Adv. Oper. Theory 3 (2018), no. 3, 639–646
https://doi.org/10.15352/aot.1801-1295
ISSN: 2538-225X (electronic)
https://projecteuclid.org/aot

WUR MODULUS AND NORMAL STRUCTURE IN BANACH SPACES

JI GAO

Communicated by T. Schlumprecht

Abstract. Let X be a Banach space. In this paper, we study the properties of wUR modulus of X, $\delta_X(\varepsilon, f)$, where $0 \leq \varepsilon \leq 2$ and $f \in S(X^*)$, and the relationship between the values of wUR modulus and reflexivity, uniform non-squareness and normal structure, respectively. Among other results, we proved that if $\delta_X(1, f) > 0$, for any $f \in S(X^*)$, then X has weak normal structure.

1. Introduction and preliminaries

Let X be a normed linear space. Let $B(X) = \{x \in X : \|x\| \leq 1\}$ and $S(X) = \{x \in X : \|x\| = 1\}$ be the unit ball and the unit sphere of X, respectively. Let X^* be the dual space of X. In 1948, Brodskiĭ and Mil'man [2] introduced the following geometric concepts:

Definition 1.1. A bounded and convex subset K of a Banach space X is said to have normal structure if every convex subset H of K that contains more than one point contains a point $x_0 \in H$ such that $\sup\{\|x_0 - y\| : y \in H\} < d(H)$, where $d(H) = \sup\{\|x - y\| : x, y \in H\}$ denotes the diameter of H. A Banach space X is said to have normal structure if every bounded and convex subset of X has normal structure. A Banach space X is said to have weak normal structure if for each weakly compact convex set K in X has normal structure. X is said to have uniform normal structure if there exists $0 < c < 1$ such that for any bounded closed convex subset K of X that contains more than one point, there exists $x_0 \in K$ such that $\sup\{\|x_0 - y\| : y \in K\} \leq c \cdot d(K)$.

Date: Received: Jan. 14, 2018; Accepted: Feb. 27, 2018.
2010 *Mathematics Subject Classification.* Primary 46B20; Secondary 47H10.
Key words and phrases. Uniform convexity, normal structure, wUR..

For a reflexive Banach space, the normal structure and weak normal structure coincide.

Let D be a nonempty subset of a Banach space X. A mapping $T : D \to D$ is called to be nonexpansive whenever $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in D$. A Banach space has fixed point property if for every bounded closed and convex subset D of X and for each nonexpansive mapping $T : D \to D$, there is a point $x \in D$ such that $x = Tx$ [8].

In 1965, Kirk [8] proved that if a Banach space X has weak normal structure, then it has weak fixed point property; that is, every nonexpansive mapping from a weakly compact and convex subset of X into itself has a fixed point.

In [1], Clarkson introduced the following modulus of convexity:
$\delta_X(\varepsilon) = \inf\{1 - \frac{1}{2}\|x + y\| : x, y \in S(X), \|x - y\| \ge \varepsilon\}$, where $0 \le \varepsilon \le 2$. It was proved that if there exists $\varepsilon > 0$ such that $\delta_X(1 + \varepsilon) > \frac{\varepsilon}{2}$, then X has uniform normal structure [3].

Many more geometric parameters were introduced after that to study the geometric properties of Banach spaces. For some of these parameters, see [5, 9].

In [11], Smulian introduced the following function $\delta_X(\varepsilon, f)$: from $[0, 2] \times S(X^*)$ to $[0.1]$ by the formula

$$\delta_X(\varepsilon, f) = \inf\{\{1\} \cup \{1 - \frac{1}{2}\|x + y\| : x, y \in S(X), | < x - y, f > | \ge \varepsilon\},$$

where $f \in S(X^*)$ and $0 \le \varepsilon \le 2$.

The reason for specifically including 1 in the set whose infimum defines the wUR modulus is to avoid the following particular situation: when f is a non-norm attaining functional, so there are no points x and y in $S(X)$ such that $| < x - y, f > | \ge 2$. Therefore $\delta_X(2, f)$ would not be well defined.

Then the $\delta_X(\varepsilon, f)$ is called the wUR modulus convexity of X. The space X is weakly uniformly rotund or weakly uniformly convex if $\delta_X(\varepsilon, f) > 0$ whenever $0 < \varepsilon \le 2$ and $f \in S(X^*)$.

Theorem 1.2. [12] *For any $f \in X^*$, $\frac{\delta_X(\varepsilon,f)}{\varepsilon}$ is an increasing function of ε in $(0, 2]$, and $\delta_X(\varepsilon, f)$ is a continuous function in $0 \le \varepsilon < 2$.*

Definition 1.3. [6] A Banach space X is called uniformly nonsquare if there exists $\delta > 0$ such that if $x, y \in S(X)$, then either $\frac{\|x+y\|}{2} \le 1 - \delta$ or $\frac{\|x-y\|}{2} \le 1 - \delta$.

Definition 1.4. [2] Let X and Y be Banach spaces. We say that Y is *finitely representable in* X if, for any $\varepsilon > 0$ and any finite dimensional subspace $N \subseteq Y$, there is an isomorphism $T : N \to X$ such that, for any $y \in N$, $(1 - \varepsilon)\|y\| \le \|Ty\| \le (1 + \varepsilon)\|y\|$.

The Banach space X is called *super-reflexive* if any space Y which is finitely representable in X is reflexive.

It is well known that if X is uniformly nonsquare, then X is supper-reflexive, and therefore X is reflexive.

Theorem 1.5. [6] *Let X be a Banach space. Then X is not reflexive if and only if, for any $0 < \varepsilon < 1$, there are two sequences $\{x_n\} \subseteq S(X)$ and $\{f_n\} \subseteq S(X^*)$ such that*

(a) $\langle x_m, f_n\rangle = \varepsilon$ *whenever* $n \leq m$;
(b) $\langle x_m, f_n\rangle = 0$ *whenever* $n > m$.

2. Main results

Theorem 2.1. *For a Banach space X, if $\delta_X(\varepsilon, f) > 1-\varepsilon$, for all $f \in S(X^*)$ and $0 < \varepsilon < 1$, then X is reflexive.*

Proof. Suppose X is not reflexive. For any $0 < \varepsilon < 1$, let the sequences $\{x_n\} \subseteq S(X)$ and $\{f_n\} \subseteq S(X^*)$ satisfy the two conditions in Theorem 1.5.

Let $m_1 < n < m_2$; we have $< x_{m_2} - x_{m_1}, f_n >= \varepsilon$.

Let $m < m_1 < m_2$; we have $< x_{m_2} + x_{m_1}, f_m >= 2\varepsilon$; therefore $\|x_{m_2} + x_{m_1}\| \geq 2\varepsilon$ and $1 - \frac{\|x_{m_2}+x_{m_1}\|}{2} \leq 1-\varepsilon$.

This implies $\delta_X(\varepsilon, f_n) = \inf\{1 - \frac{\|x+y\|}{2}, < x-y, f_n >\geq \varepsilon\} \leq 1 - \frac{\|x_{m_2}+x_{m_1}\|}{2} \leq 1-\varepsilon$ for this fixed $f_n \in S(X^*)$. □

For a Banach space X, we define $\nabla_x \subset S(X^*)$ to be the set of norm 1 supporting functionals of $S(X)$ at x; that is, $f_x \in \nabla_x \iff \langle x, f_x\rangle = 1$. For $x_1, x_2 \in B(X)$, we use $[x_1, x_2]$ to denote the line segment connecting x_1 and x_2 in X. Let X_2 be a two-dimensional subspace of X; for $x_1, x_2 \in S(X_2)$, we use $\widetilde{x_1, x_2}$ to denote the curve on $S(X_2)$ from x_1 to x_2 clockwise.

Lemma 2.2. [4] *If $x_1, x_2 \in B(X)$ and $0 < \epsilon < 1$ are such that $\frac{\|x_1+x_2\|}{2} > 1-\epsilon$, then, for all $0 \leq c \leq 1$ and $z = cx_1 + (1-c)x_2 \in [x_1, x_2]$, the line segment connecting x_1 and x_2 follows that $\|z\| > 1-2\epsilon$.*

Theorem 2.3. *For a Banach space X, if $\delta_X(\varepsilon, f) > \frac{1}{2} - \frac{\varepsilon}{4}$, for all $f \in S(X^*)$ and $0 < \varepsilon < 2$, then X is uniform nonsquare.*

Proof. Suppose X is not uniform nonsquare. For any $0 < \varepsilon < 2$, let $x, y \in S(X)$ such that both $\|x+y\| \geq 1 + \frac{\varepsilon}{2}$ and $\|x-y\| \geq 1+\frac{\varepsilon}{2}$.

So we have $\frac{\|x+y\|}{2} \geq \frac{1}{2} + \frac{\varepsilon}{4}$ and $\frac{\|x-y\|}{2} \geq \frac{1}{2}+\frac{\varepsilon}{4}$.

This implies $\frac{\|x+y\|}{2} \geq 1 - (\frac{1}{2} - \frac{\varepsilon}{4})$ and $1 - \frac{\|x-y\|}{2} \leq \frac{1}{2} - \frac{\varepsilon}{4}$.

Consider the two-dimensional subspace X_2 of X spanned by x and y, and x and y are clockwise located on $\widetilde{x, y} \subseteq S(X_2)$.

Let $f_x \in \nabla_x$ and $f_y \in \nabla_y$, respectively, we have
$1 =< x, f_x >\geq< y, f_x >$ and $1 =< y, f_y >\geq< x, f_y >$.
There must be a $z \in \widetilde{x, y} \subseteq S(X_2)$ such that $< x, f_z >=< y, f_z >$.
Let $z = \alpha z'$ such that $z' \in [x, y]$; then $\|z'\| > 1 - 2(\frac{1}{2} - \frac{\varepsilon}{4}) = \frac{\varepsilon}{2}$, from Lemma 2.2.

We have $< x, f_z >=< y, f_z >=< z', f_z >= \|z'\| \geq \frac{\varepsilon}{2}$.

This implies $< x+y, f_z >\geq \varepsilon$.

Use Hahn–Banach theorem to extend f_z to X. Let $f = f_z \in \nabla_z$ in $S(X^*)$.

We have $\delta_X(\varepsilon, f) = \inf\{1 - \frac{\|x-y\|}{2}, < x+y, f >\geq \varepsilon\} \leq \frac{1}{2} - \frac{\varepsilon}{4}$ for this fixed $f \in S(X^*)$.

□

Lemma 2.4. [4] *Let X be a Banach space without weak normal structure; then for any $0 < \epsilon < 1$, there exists a sequence $\{z_n\}_{k=1}^{\infty} \subseteq S(X)$ with $z_n \to^w 0$ such* *that*

$$1-\epsilon < \|z_{n+1} - z\| < 1+\epsilon$$

for sufficiently large n and any $z \in co\{z_k\}_{k=1}^n$.

Theorem 2.5. *For a Banach space X, if $\delta_X(1,f) > 0$, for all $f \in S(X^*)$, then X has weak normal structure.*

Proof. For any $\eta > 0$, let $\{z_k\}_{k=1}^\infty \subseteq S(X)$ be chosen as in above Lemma 2.4.

Since $z_k \to^w 0$, 0 is in the w-closed convex hull of $\{z_k\}_{k=1}^\infty$ which equals to the norm closed convex hull, $co\{z_k\}_{k=1}^\infty$.

So there exist n_0 and $y \in co\{z_k\}_{k=1}^{n_0}$ with $\|y\| < \eta$.

We may assume that n_0 also satisfies

$$1-\eta \le \|z_n - z\| \le 1+\eta,$$

for $n > n_0$, and that $z \in co\{z_k\}_{k=1}^{n_0}$ as in above Lemma 2.4.

We therefore have, for $n > n_0$,

$\|z_n - \frac{z_1}{2}\| \ge \|z_n - \frac{y+z_1}{2}\| - \|\frac{y}{2}\| \ge (1-\eta) - \frac{\eta}{2} > 1-2\eta$ and

$\|z_n - \frac{z_1}{2}\| \le \|z_n - \frac{y+z_1}{2}\| + \|\frac{y}{2}\| \le (1+\eta) + \frac{\eta}{2} < 1+2\eta$.

So,

$$1-2\eta \le \|z_n - \frac{z_1}{2}\| \le 1+2\eta.$$

Since $z_k \to^w 0$, take an $f_1 \in \nabla_{z_1}$. We may assume, for this fixed $f_1 \in \nabla_{z_1}$, that $| < z_{n_0}, f_1 > | < \eta$ and $1-\eta < \|z_{n_0} - z_1\|, \|z_{n_0} - \frac{z_1}{2}\| < 1+\eta$.

Let $x = \frac{-z_1+z_{n_0}}{\|z_1-z_{n_0}\|}$. Then

$$\begin{aligned} \|x + z_{n_0}\| &\ge \|z_{n_0} - (z_1 - z_{n_0})\| - \|(z_1 - z_{n_0}) + x\| \\ &= 2\|z_{n_0} - \frac{z_1}{2}\| - \|z_1 - z_{n_0}\||1 - \frac{1}{\|z_1 - z_{n_0}\|}| \\ &\ge 2(1-\eta) - (1+\eta)\frac{\eta}{1-\eta}. \end{aligned}$$

So, $1 - \frac{\|x+z_{n_0}\|}{2} \le \eta + \frac{1}{2}(1+\eta)\frac{\eta}{1-\eta}$.

But,

$$\begin{aligned} | < x - z_{n_0}, f_1 > | &= | < \frac{z_1 - z_{n_0}}{\|z_1 - z_{n_0}\|} + z_{n_0}, f_1 > | \\ &= \frac{1}{\|z_1 - z_{n_0}\|} | < z_1 - z_{n_0} + \|z_1 - z_{n_0}\| z_{n_0}, f_1 > | \\ &= \frac{1}{\|z_1 - z_{n_0}\|} | < z_1 - (1 - \|z_1 - z_{n_0}\|) z_{n_0}, f_1 > | \ge \frac{1-\eta}{1+\eta}. \end{aligned}$$

Since η can be arbitrarily small and $\delta_X(\varepsilon, f_1)$ is a continuous function in $0 \le \varepsilon < 2$, for any $f_1 \in S(X^*)$, we have $\delta_X(1, f_1) = 0$.

$\square$

We consider the uniform normal structure.

Let $\mathcal{F}$ be a filter of an index set I, and let $\{x_i\}_{i\in I}$ be a subset of a Hausdorff topological space X. Then $\{x_i\}_{i\in I}$ is said to *converge to x with respect to $\mathcal{F}$*, denoted by $\lim_{\mathcal{F}} x_i = x$, if for each neighborhood U of x, $\{i \in I : x_i \in U\} \in \mathcal{F}$.

A filter $\mathcal{U}$ on I is called an *ultrafilter* if it is maximal with respect to the ordering of the set inclusion. An ultrafilter is called *trivial* if it is of the form $\{A : A \subseteq I, i_0 \in A\}$ for some $i_0 \in I$. We will use the fact that if $\mathcal{U}$ is an ultrafilter, then

(i) for any $A \subseteq I$, either $A \subseteq U$ or $I - A \subseteq U$;

(ii) if $\{x_i\}_{i\in I}$ has a cluster point x, then $\lim_{\mathcal{U}} x_i$ exists and equals to x.

Let $\{X_i\}_{i\in I}$ be a family of Banach spaces, and let $l_\infty(I, X_i)$ denote the subspace of the product space equipped with the norm $\|(x_i)\| = \sup_{i\in I} \|x_i\| < \infty$.

Definition 2.6. [10] Let $\mathcal{U}$ be an ultrafilter on I, and let $N_U = \{(x_i) \in l_\infty(I, X_i) : \lim_{\mathcal{U}} \|x_i\| = 0\}$. The *ultraproduct* of $\{X_i\}_{i\in I}$ is the quotient space $l_\infty(I, X_i)/N_{\mathcal{U}}$ equipped with the quotient norm.

We will use $(x_i)_{\mathcal{U}}$ to denote the element of the ultraproduct. It follows from remark (ii) above and the definition of quotient norm that

$$\|(x_i)_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|x_i\|. \tag{2.1}$$

In the following we will restrict our index set I to be $\mathbb{N}$, the set of natural numbers, and let $X_i = X, i \in \mathbb{N}$ for some Banach space X. For an ultrafilter $\mathcal{U}$ on $\mathbb{N}$, we use $X_{\mathcal{U}}$ to denote the ultraproduct. Note that if $\mathcal{U}$ is nontrivial, then X can be embedded into $X_{\mathcal{U}}$ isometrically.

Lemma 2.7. [10] *Suppose that $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$ and that X is a Banach space. Then $(X^*)_{\mathcal{U}} \cong (X_{\mathcal{U}})^*$ if and only if X is super-reflexive, and in this case, the mapping J defined by*

$$\langle (x_i)_{\mathcal{U}}, J((f_i)_{\mathcal{U}})\rangle = \lim_{\mathcal{U}} \langle x_i, f_i\rangle, \quad \text{for all } (x_i)_{\mathcal{U}} \in X_{\mathcal{U}},$$

is the canonical isometric isomorphism from $(X^*)_{\mathcal{U}}$ onto $(X_{\mathcal{U}})^*$.

Theorem 2.8. *Let X be a super-reflexive Banach space. Then, for any nontrivial ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and for any $0 < \varepsilon < 2$, we have $\delta_{X_{\mathcal{U}}}(\varepsilon, (f_i)_{\mathcal{U}}) > a$, for all $(f_i)_{\mathcal{U}} \in X^*_{\mathcal{U}}$, if and only if $\delta_X(\varepsilon, f) > a$ for all $f \in X^*$.*

Proof. Since X can be embedded into $X_{\mathcal{U}}$ isometrically, we may consider X as a subspace of $X_{\mathcal{U}}$. From the definition of $\delta_X(\varepsilon, f)$, we have
$\delta_{X_{\mathcal{U}}}(\varepsilon, (f_i)_{\mathcal{U}}) > a$, for all $(f_i)_{\mathcal{U}} \in S(X^*_{\mathcal{U}})$, implies $\delta_X(\varepsilon, f) > a$ for all $f \in S(X^*)$.

We prove the reverse inequality.

Suppose there is a fixed $(f_i)_{\mathcal{U}} \in S(X^*_{\mathcal{U}})$ such that $\delta_{X_{\mathcal{U}}}(\varepsilon, (f_i)_{\mathcal{U}}) \le a$. From the definition of $\delta_{X_{\mathcal{U}}}(\varepsilon, (f_i)_{\mathcal{U}})$, there are $(x_i)_{\mathcal{U}} \in S(X_{\mathcal{U}})$ and $(y_i)_{\mathcal{U}} \in S(X_{\mathcal{U}})$ such that $1 - \frac{\|(x_i)_{\mathcal{U}} + (y_i)_{\mathcal{U}}\|}{2} \le a$, but $< (x_i)_{\mathcal{U}} - (y_i)_{\mathcal{U}}, (f_i)_{\mathcal{U}} > \ge \varepsilon$.

From (2.1), for any $\eta > 0$ we may assume the subsets

$$A = \{i : 1 - \eta < \|x_i\| < 1 + \eta\},$$
$$B = \{i : 1 - \eta < \|y_i\| < 1 + \eta\},$$

and

$$C = \{i : 1 - \eta < \|f_i\| < 1 + \eta\}$$

are all in $\mathcal{U}$.

From the property of ultraproduct, we know the subsets

$$P = \{i :< x_i - y_i, f_i >\geq \varepsilon - \eta\}$$

and

$$Q = \{i : 1 - \frac{\|x_i + y_i\|}{2} \leq a + \eta\}$$

are all in $\mathcal{U}$.

So the intersection $A \cap B \cap C \cap P \cap Q$ is in $\mathcal{U}$ too, and is hence not empty.

Let $i \in A \cap B \cap C \cap P \cap Q$. For this fixed i, we have

$$1 - \eta < \|x_i\| < 1 + \eta,$$

$$1 - \eta < \|y_i\| < 1 + \eta,$$

$$1 - \eta < \|f_i\| < 1 + \eta,$$

$$< x_i - y_i, f_i >\geq \varepsilon - \eta,$$

and

$$1 - \frac{\|x_i + y_i\|}{2} \leq a + \eta.$$

So, $\frac{\|x_i+y_i\|}{2} \geq 1 - a - \eta$.

Consider $x_i' = \frac{x_i}{\|x_i\|}, y_i' = \frac{y_i}{\|y_i\|}$, and $f_i' = \frac{f_i}{\|f_i\|}$.

Then $x_i', y_i' \in S(X)$, $f_i' \in S(X^*)$, $\|x_i' - x_i\| < \eta$, $\|y_i' - y_i\| < \eta$, and $\|f_i' - f_i\| < \eta$.

We have

$$\begin{aligned} < x_i' - y_i', f_i' > &= < x_i - y_i, f_i > + < (x_i' - x_i) - (y_i' - y_i), f_i > \\ &\quad + < x_i' - y_i', f_i' - f_i > \\ &\geq < x_i - y_i, f_i > - \|x_i' - x_i\| - \|y_i' - y_i\| - \|x_i' - y_i'\|\|f_i' - f_i\| \\ &\geq \varepsilon - \eta - 3\eta = \varepsilon - 4\eta. \end{aligned}$$

From

$$\begin{aligned} \|x_i' + y_i'\| &= \|(x_i + y_i) + (x_i' - x_i) + (y_i' - y_i)\|) \\ &\geq \|x_i + y_i\| - \|x_i' - x_i\| - \|y_i' - y_i\|) \\ &\geq 2 - 2a - 2\eta - \eta - \eta = 2 - 2a - 4\eta. \end{aligned}$$

So, $1 - \frac{\|x_i'+y_i'\|}{2} \leq a + 2\eta$.

This implies $\delta_X(\varepsilon - 4\eta, f_i') \leq a + 2\eta$.

Since $\eta > 0$ can be arbitrarily small, it is impossible to have $\delta_X(\varepsilon, f) > a$ for all $f \in S(X^*)$. So, $\delta_X(\varepsilon, f) > a$ for all $f \in S(X^*)$ implies $\delta_{X_\mathcal{U}}(\varepsilon, (f_i)_\mathcal{U}) > a$ for all $(f_i)_\mathcal{U} \in S(X_\mathcal{U}^*)$.

□

Lemma 2.9. [7] *If X is a super-reflexive Banach space, then X has uniform normal structure if and only if $X_\mathcal{U}$ has normal structure.*

Theorem 2.10. *For a Banach space X, if $\delta_X(\varepsilon, f) > \frac{1}{2} - \frac{\varepsilon}{4}$, for all $f \in S_{X^*}$ and $0 < \varepsilon < 2$, then X is uniform nonsquare and has uniform normal structure.*

Proof. The inequality $\delta_X(\varepsilon, f) > \frac{1}{2} - \frac{\varepsilon}{4}$, for all $f \in S(X^*)$ and $0 < \varepsilon < 2$, implies $\delta_X(1, f) > 0$ for all $f \in S(X^*)$ and $0 < \varepsilon < 2$; then X has weak normal structure from Theorem 2.5. The inequality $\delta_X(\varepsilon, f) > \frac{1}{2} - \frac{\varepsilon}{4}$, for all $f \in S(X^*)$ and $0 < \varepsilon < 2$ implies X is uniformly nonsquare from Theorem 2.3. So, X is super-reflexive.

Then the result follows directly from Theorem 2.8 and Lemma 2.9.

□

Example 2.11. Let H be an Hilbert space. Then $\delta_H(\varepsilon, f) = 1 - \frac{\sqrt{2(2-\varepsilon)}}{2}$ for all $f \in S(X^*)$ and $0 \le \varepsilon \le 2$.

Proof. For any $f \in S(H^*)$, let $x \in S(H)$ such that $< x, f >= 1$.

For any $0 \le \varepsilon \le 2$, let HP be the Hyperplane of $H : HP = \{z :< z, f >= -1 + \varepsilon, z \in X\}$.

It is easy to see that $\delta_X(\varepsilon, f)$ is obtained at $-x$ and any $y \in HP \cap S(H)$.

We have $< x + y, f >= 1 + (-1 + \varepsilon) = \varepsilon$.

But $\|x - y\| = \sqrt{2(2-\varepsilon)}$, so $1 - \frac{\|x-y\|}{2} = 1 - \frac{\sqrt{2(2-\varepsilon)}}{2}$.

We have, $\delta_H(\varepsilon, f) = 1 - \frac{\sqrt{2(2-\varepsilon)}}{2}$.

□

Acknowledgements. The author would like to thank the referees for their many valuable recommendations and suggestions.

References

1. J. A. Clarkson, *Uniformly convex spaces*, Trans. Amer. Math. Soc. **40** (1936), no. 3, 396–414.
2. J. Diestel, *Geometry of Banach spaces–selected topics*, Lecture Notes in Mathematics Vol. 485. Springer-Verlag, Berlin-New York, 1975.
3. J. Gao, *Modulus of convexity in Banach spaces*, Appl. Math. Lett. **16** (2003), no. 3, 273–278.
4. J. Gao and K.S. Lau, *On two classes of Banach spaces with uniform normal structure*, Studia Math. **99** (1991), no. 1, 41–56.
5. J. Gao and S. Saejung, *Some geometric parameters and normal structure in Banach spaces*, Nonlinear Func. Anal. and Appl. **15** (2010), no. 2, 185–192.
6. R. C. James, *Uniformly non-square Banach spaces*, Ann. of Math. (2) **80** (1964), 542–550.
7. M. A. Khamsi, *Uniform smoothness implies super-normal structure property*, Nonlinear Anal. **19** (1992), no. 11, 1063–1069.
8. W. A. Kirk, *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly **72** (1965), 1004–1006.
9. S. Saejung and J. Gao, *Normal Structure and Polygons in Banach Spaces*, Nonlinear Func. Anal. Appl. **19** (2014), no. 1, 131–143.
10. B. Sims, *"Ultra"-techniques in Banach space theory*, Queen's Papers in Pure and Applied Mathematics, 60. Queen's University, Kingston, ON, 1982.
11. V. L. Smulian, *On the principle of inclusion in the space of the type* (B), (Russian) Rec. Math. [Mat. Sbornik] N.S. **5(47)**, (1939), 317–328.
12. A. Ullán de Celis, *Mdulos de convexidad y lisura en espacios normados*, (Spanish) [Moduli of convexity and smoothness in normed spaces], Dissertation, Universidad de Extremadura, Badajoz, 1990. Publicaciones del Departamento de Matemticas, Universidad de Extremadura [Publications of the Mathematics Department of the University of Extremadura], 27. Universidad de Extremadura, Facultad de Ciencias, Departamento de Matemticas, Badajoz, 1991.

Department of Mathematics,Community College of Philadelphia, PA 19130-3991, USA.
E-mail address: jgao@ccp.edu