+_A(E)$ for the set $Sph(E;S(A^+))$. For each element a in A, we shall write $Sph^+(a)$ instead of $Sph^+(\{a\})$.

Let a be a positive norm-one element in $B(\ell_2^n) = M_n(\mathbb{C})$. The commented characterization established by Nagy proves that the following two statements are equivalent:

$$\begin{aligned} &(i)\ a \text{ is a projection;}\\ &(ii)\ Sph^+_{M_n(\mathbb{C})}\left(Sph^+_{M_n(\mathbb{C})}(a)\right) = \{a\}, \end{aligned} \tag{1.1}$$

(see the final paragraph in the proof of [10, Claim 1]). As remarked by G. Nagy in [10, §3], the previous characterization (and the whole statement in [10, Claim 1]) remains as an open problem when H is an arbitrary complex Hilbert space. This is an interesting problem to be considered in operator theory and in the wider setting of general C^*-algebras.

In this note we extend the characterization in (1.1) to the case in which H is an arbitrary complex Hilbert space. In a first result we prove that, for any positive element a in the unit sphere of a C^*-algebra A, the equality $Sph^+_A\left(Sph^+_A(a)\right) = \{a\}$ is a sufficient condition to guarantee that a is a projection in A (see Proposition 2.2). In Theorem 2.3 we extend Nagy's characterization to the setting of atomic von Neumann algebras by showing that the following statements are equivalent for every positive norm-one element a in an atomic von Neumann algebra M (in particular when $M = B(H)$, where H is an arbitrary complex Hilbert space):

(a) a is a projection;
(b) $Sph^+_M\left(Sph^+_M(a)\right) = \{a\}$.

We shall also explore whether the above characterization also holds when M is replaced with $K(H)$, the space of all compact operators on a complex Hilbert space H. Our conclusion in this case is the following: Let H_2 be a separable complex Hilbert space, and suppose that a is a positive norm-one element in $K(H_2)$. Then the following statements are equivalent:

(a) a is a projection;
(b) $Sph^+_{K(H_2)}\left(Sph^+_{K(H_2)}(a)\right) = \{a\}$.

When H is a finite-dimensional complex Hilbert space, Nagy computed in [10] the second unit sphere around a positive element in the unit sphere of $B(H)^+$ and showed that the identity

$$Sph^+_{B(H)}\left(Sph^+_{B(H)}(a)\right) = \left\{ b \in S(B(H)^+) : \begin{array}{c} \mathrm{Fix}(a) \subseteq \mathrm{Fix}(b), \\ \text{and } \ker(a) \subseteq \ker(b) \end{array} \right\}$$

holds for every element a in $S(B(H)^+)$, where for each a in $S(B(H)^+)$ we set $\mathrm{Fix}(a) = \{\xi \in H : a(\xi) = \xi\}$, (see the beginning of the proof of [10, Claim 1]). In Theorem 2.8 we establish a generalization of this fact to the setting of compact operators. We prove that if H_2 is a separable infinite-dimensional complex Hilbert space, then the identity

$$Sph^+_{K(H_2)}\left(Sph^+_{K(H_2)}(a)\right) = \left\{ b \in S(K(H_2)^+) : \begin{array}{c} s_{_{K(H_2)}}(a) \leq s_{_{K(H_2)}}(b), \text{ and} \\ \mathbf{1} - r_{_{B(H_2)}}(a) \leq \mathbf{1} - r_{_{B(H_2)}}(b) \end{array} \right\},$$

holds for every a in the unit sphere of $K(H_2)^+$, where $r_{_{B(H_2)}}(a)$ and $s_{_{K(H_2)}}(a)$ stand for the range and support projections of a in $B(H_2)$ and $K(H_2)$, respectively.

As we have already commented at the beginning of this introduction, the characterization obtained by Nagy in (1.1) is one of the key results to establish that every surjective isometry $\Delta : S(M_n(\mathbb{C})^+) \to S(M_n(\mathbb{C})^+)$ admits an extension to a surjective real linear or complex linear isometry on $M_n(\mathbb{C})$ (see [10, Theorem]). Another related results are known when $M_n(\mathbb{C}) = B(\ell_2^n)$ is replaced with the space $(C_p(H), \|\cdot\|_p)$ of all p-Schatten–von Neumann operators on a complex Hilbert space H, with $1 \leq p < \infty$. L. Molnár and W. Timmermann proved that for every complex Hilbert space H, every surjective isometry $\Delta : S(C_1(H)^+) \to S(C_1(H)^+)$ can be extended to a surjective complex linear isometry on $C_1(H)$ (see [7]). Nagy showed in [9, Theorem 1] that the same conclusion remains true for every $1 < p < \infty$.

The results commented in the previous paragraph are subtle variants of the so-called Tingley's problem. This problem asks whether every surjective isometry between the unit spheres of two Banach spaces X and Y admits an extension to a surjective real linear isometry from X onto Y. Tingley's problem remains open after thirty years. However, in what concerns operator algebras, certain positive solutions to this problem have been recently established in the setting of finite-dimensional C^*-algebras and finite von Neumann algebras [16, 17], spaces of compact linear operators and compact C^*-algebras [13], $B(H)$ spaces [4] (see also [3]), a wide family of von Neumann algebras [6], spaces of trace class operators [1], preduals of von Neumann algebras [8], and spaces of p-Schatten von Neumann operators with $2 < p < \infty$ [2]. The reader is referred to the survey [12] for additional details.

After completing the description of all surjective isometries on $S(M_n(\mathbb{C})^+)$, Nagy conjectured that a similar result should also hold for surjective isometries on $S(B(H)^+)$, where H is an arbitrary complex Hilbert space (see [10, §3]). The results presented in this note are a first step towards a proof of Nagy's conjecture.

2. The results

Let us fix some notation. Along the paper, the closed unit ball and the dual space of a Banach space X will be denoted by $\mathcal{B}_{_X}$ and X^*, respectively. Given a subset $B \subset X$, we shall write $\mathcal{B}_{_B}$ for $\mathcal{B}_{_X} \cap B$.

The cone of positive elements in a C^*-algebra A will be denoted by A^+, while the symbol $(A^*)^+$ will stand for the set of positive functionals on A. A *state* of A is a positive functional in $S(A^*)$. The set of states of A will be denoted by $\mathcal{S}_A$. It is well known that $\mathcal{B}_{(A^*)^+} = \mathcal{B}_{A^*} \cap (A^*)^+$ is a weak*-closed convex subset of $\mathcal{B}_{A^*}$. The set of *pure states* of A is precisely the set $\partial_e(\mathcal{B}_{(A^*)^+})$ of all extreme points of $\mathcal{B}_{(A^*)^+}$ (see [11, §3.2]).

Suppose a is a positive element in the unit sphere of a von Neumann algebra M. The *range projection* of a in M (denoted by $r(a)$) is the smallest projection p in M satisfying $ap = a$. It is known that the sequence $((1/n\mathbf{1} + a)^{-1}a)_n$ is monotone increasing to $r(a)$, and hence it converges to $r(a)$ in the weak*-topology of M. Actually, $r(a)$ also coincides with the weak*-limit of the sequence $(a^{1/n})_n$ in M (see [11, 2.2.7]). It is also known that the sequence $(a^n)_n$ converges to a projection $s(a) = s_{_M}(a)$ in M, which is called the *support projection* of a in M. Unfortunately, the support projection of a norm-one element in M might be zero. For example, let $\{\xi_n : n \in \mathbb{N}\}$ denote an orthonormal basis of ℓ_2, and let a be the positive operator in $B(\ell_2)$ given by $a = \displaystyle\sum_{m=1}^{\infty} \frac{m}{m+1} p_m$, where, for each m, p_m is the rank one projection $\xi_m \otimes \xi_m$. It is not hard to check that $s_{B(\ell_2)}(a) = 0$.

Elements a and b in a C^*-algebra A are called orthogonal (written $a \perp b$) if $ab^* = b^*a = 0$. It is known that $\|a + b\| = \max\{\|a\|, \|b\|\}$ for every $a, b \in A$ with $a \perp b$. Clearly, self-adjoint elements $a, b \in A$ are orthogonal if and only if $ab = 0$.

We recall some geometric properties of C^*-algebras. Let p be a projection in a unital C^*-algebra A. Suppose that $x \in S(A)$ satisfies $pxp = p$; then

$$x = p + (\mathbf{1} - p)x(\mathbf{1} - p), \tag{2.1}$$

(see, for example, [5, Lemma 3.1]). Another property needed later reads as follows: Suppose that $b \in A^+$ satisfies $pbp = 0$; then

$$pb = bp = 0, \text{ equivalently, } p \perp b. \tag{2.2}$$

To see this property let us take a positive $c \in A$ satisfying $c^2 = b$. The identity $0 \le (pc)(pc)^* = pc^2p = pbp = 0$ and the Gelfand-Naimark axiom imply that $pc = cp = 0$, and hence $pb = pc^2 = 0 = c^2p = bp$.

A nonzero projection p in a C^*-algebra A is called minimal if $pAp = \mathbb{C}p$. A von Neumann algebra M is called atomic if it coincides with the weak* closure of the linear span of its minimal projections. It is known from the structure theory of von Neumann algebras that every atomic von Neumann algebra M can be written in the form $M = \displaystyle\bigoplus_j^{\ell_\infty} B(H_j)$, where each H_j is a complex Hilbert space (compare [14, §2.2] or [15, §V.1]).

Let p be a nonzero projection in an atomic von Neumann algebra $M = \displaystyle\bigoplus_j^{\ell_\infty} B(H_j)$. In this case we can always find a family (q_λ) of mutually orthogonal minimal projections in M such that $p = \text{w}^*\text{-}\displaystyle\sum_\lambda q_\lambda$ (compare [14, Definition 1.13.4]). Furthermore, p is the least upper bound of the set of all minimal projections in M which are smaller than or equal to p.

The bidual, A^{**}, of a C^*-algebra A is a von Neumann algebra whose predual contains an abundant collection of pure states of A. This geometric advantage implies that the support projection in A^{**} of every element in $S(A^+)$ is a nonzero projection. Namely, if a lies in $S(A^+)$ it is well known that we can find a pure state $\phi \in \partial_e(\mathcal{B}_{(A^*)^+})$ satisfying $\phi(a) = 1$. Pure states in A^* are in one-to-one correspondence with minimal projections in A^{**}; more concretely, for each $\phi \in \partial_e(\mathcal{B}_{(A^*)^+})$ there exists a unique minimal partial isometry $p_\phi \in A^{**}$ satisfying $\phi(p_\phi) = 1$ and $p_\phi x p_\phi = \phi(x) p_\phi$ for all $x \in M$ (see [11, Proposition 3.13.6]). The projection p_ϕ is called the *support projection* of ϕ. Since A is weak*-dense in A^{**} and the product of the latter von Neumann algebra is separately weak*-continuous (see [11, Proposition 3.6.2 and Remark 3.6.5] or [14, Theorem 1.7.8]), it can be easily seen that every minimal projection in A is minimal in A^{**}.

Let a be a positive norm-one element in a C^*-algebra A. Let us take a state $\phi \in \mathcal{S}_A$ satisfying $\phi(a) = 1$ (compare [14, Proposition 1.5.4 and its proof]). The set $\{\psi \in \mathcal{B}_{(A^*)^+} : \psi(a) = 1\}$ is a nonempty weak* closed convex subset of $\mathcal{B}_{A^*}$. By the Krein–Milman theorem there exists $\varphi \in \partial_e(\mathcal{B}_{(A^*)^+})$ belonging to the previous set, and hence $\varphi(a) = 1$. We consider the support projection p_φ of φ in A^{**}, which is a minimal projection. The condition $\varphi(a) = 1$ implies $p_\varphi = p_\varphi a p_\varphi$, and (2.1) assures that $a = p_\varphi + (\mathbf{1} - p_\varphi) a (\mathbf{1} - p_\varphi)$, and thus $0 \neq p_\varphi \leq s_{A^{**}}(a)$. We can therefore deduce that

$$s_{A^{**}}(a) \neq 0 \qquad \text{for all } a \in S(A^+). \tag{2.3}$$

In order to recall the connections with Nagy's paper, we observe that, given a norm-one positive operator a in $B(H)$, we denote $\operatorname{Fix}(a) = \{\xi \in H : a(\xi) = \xi\}$, and we write p_a for the projection of H onto $\operatorname{Fix}(a)$. Since $a = p_a + (\mathbf{1} - p_a) a (\mathbf{1} - p_a)$, it follows that p_a is smaller than or equal to the support projection of a in $B(H)^{**}$. In some cases, p_a may be zero while $s_{B(H)^{**}}(a) \neq 0$. When H is finite dimensional p_a and $s(a)$ coincide. If we take a positive norm-one element in the space $K(H)$ of all compact operators on H, the element $s_{B(H)}(a) = s_{K(H)^{**}}(a) = p_a$ is a (nonzero) finite rank projection and lies in $K(H)$. We shall write $s_{K(H)}(a)$ for the projection $s_{B(H)}(a)$.

If p is a nonzero projection in a C^*-algebra A, then

$$\text{for each } a \text{ in } S(A^+) \text{ such that } p \leq a, \text{ we have } a = p + (\mathbf{1} - p) a (\mathbf{1} - p).$$

Namely, under the above hypothesis, we also have $p \leq a$ in the von Neumann algebra A^{**}. It follows that $p \leq s_{A^{**}}(a) \leq a$, and hence $s_{A^{**}}(a) - p$ is a projection in A^{**} which is orthogonal to p. Since $a = s_{A^{**}}(a) + (\mathbf{1} - s_{A^{**}}(a)) a (\mathbf{1} - s_{A^{**}}(a))$, we have $pap = p s_{A^{**}}(a) p = p$, and thus $a = p + (\mathbf{1} - p) a (\mathbf{1} - p)$ (compare (2.1)).

It is part of the folklore in the theory of C^*-algebras that the distance between two positive elements a and b in the closed unit ball of a C^*-algebra A is bounded by one. Namely, since $-\mathbf{1} \leq -b \leq a - b \leq a \leq \mathbf{1}$, we deduce that $\|a - b\| \leq 1$.

In our first result, which is an infinite-dimensional version of [10, Corollary], we establish a precise description of those pairs of elements in $S(A^+)$ whose distance is exactly one.

Lemma 2.1. *Let A be a C^*-algebra, and let a and b be elements in $S(A^+)$. Then $\|a-b\|=1$ if and only if there exists a minimal projection e in A^{**} such that one of the following statements holds:*

(a) $e \leq a$ and $e \perp b$ in A^{**};
(b) $e \leq b$ and $e \perp a$ in A^{**}.

Proof. Let us first assume that $\|a-b\|=1$. Arguing as in the proof of (2.3), we can find $\varphi \in \partial_e(\mathcal{B}_{(A^*)^+})$ such that $|\varphi(a-b)|=1$. Since $0 \leq \varphi(a), \varphi(b) \leq 1$, we can deduce that precisely one of the following holds:

(a) $\varphi(a)=1$ and $\varphi(b)=0$;
(b) $\varphi(b)=1$ and $\varphi(a)=0$.

Let $e=p_\varphi$ be the minimal projection in A^{**} associated with the pure state φ. In case (a) we know that $eae=e$ and $ebe=0$. Thus, by (2.1) and (2.2) it follows that $a=e+(\mathbf{1}-e)a(\mathbf{1}-e) \geq e$ and $b \perp e$ in A^{**}. Similar arguments show that in case (b) we get $e \leq b$ and $e \perp a$ in A^{**}.

Suppose now that we can find a minimal projection e in A^{**} satisfying (a) or (b) in the statement of the lemma. We shall only consider the case in which statement (a) holds, the other case is identical. Let φ be the pure state in A^* associated with e. Since $a=e+(\mathbf{1}-e)a(\mathbf{1}-e)$ and $b=(\mathbf{1}-e)b(\mathbf{1}-e)$ in A^{**}, we obtain $\varphi(a-b)=\varphi(e)=1 \leq \|a-b\| \leq 1$. $\square$

We are now in position to establish a sufficient condition in terms of the set $Sph_A^+\left(Sph_A^+(a)\right)$, to guarantee that a positive norm-one element a in a C^*-algebra A is a projection.

Proposition 2.2. *Let A be a C^*-algebra, and let a be a positive norm-one element in A. Suppose $Sph_A^+\left(Sph_A^+(a)\right)=\{a\}$. Then a is a projection.*

Proof. Let $\sigma(a)$ denote the spectrum of a. We identify the C^*-subalgebra of A generated by a with the commutative C^*-algebra $C_0(\sigma(a))$ of all continuous functions on $\sigma(a)\cup\{0\}$ vanishing at 0. Fix an arbitrary function $c \in C_0(\sigma(a))$ with $0 \leq c \leq 1$, $c(0)=0$, and $c(1)=1$. We claim that any such element c satisfies the following properties:

$(P1)$ If q is a minimal projection in A^{**} with $q \leq a$, then $q \leq c$ in A^{**};
$(P2)$ If q is a projection in A^{**} with $q \perp a=0$, then $qc=0$.

We shall next prove the claim. $(P1)$ Let q be a minimal projection in A^{**} with $q \leq a$. Let $\varphi \in \partial_e(\mathcal{B}_{(A^*)^+})$ be a pure state of A satisfying $\varphi(q)=1$. In this case $a=q+(\mathbf{1}-q)a(\mathbf{1}-q)$ in A^{**}. This proves that $s_{A^{**}}(a)=q+s_{A^{**}}((\mathbf{1}-q)a(\mathbf{1}-q)) \geq q$ in A^{**}. The element c has been defined to satisfy $s_{C_0(\sigma(a))^{**}}(a) \leq s_{C_0(\sigma(a))^{**}}(c)$. Since $C_0(\sigma(a))^{**}$ can be identified with the weak* closure of $C_0(\sigma(a))^{**}$ in A^{**}, we can actually conclude that $q \leq s_{A^{**}}(a)=s_{C_0(\sigma(a))^{**}}(a) \leq s_{C_0(\sigma(a))^{**}}(c)=s_{A^{**}}(c)$. This implies that $\varphi(c)=1$ and hence $q \leq c$ in A^{**}.

$(P2)$ Any element in A^{**}, which is orthogonal to a, must be orthogonal to every element in $C_0(\sigma(a))$, because the latter is the C^*-subalgebra of A generated by a. This finishes the proof of the claim.

By Lemma 2.1, an element x lies in $Sph^+_A(a)$ if and only if there exists a minimal projection e in A^{**} such that one of the following statements holds:
(a) $e \leq a$ and $e \perp x$ in A^{**};
(b) $e \leq x$ and $e \perp a$ in A^{**}.

In case (a), $e \perp x$ and $e \leq c$ by $(P1)$, and Lemma 2.1 implies that $\|x - c\| = 1$.

In case (b), $e \leq x$ and $e \perp a$, and hence $e \perp c$ by $(P2)$. Lemma 2.1 implies that $\|x - c\| = 1$.

We have proved that, any function $c \in C_0(\sigma(a))$ with $0 \leq c \leq 1$, $c(0) = 0$, and $c(1) = 1$ belongs to $Sph^+_A\left(Sph^+_A(a)\right) = \{a\}$, which forces to $\sigma(a) = \{0, 1\}$, and hence a is a projection. $\square$

The promised characterization of nonzero projections in an atomic von Neumann algebra is established next.

Theorem 2.3. *Let M be an atomic von Neumann algebra, and let a be a positive norm-one element in M. Then the following statements are equivalent:*
(a) a is a projection;
(b) $Sph^+_M\left(Sph^+_M(a)\right) = \{a\}$.

Proof. $(a) \Rightarrow (b)$ Suppose $a = p$ is a projection. Clearly

$$\{p\} \subseteq Sph^+_M\left(Sph^+_M(p)\right).$$

Let us take b in the set $Sph^+_M\left(Sph^+_M(p)\right)$. We shall first prove that $\mathbf{1} - p \perp b$. If $\mathbf{1} - p = 0$ there is nothing to prove. Otherwise, let e be a minimal projection in M with $e \leq \mathbf{1} - p$. Since $\|e + \frac{1}{2}(\mathbf{1} - e) - p\| = 1$, we deduce that $\|e + \frac{1}{2}(\mathbf{1} - e) - b\| = 1$.

Lemma 2.1 proves the existence of a minimal projection $q \in M^{**}$ such that one of the next statements holds:
(1) $q \leq e + \frac{1}{2}(\mathbf{1} - e)$ and $q \perp b$ in M^{**};
(2) $q \leq b$ and $q \perp e + \frac{1}{2}(\mathbf{1} - e)$ in M^{**}.

We claim that case (2) is impossible. Indeed, $q \perp e + \frac{1}{2}(\mathbf{1} - e)$ is equivalent to $q \perp r_{M^{**}}(e + \frac{1}{2}(\mathbf{1} - e)) = \mathbf{1}$, which is impossible. Therefore, only case (1) holds, and thus $q \leq e$. Since e also is a minimal projection in M^{**}, we deduce from the minimality of q that $e = q \perp b$.

We have shown that for every minimal projection e in M with $e \leq \mathbf{1} - p$ we have $e \perp b$. Since $\mathbf{1} - p$ is the least upper bound of all minimal projections q in M with $q \leq \mathbf{1} - p$ (actually $\mathbf{1} - p = \displaystyle\sum_j e_j$, where $\{e_j\}$ is a family of mutually orthogonal minimal projections in M), it follows that $\mathbf{1} - p \perp b$ (equivalently, $pb = bp = b$).

We shall next prove that b is a projection and $p = b$. Let $\sigma(b)$ be the spectrum of b, let $\mathcal{C}$ denote the C^*-subalgebra of M generated by b and p, and let us identify $\mathcal{C}$ with $C(\sigma(b))$, b with the function $t \mapsto t$, and p with the unit of $\mathcal{C}$. We shall distinguish two cases:

(i) $0 \notin \sigma(b)$ (that is, b is invertible in $\mathcal{C}$);
(ii) $0 \in \sigma(b)$ (that is, b is not invertible in $\mathcal{C}$).

We deal first with case (i). If $0 \notin \sigma(b)$, let m_0 be the minimum of $\sigma(b)$. If $0 < m_0 < 1$, we consider the function $d \in \mathcal{C} \equiv C(\sigma(b))$ defined by $d(t) = \frac{1}{1-m_0}(t-m_0)$ $(t \in \sigma(b))$. It is not hard to check that $0 \leq \|b-d\| = m_0 < 1$ and $\|p-d\| = 1$, which contradicts that $b \in Sph_M^+\left(Sph_M^+(p)\right)$. Therefore $m_0 = 1$, and hence b is invertible with $\sigma(b) = \{1\}$, witnessing that $\mathbf{1} = b \leq p \leq \mathbf{1}$. We have proved that $b = p = \mathbf{1}$.

In case (ii), $0 \in \sigma(b)$. If there exists $t_0 \in \sigma(b) \cap (0,1)$, the function

$$c(t) = \begin{cases} 0 & \text{if } t \in \sigma(b) \cap [0, t_0]; \\ \frac{1+t_0}{1-t_0}(t-t_0) & \text{if } t \in \sigma(b) \cap [t_0, \frac{1+t_0}{2}]; \\ t & \text{if } t \in \sigma(b) \cap [\frac{1+t_0}{2}, 1], \end{cases}$$

defines a positive norm-one element in $c \in C(\sigma(b))$ such that $\|p-c\| = 1$ and $\|b-c\| = t_0 < 1$. This contradicts that $b \in Sph_M^+\left(Sph_M^+(p)\right)$. Therefore, $\sigma(b) \subseteq \{0,1\}$, and hence b is a projection. If $b < p$, we get $\|b-b\| = 0$ and $\|p-b\| = 1$, contradicting that $b \in Sph_M^+\left(Sph_M^+(p)\right)$. Therefore $p = b$.

We have shown that $Sph_M^+\left(Sph_M^+(p)\right) = \{p\}$.

The implication $(b) \Rightarrow (a)$ follows from Proposition 2.2. $\square$

The next result is a clear consequence of our previous theorem and extends the characterization of projections in $M_n(\mathbb{C})$ established by G. Nagy in the final paragraph of the proof of [10, Claim 1] (compare (1.1)).

Corollary 2.4. *Let H be an arbitrary complex Hilbert space, and let a be a positive norm-one element in $B(H)$. Then the following statements are equivalent:*

(a) a is a projection;

(b) $Sph_{B(H)}^+\left(Sph_{B(H)}^+(a)\right) = \{a\}$. $\square$

It seems natural to ask whether the above corollary remains true if $B(H)$ is replaced with $K(H)$. For an infinite-dimensional separable complex Hilbert space H_2, the conclusion of Theorem 2.3 and Corollary 2.4 can be also extended to projections in the space $K(H_2)$. The arguments in the proof of Theorem 2.3 actually require a subtle adaptation.

Theorem 2.5. *Let a be a positive norm-one element in $K(H_2)$, where H_2 is a separable complex Hilbert space. Then the following statements are equivalent:*

(a) a is a projection;

(b) $Sph_{K(H_2)}^+\left(Sph_{K(H_2)}^+(a)\right) = \{a\}$.

Proof. When H_2 is finite-dimensional, the equivalence is proved in [10, final paragraph of the proof of Claim 1]. We can therefore assume that H_2 is infinite-dimensional.

$(a) \Rightarrow (b)$ We assume first that $a = p \in K(H_2)$ is a projection. We can find a family $\{q_1, \ldots, q_n\}$ of mutually orthogonal minimal projections in $K(H)$ such that $p = \sum_{j=1}^{n} q_j$. As before, the inclusion

$$\{p\} \subseteq Sph^+_{K(H_2)}\left(Sph^+_{K(H_2)}(p)\right)$$

always holds. Let us take b in the set $Sph^+_{K(H_2)}\left(Sph^+_{K(H_2)}(p)\right)$. Clearly $0 \neq \mathbf{1} - p \notin K(H_2)$. Let e be a minimal projection in $K(H_2)$ with $e \leq \mathbf{1} - p$ in $B(H_2)$. Since H_2 is separable, we can pick a maximal family $\{v_n : n \in \mathbb{N}\}$ of mutually orthogonal minimal projections in $(\mathbf{1} - e)K(H_2)(\mathbf{1} - e)$ with $\mathbf{1} - e = \sum_{n=1}^{\infty} v_n$. The element $e + \sum_{n=1}^{\infty} \frac{1}{2n} v_n$ lies in $S(K(H_2)^+)$ and $\left\| e + \sum_{n=1}^{\infty} \frac{1}{2n} v_n - p \right\| = 1$; thus, the hypothesis on b implies that $\left\| e + \sum_{n=1}^{\infty} \frac{1}{2n} v_n - b \right\| = 1$. Lemma 2.1 proves the existence of a minimal projection $q \in K(H_2)^{**} = B(H_2)$ such that one of the next statements holds:

(1) $q \leq e + \sum_{n=1}^{\infty} \frac{1}{2n} v_n$ and $q \perp b$ in $K(H_2)^{**} = B(H_2)$;

(2) $q \leq b$ and $q \perp e + \sum_{n=1}^{\infty} \frac{1}{2n} v_n$ in $K(H_2)^{**} = B(H_2)$.

In case (2), $q \perp e + \sum_{n=1}^{\infty} \frac{1}{2n} v_n$, and hence $q \perp e, v_n$ for all n, which proves that $q \perp e + \sum_{n=1}^{\infty} v_n = \mathbf{1}$ in $B(H_2)$, which is impossible. Therefore, case (1) holds, and thus $q \leq e$. Since e is a minimal projection in $K(H_2)^{**} = B(H_2)$, we deduce from the minimality of q that $e = q \perp b$.

We have shown that for every minimal projection e in $B(H_2)$ with $e \leq \mathbf{1} - p$ we have $e \perp b$, and then $\mathbf{1} - p \perp b$ (equivalently, $pb = bp = b$).

The above arguments show that $b, p \in pK(H_2)p \cong M_n(\mathbb{C})$. Furthermore, every $x \in Sph^+_{pK(H_2)p}(a)$ lies in $Sph^+_{K(H_2)}(a)$, and hence $\|b - x\| = 1$; therefore b lies in $Sph^+_{pK(H_2)p}(Sph^+_{pK(H_2)p}(p))$. It follows from [10, final paragraph of the proof of Claim 1] (see also (1.1)) that $Sph^+_{pK(H_2)p}(Sph^+_{pK(H_2)p}(p)) = \{p\}$, and hence $b = p$. Therefore, $Sph^+_{K(H_2)}\left(Sph^+_{K(H_2)}(p)\right) = \{p\}$.

The implication $(b) \Rightarrow (a)$ follows from Proposition 2.2. □

Many consequences can be expected from the characterizations established in Theorem 2.3 and Corollary 2.4. We shall conclude this note with a first application. For a C^*-algebra A, let $\mathcal{P}roj(A)^*$ denote the set of all nonzero projections in A. The next result is an infinite-dimensional version of [10, Claim 1] which proves one of the conjectures posed at the end of the just quoted paper.

Corollary 2.6. *Let $\Delta : S(M^+) \to S(N^+)$ be a surjective isometry, where M and N are atomic von Neumann algebras. Then Δ maps $\mathcal{P}roj(M)^*$ onto $\mathcal{P}roj(N)^*$, and the restriction $\Delta|_{\mathcal{P}roj(M)^*} : \mathcal{P}roj(M)^* \to \mathcal{P}roj(N)^*$ is a surjective isometry.*

Proof. Let p be a nonzero projection in M. Applying Theorem 2.3 we have $Sph^+_M\left(Sph^+_M(p)\right) = \{p\}$. Since Δ is a surjective isometry, the sphere around a set $E \subset S(M^+)$, $Sph^+_M(E)$, is always preserved by Δ; that is, $\Delta\left(Sph^+_M(E)\right) = Sph^+_N(\Delta(E))$. We consequently have

$$\{\Delta(p)\} = \Delta(\{p\}) = \Delta\left(Sph^+_M\left(Sph^+_M(p)\right)\right) = Sph^+_N\left(Sph^+_N(\Delta(p))\right),$$

and a new application of Theorem 2.3 assures that $\Delta(p)$ is a projection in N.

We have shown that $\Delta(\mathcal{P}roj(M)^*) \subseteq \mathcal{P}roj(N)^*$. Since Δ^{-1} also is a surjective isometry, we get $\Delta(\mathcal{P}roj(M)^*) = \mathcal{P}roj(N)^*$. Clearly $\Delta|_{\mathcal{P}roj(M)^*} : \mathcal{P}roj(M)^* \to \mathcal{P}roj(N)^*$ is a surjective isometry. $\square$

When in the previous proof we replace Theorem 2.3 with Theorem 2.5 the same arguments are valid to prove the following:

Corollary 2.7. *Let H_2 and H_3 be separable complex Hilbert spaces, and let us assume that $\Delta : S(K(H_2)^+) \to S(K(H_3)^+)$ is a surjective isometry. Then Δ maps $\mathcal{P}roj(K(H_2))^*$ to $\mathcal{P}roj(K(H_3))^*$, and the restriction*

$$\Delta|_{\mathcal{P}roj(K(H_2))^*} : \mathcal{P}roj(K(H_2))^* \to \mathcal{P}roj(K(H_3))^*$$

is a surjective isometry. $\square$

Another result established by G. Nagy in [10] asserts that for a finite-dimensional complex Hilbert space H, the equality

$$Sph^+_{B(H)}\left(Sph^+_{B(H)}(a)\right) = \left\{ b \in S(B(H)^+) : \begin{array}{c} \mathrm{Fix}(a) \subseteq \mathrm{Fix}(b), \\ \text{and } \ker(a) \subseteq \ker(b) \end{array} \right\}$$

holds for every element a in $S(B(H)^+)$ (see the beginning of the proof of [10, Claim 1]). Our next result is an abstract version of Nagy's result to the space of compact operators.

Theorem 2.8. *Let H_2 be a separable infinite-dimensional complex Hilbert space. Then the identity*

$$Sph^+_{K(H_2)}\left(Sph^+_{K(H_2)}(a)\right) = \left\{ b \in S(K(H_2)^+) : \begin{array}{c} s_{_{K(H_2)}}(a) \leq s_{_{K(H_2)}}(b), \ \textit{and} \\ \mathbf{1} - r_{_{B(H_2)}}(a) \leq \mathbf{1} - r_{_{B(H_2)}}(b) \end{array} \right\},$$

holds for every a in the unit sphere of $K(H_2)^+$.

Proof. ($\supseteq$) We recall that, for each $b \in S(K(H_2)^+)$ we have $s_{_{K(H_2)}}(b) = p_b \in K(H_2)$. Let $b \in S(K(H_2)^+)$ be with $s_{_{K(H_2)}}(a) \leq s_{_{K(H_2)}}(b)$, and let $\mathbf{1} - r_{_{B(H_2)}}(a) \leq \mathbf{1} - r_{_{B(H_2)}}(b)$. We pick an arbitrary $c \in Sph^+_{K(H_2)}(a)$. Since $\|a - c\| = 1$, Lemma 2.1 implies the existence of a minimal projection e in $B(H_2)$ such that one of the following statements holds:

(a) $e \leq a$ and $e \perp c$ in $K(H_2)^{**} = B(H_2)$;
(b) $e \leq c$ and $e \perp a$ in $K(H_2)^{**} = B(H_2)$.

In case (a), we have $e \leq s_{_{K(H_2)}}(a) \leq s_{_{K(H_2)}}(b)$ and $e \perp c$. Lemma 2.1 implies that $\|c-b\| = 1$.

In case (b), the condition $e \perp a$ implies that $e \leq \mathbf{1} - r_{_{B(H_2)}}(a) \leq \mathbf{1} - r_{_{B(H_2)}}(b)$, and thus $e \perp b$. Since $e \leq c$, Lemma 2.1 assures that $\|c-b\| = 1$.

We have shown that $\|c-b\| = 1$ for all $c \in Sph^+_{K(H_2)}(a)$, and thus b lies in $Sph^+_{K(H_2)}(Sph^+_{K(H_2)}(a))$.

$(\subseteq)$ Let us take $b \in Sph^+_{K(H_2)}\left(Sph^+_{K(H_2)}(a)\right)$.

We shall first prove that $\mathbf{1} - r_{_{B(H_2)}}(a) \leq \mathbf{1} - r_{_{B(H_2)}}(b)$. If $\mathbf{1} - r_{_{B(H_2)}}(a) = 0$ there is nothing to prove. Otherwise, let e be a minimal projection in $K(H_2)$ with $e \leq \mathbf{1} - r_{_{B(H_2)}}(a)$. Let (e_n) be a maximal family of mutually orthogonal minimal projections in $K(H_2)$ such that $\mathbf{1} - e = \sum\limits_{n=1}^{\infty} e_n$ (here we apply that H_2 is separable). Since $\left\| e + \sum\limits_{n=1}^{\infty} \frac{1}{2n} e_n - a \right\| = 1$ and $e + \sum\limits_{n=1}^{\infty} \frac{1}{2n} e_n \in K(H_2)$, we deduce that $\left\| e + \sum\limits_{n=1}^{\infty} \frac{1}{2n} e_n - b \right\| = 1$. Lemma 2.1 proves the existence of a minimal projection $q \in B(H_2)$ such that one of the next statements holds:

(a) $q \leq e + \sum\limits_{n=1}^{\infty} \frac{1}{2n} e_n$ and $q \perp b$ in $B(H_2)$;

(b) $q \leq b$ and $q \perp e + \sum\limits_{n=1}^{\infty} \frac{1}{2n} e_n$ in $B(H_2)$.

We claim that case (b) is impossible. Indeed, $q \perp e + \sum\limits_{n=1}^{\infty} \frac{1}{2n} e_n$ is equivalent to $q \perp r_{_{B(H_2)}}\left(e + \sum\limits_{n=1}^{\infty} \frac{1}{2n} e_n\right) = \mathbf{1}$, which is impossible. Therefore, only case (a) holds, and by the minimality of q, q coincides with e, and $e = q \perp b$ assures that $q = e \leq \mathbf{1} - r_{_{B(H_2)}}(b)$.

We have shown that for every minimal projection e in $B(H_2)$ with $e \leq \mathbf{1} - r_{_{B(H_2)}}(a)$ we have $q \leq \mathbf{1} - r_{_{B(H_2)}}(b)$. Since in $B(H_2)$ every projection is the least upper bound of all minimal projections smaller than or equal to it, we deduce that

$$\mathbf{1} - r_{_{B(H_2)}}(a) \leq \mathbf{1} - r_{_{B(H_2)}}(b).$$

Our next goal is to show that $s_{_{K(H_2)}}(a) \leq s_{_{K(H_2)}}(b)$. If $r_{_{B(H_2)}}(a) - s_{_{B(H_2)}}(a) = 0$, we have $s_{_{K(H_2)}}(a) = a = r_{_{B(H_2)}}(a) \geq r_{_{B(H_2)}}(b) \geq s_{_{B(H_2)}}(b)$. In particular, a is a projection in $K(H_2)$. We shall prove that b is a projection and $a = b$. Let $\sigma(b)$ be the spectrum of b, let $\mathcal{C}$ denote the C^*-subalgebra of $K(H_2)$ generated by b and $a = r_{_{K(H_2)}}(a)$, and let us identify $\mathcal{C}$ with $C(\sigma(b))$ and b with the identity function on $\sigma(b)$. If there exists $t_0 \in \sigma(b) \cap (0,1)$, then the function

$$c(t) = \begin{cases} 0 & \text{if } t \in \sigma(b) \cap [0, t_0]; \\ \frac{1+t_0}{1-t_0}(t - t_0) & \text{if } t \in \sigma(b) \cap [0, t_0]; \\ t & \text{if } t \in \sigma(b) \cap [\frac{1+t_0}{2}, 1], \end{cases} \tag{2.4}$$

defines a positive, norm-one element in $c \in C(\sigma(b)) \subset K(H_2)$ such that $\|a - c\| = 1$ and $\|b - c\| < 1$. This contradicts that $b \in Sph^+_{K(H_2)}\left(Sph^+_{K(H_2)}(a)\right)$. Therefore, $\sigma(b) \subseteq \{0,1\}$, and hence b is a projection. If $s_{B(H_2)}(b) = b < s_{K(H_2)}(a) = a$, we get $\|b - s_{K(H_2)}(b)\| = 0$, and $\|a - b\| = \|a - s_{K(H_2)}(b)\| = \|s_{K(H_2)}(a) - s_{K(H_2)}(b)\| = 1$, contradicting that $b \in Sph^+_{K(H_2)}\left(Sph^+_{K(H_2)}(a)\right)$. Therefore $a = b$ is a projection and $s_{K(H_2)}(b) = b = a = s_{K(H_2)}(a)$.

We assume next that $r_{B(H_2)}(a) - s_{K(H_2)}(a) \neq 0$. We first prove the following property.

Property ($\checkmark$.1): for each pair of minimal projections $v, q \in B(H_2)$ with $v \leq s_{K(H_2)}(a)$ and $q \leq r_{B(H_2)}(a) - s_{K(H_2)}(a)$ one of the following statements holds:

(1) $q \perp b$, or equivalently, $q \leq \mathbf{1} - r_{B(H_2)}(b)$;

(2) $v \leq s_{B(H_2)}(b) \leq b$.

To prove the property, we consider a family (v_n) of mutually orthogonal minimal projections in $K(H_2)$ satisfying $\mathbf{1} - v - q = \sum_{n=1}^{\infty} v_n$, and the element $q + \sum_{n=1}^{\infty} \frac{1}{2n} v_n \in S(K(H_2)^+)$. Clearly, v is a minimal projection in $B(H_2)$ satisfying $v \leq a$ and $v \perp q, \mathbf{1} - v$, and hence $v \perp q + \sum_{n=1}^{\infty} \frac{1}{2n} v_n$. Lemma 2.1 assures that $\left\| a - \left(q + \sum_{n=1}^{\infty} \frac{1}{2n} v_n \right) \right\| = 1$, and by hypothesis $\left\| b - \left(q + \sum_{n=1}^{\infty} \frac{1}{2n} v_n \right) \right\| = 1$. A new application of Lemma 2.1 assures the existence of a minimal projection $e \in B(H_2)$ such that one of the following statements holds:

(a) $e \leq b$ and $e \perp q + \sum_{n=1}^{\infty} \frac{1}{2n} v_n$ in $B(H_2)$;

(b) $e \leq q + \sum_{n=1}^{\infty} \frac{1}{2n} v_n$ and $e \perp b$ in $B(H_2)$.

In the second case $e = q \perp b$; equivalently, $q \leq \mathbf{1} - r_{B(H_2)}(b)$. In the first case $e \leq b \leq r_{B(H_2)}(b) \leq r_{B(H_2)}(a)$, and $e \perp q, \mathbf{1} - v$. Since $e \leq r_{B(H_2)}(a)$ and $r_{B(H_2)}(a) = (r_{B(H_2)}(a) - v) + v$, we deduce that $e \leq v$. The minimality of e and v proves that $e = v \leq b$, and thus $v \leq s_{B(H_2)}(b) \leq b$. This finishes the proof of *Property* ($\checkmark$.1).

We discuss now the following dichotomy:

- There exists a minimal projection v in $B(H_2)$ with $v \leq s_{K(H_2)}(a)$ and $v \nleq s_{K(H_2)}(b)$;
- For every minimal projection v in $B(H_2)$ with $v \leq s_{K(H_2)}(a)$ we have $v \leq s_{K(H_2)}(b)$.

In the first case, let v be a minimal projection in $K(H_2)$ with $v \leq s_{K(H_2)}(a)$ and $v \nleq s_{K(H_2)}(b)$. *Property* ($\checkmark$.1) implies that for every minimal projection $q \in B(H_2)$ with $q \leq r_{B(H_2)}(a) - s_{K(H_2)}(a)$ we have $q \leq \mathbf{1} - r_{B(H_2)}(b)$. This proves that

$$r_{B(H_2)}(a) - s_{K(H_2)}(a) \leq \mathbf{1} - r_{B(H_2)}(b).$$

We have therefore shown that

$$\mathbf{1} - s_{K(H_2)}(a) = (\mathbf{1} - r_{B(H_2)}(a)) + (r_{B(H_2)}(a) - s_{K(H_2)}(a)) \leq \mathbf{1} - r_{B(H_2)}(b),$$

and thus $r_{B(H_2)}(b) \leq s_{K(H_2)}(a)$. In this case we have $0 \leq b \leq r_{B(H_2)}(b) \leq s_{K(H_2)}(a)$, and then $ab = ba = b$. If $\sigma(b) \cap (0,1) \neq \emptyset$, by considering the C^*-subalgebra of $K(H_2)$ generated by b and the definition in (2.4), we can find an element c in $S(K(H_2)^+)$ such that $\|a - c\| = 1$ and $\|b - c\| < 1$, contradicting that $b \in Sph^+_{K(H_2)}\left(Sph^+_{K(H_2)}(a)\right)$. Therefore $\sigma(b) \subseteq \{0,1\}$, and hence b is a projection with $b \leq s_{K(H_2)}(a)$. If $b < s_{K(H_2)}(a)$, we have $\|b - b\| = 0$ and $\|a - b\| = 1$ contradicting, again, that $b \in Sph^+_{K(H_2)}\left(Sph^+_{K(H_2)}(a)\right)$. We have shown that in this case $b = s_{K(H_2)}(b) = s_{K(H_2)}(a)$.

In the second case of the above dichotomy, having in mind that $s_{K(H_2)}(a)$ can be written as a finite sum of mutually orthogonal minimal projections in $K(H_2)$, we have $s_{K(H_2)}(a) \leq s_{K(H_2)}(b)$ as desired. $\square$

Remark 2.9. Let us remark that Theorem 2.5 can be derived as a straight consequence of our previous Theorem 2.8. Namely, let H_2 be a separable complex Hilbert space, and let a be an element in $S(K(H_2)^+)$. Applying Theorem 2.8 we get

$$Sph^+_{K(H_2)}\left(Sph^+_{K(H_2)}(a)\right) = \left\{ b \in S(K(H_2)^+) : \begin{array}{c} s_{K(H_2)}(a) \leq s_{K(H_2)}(b), \text{ and} \\ \mathbf{1} - r_{B(H_2)}(a) \leq \mathbf{1} - r_{B(H_2)}(b) \end{array} \right\}.$$

If a is a projection, then $s_{K(H_2)}(a) = r_{B(H_2)}(a) = a$ and hence

$$Sph^+_{K(H_2)}\left(Sph^+_{K(H_2)}(a)\right) = \{a\}.$$

If, on the other hand, $Sph^+_{K(H_2)}\left(Sph^+_{K(H_2)}(a)\right) = \{a\}$, having in mind that $s_{K(H_2)}(a)$ belongs to $S(K(H_2)^+)$ and $s_{K(H_2)}(a) \leq r_{B(H_2)}(a)$, we deduce that $s_{K(H_2)}(a)$ lies in the set $Sph^+_{K(H_2)}\left(Sph^+_{K(H_2)}(a)\right) = \{a\}$, and hence $s_{K(H_2)}(a) = a$ is a projection.

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References

1. F. J. Fernández-Polo, J. J. Garcés, A. M. Peralta, and I. Villanueva, *Tingley's problem for spaces of trace class operators*, Linear Algebra Appl. **529** (2017), 294–323.
2. F. J. Fernández-Polo, E. Jordá, and A.M. Peralta, *Tingley's problem for p-Schatten von Neumann classes for* $2 < p < \infty$, preprint, arXiv:1803.00763v1
3. F. J. Fernández-Polo and A. M. Peralta, *Tingley's problem through the facial structure of an atomic JBW*-triple*, J. Math. Anal. Appl. **455** (2017), 750–760.
4. F. J. Fernández-Polo and A. M. Peralta, *On the extension of isometries between the unit spheres of a* C^**-algebra and* $B(H)$, Trans. Amer. Math. Soc. **5** (2018), 63–80.
5. F.J. Fernández-Polo and A .M. Peralta, *Partial Isometries: a survey*, Adv. Oper. Theory **3** (2018), no. 1, 87–128.
6. F.J. Fernández-Polo and A. M. Peralta, *On the extension of isometries between the unit spheres of von Neumann algebras*, preprint, arXiv:1709.08529v1
7. L. Molnár and W. Timmermann, *Isometries of quantum states*, J. Phys. A: Math. Gen. **36** (2003), 267–273.
8. M. Mori, *Tingley's problem through the facial structure of operator algebras*, preprint 2017. arXiv:1712.09192v1
9. G. Nagy, *Isometries on positive operators of unit norm*, Publ. Math. Debrecen **82** (2013), 183–192.
10. G. Nagy, *Isometries of spaces of normalized positive operators under the operator norm*, Publ. Math. Debrecen **92** (2018), 243–254.
11. G. K. Pedersen, C^**-algebras and their automorphism groups*, London Mathematical Society Monographs Vol. 14, Academic Press, London, 1979.
12. A. M. Peralta, *A survey on Tingley's problem for operator algebras*, Acta Sci. Math. Szeged (to appear), arXiv:1801.02473v1
13. A. M. Peralta and R. Tanaka, *A solution to Tingley's problem for isometries between the unit spheres of compact* C^**-algebras and JB*-triples*, Sci. China Math. (to appear), arXiv:1608.06327v1.
14. S. Sakai, C^**-algebras and* W^**-algebras*, Springer, Berlin, 1971.
15. M. Takesaki, *Theory of operator algebras I*, Springer, New York, 2003.
16. R. Tanaka, *Spherical isometries of finite dimensional* C^**-algebras*, J. Math. Anal. Appl. **445** (2017), no. 1, 337–341.
17. R. Tanaka, *Tingley's problem on finite von Neumann algebras*, J. Math. Anal. Appl. **451** (2017), 319–326.

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