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ON NEUGEBAUER'S COVERING THEOREM

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Abstract. We present a new proof of a covering theorem of C. J. Neugebauer, stated in a slightly more general form than the original version; we also give an application to restricted weak type (1,1) inequalities for the uncentered maximal operator.

1. Introduction

Covering Theorems are often crucial in order to obtain boundedness results for maximal operators. The classical Vitali covering lemma can be extended far beyond the setting of euclidean space $\mathbb{R}^d$, provided the underlying measure is doubling. In recent years a growing interest has developed around measures for which the doubling condition may fail, motivated, among other reasons, by the study of the boundedness properties of the Cauchy transform on Lipschitz curves, and resulting in the development of a Calderón–Zygmund theory for measures that satisfy certain polynomial growth conditions, more precisely, measures μ for which there exist constants c, s with $\mu(B(x,r)) \le cr^s$ for every point x and every radius r (see, for instance, [4], and the references therein).

By a cube we mean an ℓ_∞ ball; that is, a cube with sides parallel to the coordinate axes. For measures defined by densities with respect to Lebesgue measure, Neugebauer proved in [3, Theorem 1] the following covering theorem (here λ^d stands for Lebesgue measure in $\mathbb{R}^d$).

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Theorem 1.1. *Let $0 \le f \in L^1(\lambda^d)$, and let μ be the absolutely continuous finite measure with density f. Fix $\varepsilon > 0$. Then there exists a constant $C = C(f, \varepsilon) > 0$ such that for every collection of closed cubes $\{Q_\alpha : \alpha \in \Lambda\}$ with $\mu(\cup\{Q_\alpha : \alpha \in \Lambda\}) \ge \varepsilon$, there is a finite disjoint subcollection $\{Q_n : 1 \le n \le N\}$ with*

$$\mu(\cup\{Q_\alpha : \alpha \in \Lambda\}) < C\mu(\cup\{Q_n : 1 \le n \le N\}).$$

The proof in [3] is by contradiction, and utilizes the Vitali covering lemma. Of course, the same argument works if one uses other norm balls instead of cubes. Also, the result remains valid if one considers open instead of closed balls.

Here we present a more direct proof, which clarifies how the constant C depends on the density f and on ε. Additionally, the result is stated in a slightly more general form, for almost uniformly distributed measures in geometrically doubling metric measure spaces. This allows one to state the applications from [3] in a more general form also, but we will avoid the repetition. As a new application, we will present a weak form of the restricted weak type (1,1) inequality for the uncentered maximal operator, where the constant depends on the norm of the function, but not on the function itself. This implies that if the restricted weak type (1,1) property fails, the only way to prove it is by working with a sequence of sets whose measures decrease to zero.

2. Definitions and notation

We will use $B(x,r) := \{y \in X : d(x,y) < r\}$ to denote open balls, and $B^{cl}(x,r) := \{y \in X : d(x,y) \le r\}$ to refer to metrically closed balls. It is always assumed that measures are not identically 0.

Definition 2.1. Let (X, d, μ) be a metric measure space, and let g be a locally integrable function on X. For each fixed $r > 0$ and each $x \in X$ such that $0 < \mu(B(x,r))$, the averaging operator $A_{r,\mu}$ is defined as

$$A_{r,\mu} g(x) := \frac{1}{\mu(B(x,r))} \int_{B(x,r)} g d\mu. \tag{2.1}$$

In addition, the centered Hardy–Littlewood maximal operator M_μ is given by

$$M_\mu g(x) := \sup_{\{r>0 : \mu B(x,r) > 0\}} A_{r,\mu}|g|(x), \tag{2.2}$$

while the uncentered Hardy–Littlewood maximal operator M_μ^u is defined via

$$M_\mu^u g(x) := \sup_{\{r>0, y \in X:\ d(x,y)<r \text{ and } \mu B(y,r)>0\}} A_{r,\mu}|g|(y). \tag{2.3}$$

Averaging operators are defined on the support of the measure, while maximal operators are defined everywhere.

Definition 2.2. A Borel measure μ on (X, d) is *doubling* if there exists a $C > 0$ such that, for all $r > 0$ and all $x \in X$, $\mu(B(x, 2r)) \le C\mu(B(x,r)) < \infty$.

Definition 2.3. A locally finite Borel measure μ on (X, d) is *uniformly distributed* if balls with the same radius have the same measure. The Borel measure μ is *almost uniformly distributed*, if there exist a function $h : (0,\infty) \to (0,\infty)$ and a constant $c \in (0,1)$ such that $c\, h(r) \le \mu B(x,r) \le h(r)$ for all $x \in X$ and $r > 0$. In this case we say that μ is $c - h$ almost uniformly distributed.

Remark 2.4. Note that if such c and h exist, then there is the smallest, nondecreasing h_s, which also satisfies $c\, h_s(r) \le \mu B(x,r) \le h_s(r)$, for all $x \in X$ and $r > 0$, namely, $h_s(r) := \sup_{x \in X} \mu B(x,r)$. We always work with $h = h_s$.

Definition 2.5. A metric space is *geometrically doubling* if there exists a positive integer D such that every ball of radius r can be covered with no more than D balls of radius $r/2$. We call the smallest such D the *doubling constant* of the space.

Given a locally finite Borel measure on a geometrically doubling metric space, the averaging operators are defined almost everywhere, by separability.

Definition 2.6. A metric space has the *approximate midpoint property* if for every $\varepsilon > 0$ and every pair of points x, y, there exists a point z such that $d(x,z), d(z,y) < \varepsilon + d(x,y)/2$.

Definition 2.7. A non-negative sublinear operator T having $L^1(\mu)$ as its domain and the extended real valued measurable functions as its codomain, is of restricted weak type (1,1) if there exists a constant $C > 0$ such that for all $t > 0$ and all measurable sets E, we have $t\mu\{T\mathbf{1}_E > t\} \le C\mu(E)$.

3. Covering Theorem of Neugebauer

The original proof from [3] considers a sequence of sets with increasingly worse associated constants, and takes the limsup of these sets to derive a contradiction. Here we use the Lebesgue differentiation theorem to give a more direct proof, along the standard argument for doubling measures. The selection process is as follows: If there are large balls, pick one (this may be optimal), otherwise, the collection must contain many small disjoint balls; since the "area" covered is large (greater than ε), so we can select a "large" disjoint collection.

Since we will use open balls and geometrically doubling metric spaces are separable, we can always assume that our collection of covering balls is countable to begin with.

Lemma 3.1. *Let μ be a Borel measure on the geometrically doubling metric space (X,d) with doubling constant D. If μ is $c - h$ almost uniformly distributed, then μ is doubling with constant D/c.*

Proof. Fix $B(x,r)$. Cover $B(x,2r)$ with at most D balls of radius r. If follows that $\mu B(x,2r) \le Dh(r)$. On the other hand, $\mu B(x,r) \ge c\, h(r) > 0$; so

$$\frac{\mu B(x,2r)}{\mu B(x,r)} \le \frac{Dh(r)}{c\, h(r)}.$$

□

Recall that the Lebesgue differentiation theorem holds for arbitrary doubling measures in metric spaces. This result is obtained using the same arguments as in the case of Lebesgue measure in $\mathbb{R}^d$, either by the Vitali Covering Lemma (see, for instance, [2, Theorem 1.6]) or by the weak type (1,1) of the uncentered Hardy–Littlewood maximal function. So from the preceding lemma, we conclude that the Lebesgue differentiation theorem always holds for almost uniformly distributed measures on geometrically doubling metric spaces.

Remark 3.2. The converse of the preceding lemma is not true. Consider $\mathbb{R}^d$ with the radial measure $d\mu(y) = \frac{dy}{\|y\|_2^\alpha}$, where $0 < \alpha < d$ is a fixed constant and $\|\cdot\|_2$ denotes the euclidean norm. It is well known that μ is doubling, but obviously $\lim_{x\to\infty} \mu B(x,r) = 0$. On the other hand, uniform distribution by itself does not entail doubling, as the example of area on the hyperbolic plane shows.

Example 3.3. The following example helps to motivate the selection process in the next theorem, the point being that sometimes one cannot do better than just pick one ball. Let $E \subset \mathbb{R}^2$ be the open rectangle $(0,1) \times (0,1/n)$, $n >> 1$, and let $f := \mathbf{1}_E$, and cover E with the ℓ_1 balls $B(x,1) := \{y \in \mathbb{R}^2 : \|x-y\|_1 < 1\}$ centered at the points $x \in (0,1) \times \{1\}$ (recall that if $x = (x_1, 1)$ and $y = (y_1, y_2)$, then $\|x-y\|_1 = |x_1 - y_1| + |1 - y_2|$). Taking μ to be the planar Lebesgue measure and $d\nu := \mathbf{1}_E d\mu$, we have that disjoint collections contain one ball, and the area of E covered is at most $1/n^2$; so for $0 < \varepsilon < 1/n$, $C(\varepsilon, f) \geq n$.

We shall use the same notation for countably infinite and for finite sequences, by choosing $1 \leq N \leq \infty$ and considering collections of open balls $\mathcal{C} := \{B(x_n, r_n) : 1 \leq n < N\}$.

Recall that by the Lebesgue theorem on differentiation of integrals, for metric spaces endowed with a doubling measure, if $f \in L^1_{loc}(\mu)$, then, for almost every $x \in X$,

$$f(x) = \lim_{\substack{x \in B(z,r) \\ r \to 0}} \frac{1}{\mu B(z,r)} \int_{B(z,r)} f d\mu.$$

The set of points x for which the limit exists and equals $f(x)$ is called the Lebesgue points of f.

Theorem 3.4. *Let μ be a $c-h$ almost uniformly distributed Borel measure on the geometrically doubling metric space (X,d), with doubling constant D. Let X have the approximate midpoint property, and let $0 \leq f \in L^1_{loc}(\mu)$, and let ν be defined by $d\nu := f d\mu$. Fix $\varepsilon > 0$, and suppose that $t > 0$ is such that the level set $\{f > t\}$ has finite ν measure. Then there exists a constant $C = C(f,t,\varepsilon) > 0$ such that for every sequence of open balls $\{B(x_n, r_n) : 1 \leq n < N\}$ with $\nu(\{f > t\} \cap (\cup\{B(x_n, r_n) : 1 \leq n < N\})) > \varepsilon$, there exists a finite disjoint subsequence $\{B(x_{n_j}, r_{n_j}) : 1 \leq j \leq M\}$ with*

$$\nu(\{f > t\} \cap (\cup\{B(x_n, r_n) : 1 \leq n < N\})) < C\nu(\cup\{B(x_{n_j}, r_{n_j}) : 1 \leq j \leq M\}).$$

Proof. Assume that the balls in $\{B(x_n, r_n) : 1 \leq n < N\}$ satisfy the conditions of the theorem. Let D_t be the set of Lebesgue points of f in the level set $\{f > t\}$. Write

$$D_{t,n} := \{x \in D_t : \text{ for all } r \in (0, 1/n], \text{ whenever } x \in B(z,r) \text{ we have}$$
$$f(x) - t/2 < \frac{1}{\mu B(z,r)} \int_{B(z,r)} f d\mu < f(x) + t/2\}.$$

Then the sets $D_{t,n}$ are increasing and $D_t = \cup_n D_{t,n}$; so there exists an $R \in \mathbb{N}$ with $\nu(D_t \setminus D_{t,R}) < \varepsilon/2$. Since almost every point of $\{f > t\}$ is a Lebesgue point, we have that

$$\nu((\cup\{B(x_n, r_n) : 1 \le n < N\}) \cap D_{t,R}) > \varepsilon/2.$$

Disregard the balls from $\{B(x_n, r_n) : 1 \le n < N\}$ that do not intersect $D_{t,R}$, and by relabeling (and perhaps changing N) use again $\{B(x_n, r_n) : 1 \le n < N\}$ to denote this "thined out" collection. Now the selection process is as follows: If there are large balls, pick one; otherwise, the collection must contain many small disjoint balls, since the "area" covered is large (greater than $\varepsilon/2$); so select a "large" (with respect to ν) disjoint collection.

More precisely, suppose that, for some index m, we have $(4R)^{-1} < r_m$. If we also have $r_m \le R^{-1}$, we work with $B(x_m, r_m)$. Otherwise, pick any $x \in B(x_m, r_m) \cap D_{t,R}$, and note that by the approximate midpoint property, given any ball $B(u, s)$ and any $v \in B(u, s)$, there exists a $w \in X$ such that $d(u, w), d(v, w) < s/2$; so $v \in B(w, s/2) \subset B(u, s)$. Using this property a finite number of times, by successively halving r_m we obtain a $z \in X$ and an $r \in (1/(2R), 1/R]$ with $x \in B(z, r) \subset B(x_m, r_m)$.

Letting B be either $B(x_m, r_m)$ or $B(z, r)$, depending on whether or not $r_m \le 1/R$, and using the fact that μ is $c-h$ almost uniformly distributed (with $h(r) := \sup_{x\in X} \mu B(x, r)$), we have

$$\frac{t}{2} < f(x) - \frac{t}{2} < \frac{1}{\mu B}\int_B f d\mu \le \frac{\nu B}{c\, h(1/(4R))} \le \frac{\nu B(x_m, r_m)}{c\, h(1/(4R))};$$

so

$$\frac{t\ c\ h(1/(4R))}{2} < \nu B(x_m, r_m).$$

Recall that R depends on ε. Taking

$$C_1 = C_1(f, t, \varepsilon) := \frac{2\nu\{f > t\}}{c\, h(1/(4R))t},$$

we have

$$\nu(\{f > t\} \cap (\cup\{B(x_n, r_n) : 1 \le n < N\})) \le \nu\{f > t\} < C_1 \nu B(x_m, r_m).$$

Next, suppose that for every n, $r_n \le (4R)^{-1}$. Select $T \in \mathbb{N}$; so that

$$\nu((\cup_{n=1}^T B(x_n, r_n)) \cap D_{t,R}) > \nu((\cup\{B(x_n, r_n) : 1 \le n < N\}) \cap D_{t,R}) - \varepsilon/4 > \varepsilon/4,$$

and, by relabeling if needed, assume that the finite set $\{B(x_1, r_1), \dots, B(x_T, r_T)\}$ is ordered by decreasing radius; so large balls appear first. Now we utilize the classical Vitali selection process.

Let $B(x_{n_1}, r_{n_1}) = B(x_1, r_1)$, and once $B(x_{n_1}, r_{n_1}), \dots, B(x_{n_k}, r_{n_k})$ have been chosen, let $B(x_{n_{k+1}}, r_{n_{k+1}})$ be the first $B(x_m, r_m)$ in the list $\{B(x_1, r_1), \dots, B(x_T, r_T)\}$ that does not intersect any of the balls $B(x_{n_1}, r_{n_1}), \dots, B(x_{n_k}, r_{n_k})$. The process terminates, say, with M, after no disjoint balls from the $B(x_m, r_m)$'s are left. To estimate how much mass we have lost, pick $x_0 \in B(x_{n_1}, r_{n_1}) \cap D_{t,R}$. Then the union of all $B(x_m, r_m)$'s that intersect $B(x_1, r_1)$ is contained in $B(x_1, 3r_1)$. Also

$$\mu B(x_1, 3r_1)(f(x_0) - t/2) < \int_{B(x_1, 3r_1)} f d\mu < \mu B(x_1, 3r_1)(f(x_0) + t/2),$$

and

$$\mu B(x_1,r_1)(f(x_0)-t/2) < \int_{B(x_1,r_1)} f d\mu < \mu B(x_1,r_1)(f(x_0)+t/2).$$

By Lemma 3.1, μ is doubling with constant D/c; so dividing and using $f(x_0) > t$, we get

$$\frac{\nu B(x_1,3r_1)}{\nu B(x_1,r_1)} < \frac{f(x_0)+t/2}{f(x_0)-t/2}\left(\frac{D}{c}\right)^2 < 3\left(\frac{D}{c}\right)^2.$$

Now repeat the argument with the remaining $B(x_{n_k},r_{n_k})$'s to conclude that

$$\nu \cup_{n=1}^{T} B(x_n,r_n) < 3\left(\frac{D}{c}\right)^2 \nu \cup_{k=1}^{M} B(x_{n_k},r_{n_k}).$$

It follows that

$$\begin{aligned}\nu(\{f>t\}\cap(\cup\{B(x_n,r_n):1\le n<N\})) &< 3\varepsilon/4+\nu((\cup_{n=1}^{T}B(x_n,r_n))\cap D_{t,R})\\ &< 4\nu((\cup_{n=1}^{T}B(x_n,r_n))\cap D_{t,R}) < 12\left(\frac{D}{c}\right)^2 \nu\cup_{k=1}^{M} B(x_{n_k},r_{n_k}).\end{aligned}$$

Finally, setting $C = \max\{C_1, 12\,(D/c)^2\}$ we obtain the result. □

Neugebauer's covering theorem is now an immediate corollary of Theorem 3.4.

Corollary 3.5. *Let $0 \le f \in L^1(\mu)$, where the Borel measure μ is almost uniformly distributed on the geometrically doubling metric space X with the approximate midpoint property, and let ν be the absolutely continuous finite measure with density f. Fix $\varepsilon > 0$. Then there exists a constant $C = C(f,\varepsilon) > 0$ such that, for every collection of open balls $\{B(x_\alpha,r_\alpha):\alpha\in\Lambda\}$ with $\nu(\cup\{B(x_\alpha,r_\alpha):\alpha\in\Lambda\}) > \varepsilon$, there exists a finite disjoint subsequence $\{B(x_{n_j},r_{n_j}):1\le j\le M\}$ with*

$$\nu(\cup\{B(x_\alpha,r_\alpha):\alpha\in\Lambda\}) < C\nu(\cup\{B(x_{n_j},r_{n_j}):1\le j\le M\}).$$

Proof. First, it follows from the separability of geometrically doubling metric spaces that $\{B(x_\alpha,r_\alpha):\alpha\in\Lambda\}$ can be replaced by a countable collection $\{B(x_n,r_n):1\le n<N\}$ (where $N\le\infty$) with the same union. Select $\delta>0$ such that $\varepsilon < \nu(\cup\{B(x_n,r_n):1\le n<N\})-\delta$. Since $f\in L^1(\mu)$, all the level sets of f have finite measure; so we can select a $t>0$ with $\nu(X\setminus\{f>t\})<\delta$. Now

$$\varepsilon < \nu(\cup\{B(x_n,r_n):1\le n<N\})-\delta < \nu(\{f>t\}\cap(\cup\{B(x_n,r_n):1\le n<N\}));$$

so we can apply the preceding result. □

If μ is complete; that is, if all the sets with outer measure zero are measurable, then the open balls can be replaced by closed balls, and the proofs of Theorem 3.4 and the preceding corollary still work. This is so because by the Vitali covering theorem for fine coverings (see, for instance, [1, Theorem 5.5.2]) arbitrary unions of closed balls are measurable, since they can be reduced to countable unions plus a set of measure zero.

As an application of Neugebauer's covering theorem, we present an inequality for ν that is weaker than the restricted weak type (1,1) inequality. It entails, as we noted in the introduction, that if the restricted weak type (1,1) inequality fails, this can only be shown by considering an infinite collection of sets $\{E_n : n \geq 1\}$ with $\lim_{n\to\infty} \nu E_n = 0$.

Corollary 3.6. *Let $0 \leq f \in L^1(\mu)$, where the Borel measure μ is almost uniformly distributed on the geometrically doubling metric space X with the approximate midpoint property, and let ν be the absolutely continuous finite measure with density f. Then there exists a decreasing function $C(\varepsilon) : (0, \infty) \to (0, \infty)$ such that, for all $t > 0$ and all measurable sets E with $\nu E \geq \varepsilon$, we have $t\, \nu\{M^u_\nu \mathbf{1}_E > t\} \leq C(\varepsilon)\nu(E)$.*

Proof. If $t \geq 1$, then $\{M^u_\nu \mathbf{1}_E > t\} = \emptyset$; while if $0 < t < 1$, then $\nu\{\mathbf{1}_E > t\} = \nu E$. Now by the Lebesgue differentiation theorem, (which can be applied since ν is defined by a density with respect to a doubling measure, see below for more details on this assertion)

$$M^u_\nu \mathbf{1}_E(x) \geq \limsup_{r\to 0} \frac{1}{\nu B(x,r)} \int_{B(x,r)} \mathbf{1}_E d\nu \geq \mathbf{1}_E(x)$$

ν-a.e.; so $\nu\{M^u_\nu \mathbf{1}_E > t\} \geq \nu E$ whenever $t < 1$. The result now follows by a standard argument. Cover $\{M^u_\nu \mathbf{1}_E > t\}$ with open balls $B(y,s)$ satisfying $t\, \nu B(y,s) < \int_{B(y,s)} \mathbf{1}_E d\nu$, and apply the preceding corollary to this collection of balls.

Regarding the well known fact mentioned above, whereby Lebesgue differentiation holds for densities of doubling measures, we want to show that, for $g \in L^1(\mu)$, we have ν-a.e. x,

$$g(x) = \lim_{\substack{x\in B(z,r)\\ r\to 0}} \frac{1}{\nu B(z,r)} \int_{B(z,r)} g(y) d\nu(y).$$

Now since μ is doubling, μ-a.e. x,

$$f(x) = \lim_{\substack{x\in B(z,r)\\ r\to 0}} \frac{\nu B(z,r)}{\mu B(z,r)}.$$

Now, on $\{f > 0\}$ we have μ-a.e. x,

$$\begin{aligned}
&\lim_{\substack{x\in B(z,r)\\ r\to 0}} \frac{1}{\nu B(z,r)} \int_{B(z,r)} g(y) d\nu(y) \\
&\quad = \lim_{\substack{x\in B(z,r)\\ r\to 0}} \frac{\mu B(z,r)}{\nu B(z,r)} \frac{1}{\mu B(z,r)} \int_{B(z,r)} g(y) f(y) d\mu(y) \\
&\quad = \frac{1}{f(x)} g(x) f(x) = g(x).
\end{aligned}$$

□

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