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FIXED POINTS OF A CLASS OF UNITARY OPERATORS

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ABSTRACT. In this paper, we consider a class of unitary operators defined on the Bergman space of the right half plane and characterize the fixed points of these unitary operators. We also discuss certain intertwining properties of these operators. Applications of these results are also obtained.

1. Introduction and preliminaries

Let $\mathbb{C}_+ = \{s = x + iy \in \mathbb{C} : x > 0\}$ be the right half plane. Let $d\widetilde{A}(s)$ denote the two dimensional area measure on $\mathbb{C}_+$. Let $L^2(\mathbb{C}_+, d\widetilde{A})$ be the space of complex-valued, absolutely square-integrable, and measurable functions on $\mathbb{C}_+$ with respect to the area measure. The Bergman space of the right half plane denoted by $L^2_a(\mathbb{C}_+)$ is the closed subspace of $L^2(\mathbb{C}_+, d\widetilde{A})$ consisting of holomorphic functions. The functions $H(s,w) = \frac{1}{(s+\overline{w})^2}$, $s \in \mathbb{C}_+$ *and* $w \in \mathbb{C}_+$, are the reproducing kernel [3] for $L^2_a(\mathbb{C}_+)$. Let $\mathbf{h}_w(s) = \frac{H(s,w)}{\sqrt{H(w,w)}} = \frac{2\mathrm{Re}w}{(s+\overline{w})^2}$. The functions $\mathbf{h}_w$, $w \in \mathbb{C}_+$, are the normalized reproducing kernels for $L^2_a(\mathbb{C}_+)$. Let $L^\infty(\mathbb{C}_+)$ be the space of complex-valued, essentially bounded, and Lebesgue measurable functions on $\mathbb{C}_+$. Define for $f \in L^\infty(\mathbb{C}_+)$, $\|f\|_\infty = \operatorname{ess\,sup}_{s\in\mathbb{C}_+} |f(s)| < \infty$. The space $L^\infty(\mathbb{C}_+)$ is a Banach space with respect to the essential supremum norm. For $\phi \in L^\infty(\mathbb{C}_+)$, we define the multiplication operator $\mathcal{M}_\phi$ from $L^2(\mathbb{C}_+, d\widetilde{A})$ into $L^2(\mathbb{C}_+, d\widetilde{A})$ by $(\mathcal{M}_\phi f)(s) = \phi(s)f(s)$; the Toeplitz operator $\mathcal{T}_\phi$ from $L^2_a(\mathbb{C}_+)$ into

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$L_a^2(\mathbb{C}_+)$ is defined by $\mathcal{T}_\phi f = P_+(\phi f)$ where P_+ is the orthogonal projection from $L^2(\mathbb{C}_+, d\widetilde{A})$ onto $L_a^2(\mathbb{C}_+)$. The Toeplitz operator $\mathcal{T}_\phi$ is bounded and $\|\mathcal{T}_\phi\| \leq \|\phi\|_\infty$. For more details see [4] and [7].

Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$. Let $dA(z)$ denote the Lebesgue area measure on $\mathbb{D}$, normalized so that the measure of $\mathbb{D}$ equals 1. Let $L^2(\mathbb{D}, dA)$ be the space of complex-valued, absolutely square-integrable, and measurable functions on $\mathbb{D}$ with respect to the normalized area measure. The Bergman space of the open unit disk denoted by $L_a^2(\mathbb{D})$ is the Hilbert space consisting of analytic functions on $\mathbb{D}$ that are also in $L^2(\mathbb{D}, dA)$. Since the point evaluation at $z \in \mathbb{D}$, is a bounded functional, there is a function K_z in $L_a^2(\mathbb{D})$ such that

$$f(z) = \langle f, K_z \rangle$$

for all f in $L_a^2(\mathbb{D})$. Let $K(z,w)$ be the function on $\mathbb{D} \times \mathbb{D}$ defined by $K(z,w) = \overline{K_z(w)}$. The function $K(z,w) = \frac{1}{(1-z\overline{w})^2}$, $z, w \in \mathbb{D}$, is called the Bergman reproducing kernel [10]. For $a \in \mathbb{D}$, let $k_a(z) = \frac{K(z,a)}{\sqrt{K(a,a)}} = \frac{1-|a|^2}{(1-\overline{a}z)^2}$ is called the normalized reproducing kernel for $L_a^2(\mathbb{D})$.

Let $Aut(\mathbb{D})$ be the Lie group of all automorphisms (biholomorphic mappings) of $\mathbb{D}$, and let G_0 be the isotropy subgroup at 0; that is, $G_0 = \{\psi \in Aut(\mathbb{D}) : \psi(0) = 0\}$. It is well known [8] that G_0 is compact and that G_0 is a subgroup of the unitary group $\mathcal{U}$ of $\mathbb{C}$. Since $\mathbb{D}$ is bounded symmetric, we can canonically define [1] for each a in $\mathbb{D}$ an automorphism ϕ_a in $Aut(\mathbb{D})$ such that

(1) $\phi_a \circ \phi_a(z) \equiv z$;
(2) $\phi_a(0) = a, \phi_a(a) = 0$;
(3) ϕ_a has a unique fixed point in $\mathbb{D}$.

Actually, the above three conditions completely characterize the ϕ_a's as the set of all (holomorphic) geodesic symmetries of $\mathbb{D}$. In fact, $\phi_a(z) = \frac{a-z}{1-\overline{a}z}$ for all a and z in $\mathbb{D}$. They are involutive Mobius transformations on $\mathbb{D}$. Given $a \in \mathbb{D}$ and f is any measurable function on $\mathbb{D}$, we define a function $U_a f$ on $\mathbb{D}$ by $U_a f(z) = k_a(z) f(\phi_a(z))$. Since $|k_a|^2$ is the real Jacobian determinant of the mapping ϕ_a (see [1]), U_a is easily seen to be a unitary operator on $L^2(\mathbb{D}, dA)$ and $L_a^2(\mathbb{D})$. For any $a \in \mathbb{D}$, let γ_a be the unique geodesic (all geodesics are taken in the Bergman metric on $\mathbb{D}$) such that $\gamma_a(0) = 0$ and $\gamma_a(1) = a$. Since $\mathbb{D}$ is Hermitian symmetric, there exists a unique $\phi_a \in Aut(\mathbb{D})$ such that $\phi_a \circ \phi_a(z) \equiv z$, $\gamma_a(\frac{1}{2})$ is an isolated fixed point of ϕ_a and ϕ_a is the geodesic symmetry at $\gamma_a(\frac{1}{2})$. In particular, $\phi_a(0) = a$ and $\phi_a(a) = 0$. If $a = 0$, then we have $\phi_a(z) = -z$ for all z in $\mathbb{D}$. We denote by m_a the geodesic midpoint $\gamma_a(\frac{1}{2})$ of 0 and a. Given $\psi \in Aut(\mathbb{D})$, let $a = \psi^{-1}(0)$; then we have

$$(\psi \circ \phi_a)(0) = \psi(a) = 0;$$

thus $\psi \circ \phi_a \in G_0$, and so there exists a unitary matrix U such that $\psi = U\phi_a$ ($U \in G_0$). If $\psi \in Aut(\mathbb{D})$ has an isolated fixed point in $\mathbb{D}$, then ψ has a unique fixed point and each ϕ_a has m_a as a unique fixed point. It is also not difficult to see that for any a and b in $\mathbb{D}$, there exists a unitary $U \in G_0$ such that $\phi_b \circ \phi_a = U\phi_{\phi_a(b)}$. This can be verified as follows:

let $U = \phi_b \circ \phi_a \circ \phi_{\phi_a(b)}$. Then $U(0) = \phi_b \circ \phi_a(\phi_a(b)) = \phi_b(b) = 0$; thus $U \in G_0$ is a unitary.

It is also not difficult to check that if $a \in \mathbb{D}$, then $m_a = \frac{1-\sqrt{1-|a|^2}}{|a|^2}a$. One can also check that $k_a(m_a) = 1$, for all $a \in \mathbb{D}$, and that $U_a k_{m_a} = 1$, for all $a \in \mathbb{D}$, and that $\phi_\lambda(m_a) = m_{\phi_\lambda(a)}$ for any $\lambda \in \mathbb{D}$ and $a \in \mathbb{D}$. Let $\mathcal{L}(H)$ be the set of all bounded linear operators from the Hilbert space H into itself. Let $I_{\mathcal{L}(H)}$ denote the identity operator in $\mathcal{L}(H)$. Recall that an operator $T \in \mathcal{L}(H)$ is said to be hyponormal if $T^*T \geq TT^*$. An operator $T \in \mathcal{L}(H)$ is an involution if $T^2 = I_{\mathcal{L}(H)}$.

The layout of this paper is as follows: In section 2, we introduce the unitary operator $V_a,\ a \in \mathbb{D}$, and prove certain elementary properties of the operator V_a. In section 3, we calculate the fixed points of a class of weighted composition operators $W_a,\ a \in \mathbb{D}$, defined on $L^2_a(\mathbb{C}_+)$. We then use it to calculate the fixed points of the unitary operators $V_a,\ a \in \mathbb{D}$. In section 4, we discuss certain intertwining properties of the operators $V_a,\ a \in \mathbb{D}$.

2. The unitary operator V_a

In this section, we shall introduce a class of unitary operators $V_a,\ a \in \mathbb{D}$, and establish certain elementary properties of these operators.

Define $M : \mathbb{C}_+ \to \mathbb{D}$ by $Ms = \frac{1-s}{1+s}$. Then M is one-one, onto, and $M^{-1} : \mathbb{D} \to \mathbb{C}_+$ is given by $M^{-1}(z) = \frac{1-z}{1+z}$. Thus M is its self-inverse. Let $W : L^2_a(\mathbb{D}) \to L^2_a(\mathbb{C}_+)$ be defined by $Wg(s) = \frac{2}{\sqrt{\pi}} g(Ms)\frac{1}{(1+s)^2}$. Then $W^{-1} : L^2_a(\mathbb{C}_+) \to L^2_a(\mathbb{D})$ is given by $W^{-1}G(z) = 2\sqrt{\pi}G(Mz)\frac{1}{(1+z)^2}$, where $Mz = \frac{1-z}{1+z}$.

Lemma 2.1. *If $a \in \mathbb{D}$ and $a = c + id,\ c, d \in \mathbb{R}$, then $t_a(s) = \frac{-ids+(1-c)}{(1+c)s+id}$ is an automorphism from $\mathbb{C}_+$ onto $\mathbb{C}_+$.*

Proof. It is not difficult to see that $t_a(s)$ is an one-one map from $\mathbb{C}_+$ onto $\mathbb{C}_+$. □

Proposition 2.2. *For $a \in \mathbb{D}$ and $s \in \mathbb{C}_+$, define $\psi_s(a) = t_a(s)$. Then the following conditions hold:*

(1) $(t_a \circ t_a)(s) = s$;
(2) $t'_a(s) = -l_a(s)$, *where* $l_a(s) = \frac{1-|a|^2}{((1+c)s+id)^2}$;
(3) $\phi_{m_a} \circ \phi_a = -\phi_{m_a}$;
(4) *The function ψ_s, as a function in a, is one-one and onto for any fixed $s \in \mathbb{C}_+$.*

Proof. One can prove (1) and (2) by direct calculations. To establish (3), let $U = \phi_{m_a} \circ \phi_a \circ \phi_{m_a}$. Then $U(0) = \phi_{m_a} \circ \phi_a(m_a) = \phi_{m_a}(m_a) = 0$. Thus $U \in G_0$ is a unitary. Moreover, $U^2 = \phi_{m_a} \circ \phi_a \circ \phi_{m_a} \circ \phi_{m_a} \circ \phi_a \circ \phi_{m_a} = \xi$ where $\xi(z) = z$, since $\phi_a \circ \phi_a = \xi$ for all $a \in \mathbb{D}$. Hence the eigenvalues of U are either 1 or -1. We shall show that all the eigenvalues of U are -1. In fact, if there exists $z \neq 0,\ z \in \mathbb{C}$, such that $Uz = z$, then $U(\epsilon z) = \epsilon z$ for all $\epsilon \in (0, 1)$. Choose ϵ small enough; so that $z_0 = \epsilon z \in \mathbb{D}$; then $Uz_0 = z_0$ implies $\phi_{m_a} \circ \phi_a \circ \phi_{m_a}(z_0) = z_0$ or $\phi_a(\phi_{m_a}(z_0)) = \phi_{m_a}(z_0)$. Therefore, $\phi_{m_a}(z_0)$ is a fixed point of ϕ_a. Since ϕ_a has m_a as a unique fixed point; hence we get $\phi_{m_a}(z_0) = m_a$. This implies $z_0 =$

$\phi_{m_a} \circ \phi_{m_a}(z_0) = \phi_{m_a}(m_a) = 0$, contradicting the fact that $z_0 = \epsilon z \neq 0$. Hence all the eigenvalues of U are -1, and we have $U = -\xi$, or $\phi_{m_a} \circ \phi_a = -\phi_{m_a}$. This proves (3). To prove (4), suppose that $\psi_s(a_1) = \psi_s(a_2)$. Then $t_{a_1}(s) = t_{a_2}(s)$. Hence $(M \circ \phi_{a_1} \circ M) = (M \circ \phi_{a_2} \circ M)$. This implies $\phi_{a_1}(z) = \phi_{a_2}(z)$ for some fixed $z \in \mathbb{D}$. We shall now show that $\phi_a(z)$, as a function in a, is one-one and onto for any fixed $z \in \mathbb{D}$. Let $w = \phi_a(z) = \frac{a-z}{1-\overline{a}z}$. Then $w - \overline{a}zw = a - z$. Taking conjugates both the sides, we obtain $\overline{w} - a\overline{z}\,\overline{w} = \overline{a} - \overline{z}$. Solving for a and $\overline{a}$ yield

$$a = \frac{w + z - z|w|^2 - w|z|^2}{1 - |zw|^2}.$$

The result (4) follows. $\square$

Suppose that $a \in \mathbb{D}$ and that $w = \frac{1-\overline{a}}{1+\overline{a}} = M\overline{a} \in \mathbb{C}_+$. Define $b_{\overline{w}}(s) = \frac{1}{\sqrt{\pi}} \frac{1+w}{1+\overline{w}} \frac{2Rew}{(s+w)^2}$.

Lemma 2.3. *The set of vectors* $\{b_{\overline{w}} : w \in \mathbb{C}_+\}$ *spans* $L^2_a(\mathbb{C}_+)$.

Proof. Suppose that $g \in L^2_a(\mathbb{D})$ and that g is orthogonal to $K_a, a \in \mathbb{D}$. Then $g(a) = \langle g, K_a \rangle = 0$ for all $a \in \mathbb{D}$; that is, $g = 0$. Hence span $\{k_a : a \in \mathbb{D}\}$ is dense in $L^2_a(\mathbb{D})$.

Let $w \in \mathbb{C}_+$, and let $\overline{w} = Ma, a \in \mathbb{D}$. Since $b_{\overline{w}} = Wk_a$ and W is an unitary operator from $L^2_a(\mathbb{D})$ onto $L^2_a(\mathbb{C}_+)$, hence $\{b_{\overline{w}} : w \in \mathbb{C}_+\}$ spans $L^2_a(\mathbb{C}_+)$. This can be verified as follows.

Let $f \in L^2_a(\mathbb{C}_+)$. Then $f = Wg$ for some $g \in L^2_a(\mathbb{D})$. Now since $g = \lim_{n\to\infty} g_n$, where the functions g_n are linear combinations of certain normalized Bergman kernels $k_a, a \in \mathbb{D}$, hence $f = Wg = \lim_{n\to\infty} Wg_n$, where Wg_n is a linear combination of certain $b_{\overline{w}},\ w \in \mathbb{C}_+$. Thus the set span$\{b_{\overline{w}} : w \in \mathbb{C}_+\}$ is dense in $L^2_a(\mathbb{C}_+)$. $\square$

For $a \in \mathbb{D}$ and $f \in L^2_a(\mathbb{C}_+)$, define V_a from $L^2_a(\mathbb{C}_+)$ into itself by $V_a f = (f \circ t_a)l_a$. In Proposition 2.4, we show that V_a is a self-adjoint unitary operator which is also an involution.

Proposition 2.4. *For* $a \in \mathbb{D}$. *The following conditions hold:*

(1) $V_a l_a = 1$.
(2) $V_a^{-1} = V_a, V_a^2 = I$.
(3) V_a *is self-adjoint.*
(4) V_a *is unitary.*
(5) $V_a P_+ = P_+ V_a$.

Proof. One can prove (1), (2), (3), and (4) by direct calculations. Notice that V_a can also defined on $L^2(\mathbb{C}_+)$ and $V_a(L^2(\mathbb{C}_+)) \subseteq L^2(\mathbb{C}_+)$. To prove (5), observe that $V_a(L^2_a(\mathbb{C}_+)) \subseteq L^2_a(\mathbb{C}_+)$ and that $V_a(L^2_a(\mathbb{C}_+))^\perp \subseteq (L^2_a(\mathbb{C}_+))^\perp$. Now let $f \in L^2(\mathbb{C}_+)$, and let $f = f_1 + f_2$, where $f_1 \in L^2_a(\mathbb{C}_+)$ and $f_2 \in (L^2_a(\mathbb{C}_+))^\perp$. Hence,

$$\begin{aligned} P_+ V_a f &= P_+ V_a(f_1 + f_2) \\ &= P_+(V_a f_1 + V_a f_2) \\ &= P_+ V_a f_1 \\ &= V_a f_1 \\ &= V_a P_+ f. \end{aligned}$$

□

Let $\mathcal{L}(L_a^2(\mathbb{C}_+))$ be the set of all bounded linear operators from the Hilbert space H into itself.

Lemma 2.5. *For any sequence $\{a_m\}_{m=1}^{\infty} \subset \mathbb{D}$ with $|a_m| \to 1$, then $V_{a_m} \to 0$ in the weak operator topology in $\mathcal{L}(L_a^2(\mathbb{C}_+))$.*

Proof. From Lemma 2.3, it follows that $\operatorname{span}\{b_{\overline{w}} : w \in \mathbb{C}_+\}$ is dense in $L_a^2(\mathbb{C}_+)$. Thus it suffices to show that for all $w_1, w_2 \in \mathbb{C}_+$, we have $\lim\limits_{m\to\infty} \langle V_{a_m} b_{\overline{w}_1}, b_{\overline{w}_2}\rangle = 0$.

Let $\overline{w}_1 = Mz_1$ and $\overline{w}_2 = Mz_2,\ z_1, z_2 \in \mathbb{D}$. Fix $w_1, w_2 \in \mathbb{C}_+$. Now, for each $m \geq 1$,

$$\begin{aligned}
\langle V_{a_m} b_{\overline{w}_1}, b_{\overline{w}_2}\rangle &= \langle W U_{a_m} W^{-1} b_{\overline{w}_1}, b_{\overline{w}_2}\rangle \\
&= \langle W U_{a_m} W^{-1} W k_{z_1}, W k_{z_2}\rangle \\
&= \langle U_{a_m} k_{z_1}, k_{z_2}\rangle \\
&= \left\langle U_{a_m} k_{z_1}, \frac{K_{z_2}}{\|K_{z_2}\|}\right\rangle \\
&= (1-|z_2|^2)(U_{a_m} k_{z_1})(z_2) \\
&= (1-|z_2|^2) k_{z_1}(\phi_{a_m}(z_2)) k_{a_m}(z_2) \\
&= \frac{(1-|z_2|^2)(1-|z_1|^2)(1-|a_m|^2)}{(1-\langle \phi_{a_m}(z_2), z_1\rangle)^2 (1-\langle z_2, a_m\rangle)^2}.
\end{aligned}$$

Since $|\langle \phi_{a_m}(z_2), z_1\rangle| \leq |z_1|$ and $|\langle z_2, a_m\rangle| \leq |z_2|$, we obtain

$$|\langle U_{a_m} k_{z_1}, k_{z_2}\rangle| \leq \frac{(1-|z_2|^2)(1-|z_1|^2)(1-|a_m|^2)}{((1-|z_1|)(1-|z_2|))^2}.$$

Hence it follows that $\lim\limits_{m\to\infty} \langle U_{a_m} k_{z_1}, k_{z_2}\rangle = 0$. Thus $\lim\limits_{m\to\infty} \langle V_{a_m} b_{\overline{w}_1}, b_{\overline{w}_2}\rangle = 0$. □

3. The fixed points of V_a

In this section, we shall first calculate the fixed points of a class of weighted composition operators $W_a,\ a \in \mathbb{D}$, defined on $L_a^2(\mathbb{C}_+)$. We then use it to calculate the fixed points of the unitary operators $V_a, a \in \mathbb{D}$. But we begin with the following proposition which will be frequently used in establishing results of the section.

Proposition 3.1. *For $a \in \mathbb{D}$ and $s \in \mathbb{C}_+$, the following equalities hold:*

(1) $(M \circ t_{m_a} \circ t_a)(s) = -(M \circ t_{m_a})(s)$;
(2) $(M' \circ t_{m_a} \circ t_a)(s) l_{m_a}(t_a(s)) l_a(s) = (M' \circ t_{m_a})(s) l_{m_a}(s)$.

Proof. Since $\phi_{m_a} \circ \phi_a = -\phi_{m_a}$, hence

$$(M \circ t_{m_a} \circ M \circ M \circ t_a \circ M)(z) = -(M \circ t_{m_a} \circ M)(z)$$

for all $z \in \mathbb{D}$, and therefore

$$(M \circ t_{m_a} \circ t_a)(s) = -(M \circ t_{m_a})(s)$$

for all $s \in \mathbb{C}_+$. This proves (1). To prove (2), consider the identity

$$\phi_{m_a} \circ \phi_a = -\phi_{m_a}. \tag{3.1}$$

Since $\phi_a'(z) = -k_a(z)$, taking derivatives both the sides in (3.1), we obtain

$$(k_{m_a} \circ \phi_a)k_a = k_{m_a}.$$

That is, $U_a k_{m_a} = k_{m_a}$ for all $a \in \mathbb{D}$. Hence, $U_a U_{m_a} 1 = U_{m_a} 1$. This implies,

$$(WU_aW^{-1})(WU_{m_a}W^{-1})\left(\frac{(-1)}{\sqrt{\pi}}M'\right) = (WU_{m_a}W^{-1})\left(\frac{(-1)}{\sqrt{\pi}}M'\right).$$

Hence

$$V_aV_{m_a}\left(\frac{(-1)}{\sqrt{\pi}}M'\right) = V_{m_a}\left(\frac{(-1)}{\sqrt{\pi}}M'\right). \tag{3.2}$$

Now, observe that for all $a \in \mathbb{D}$, $V_a b_{\overline{w}} = \frac{(-1)}{\sqrt{\pi}}M'$, where $w = M\overline{a}$. Thus,

$$V_{m_a} b_{Mm_a} = \left(\frac{-1}{\sqrt{\pi}}\right) M'.$$

In other words,

$$\begin{aligned} b_{Mm_a} &= V_{m_a}^{-1}\left(\frac{(-1)}{\sqrt{\pi}}M'\right) \\ &= V_{m_a}\left(\frac{(-1)}{\sqrt{\pi}}M'\right). \end{aligned}$$

Thus from (3.2), it follows that

$$V_a b_{Mm_a} = b_{Mm_a} \text{ and } V_a\left(\frac{(-1)}{\sqrt{\pi}}(M' \circ t_{m_a})l_{m_a}\right) = \frac{(-1)}{\sqrt{\pi}}(M' \circ t_{m_a})l_{m_a}.$$

This implies that

$$V_a[(M' \circ t_{m_a})l_{m_a}] = (M' \circ t_{m_a})l_{m_a}.$$

Thus,

$$(M' \circ t_{m_a} \circ t_a)(l_{m_a} \circ t_a)l_a = (M' \circ t_{m_a})l_{m_a}.$$

This proves the proposition. □

For $a \in \mathbb{D}$, define the weighted composition operator W_a on $L^2_a(\mathbb{C}_+)$ by $W_a f = (f \circ t_a)\frac{M'}{M' \circ t_a}$. In the following theorem, we describe the fixed points of the weighted composition operator W_a which will enable us to obtain the fixed points of the unitary operator V_a.

Theorem 3.2. *Given $a \in \mathbb{D}$ and a function $f \in L^2(\mathbb{C}_+)$, we have $W_a f = f$ if and only if there exists an even function g on $\mathbb{C}_+$ (i.e, $g(z) = g(-z)$) such that $f = (g \circ M \circ t_{m_a})M'$. Further, $W_a f = -f$ if and only if there exists an odd function g on $\mathbb{C}_+$ (i.e. $g(z) = -g(-z)$) such that $f = (g \circ M \circ t_{m_a})M'$.*

Proof. We shall prove the first assertion. The second one has a similar proof. Suppose that $g(z) = g(-z)$ and that $f = (g \circ M \circ t_{m_a})M'$. Then by Proposition 3.1, we obtain

$$\begin{aligned} W_a f &= (f \circ t_a)\frac{M'}{M' \circ t_a} \\ &= (g \circ M \circ t_{m_a} \circ t_a)(M' \circ t_a)\frac{M'}{M' \circ t_a} \\ &= (g \circ M \circ t_{m_a} \circ t_a)M' \\ &= g(-(M \circ t_{m_a}))M' \\ &= g(M \circ t_{m_a})M' \\ &= (g \circ M \circ t_{m_a})M' = f. \end{aligned}$$

Conversely, suppose that $W_a f = f$; that is, $(f \circ t_a)\frac{M'}{M' \circ t_a} = f$. Suppose that $g = \left(\frac{f}{M'} \circ t_a \circ t_{m_a} \circ M\right)$. Then

$$\begin{aligned} (g \circ M \circ t_{m_a})M' &= \left(\frac{f}{M'} \circ t_a \circ t_{m_a} \circ M \circ M \circ t_{m_a}\right) M' \\ &= \left(\frac{f}{M'} \circ t_a \circ t_{m_a} \circ t_{m_a}\right) M' \\ &= \left(\frac{f}{M'} \circ t_a\right) M' \\ &= \left(\frac{f \circ t_a}{M' \circ t_a}\right) M' \\ &= (f \circ t_a)\frac{M'}{M' \circ t_a} \\ &= f. \end{aligned}$$

Hence

$$\begin{aligned} (g \circ M \circ t_{m_a})(z)M'(z) = f(z) &= (f \circ t_a)(z)\frac{M'}{M' \circ t_a}(z) \\ &= (g \circ M \circ t_{m_a} \circ t_a)(z)(M' \circ t_a)(z)\frac{M'(z)}{M' \circ t_a(z)} \\ &= g(-(M \circ t_{m_a}))(z)M'(z). \end{aligned}$$

Thus $g((M \circ t_{m_a})(z)) = g(-(M \circ t_{ma})(z))$. Putting $(t_{m_a} \circ M)(z)$ in place of z, we obtain $g(z) = g((M \circ t_{m_a} \circ t_{m_a} \circ M)(z)) = g(-(M \circ t_{m_a} \circ t_{m_a} \circ M)(z)) = g(-z)$. Thus g is an even function, and $f = (g \circ M \circ t_{m_a})M'$. This completes the proof of theorem. □

Theorem 3.3. *Let $a \in \mathbb{D}$, and let $f \in L^2_a(\mathbb{C}_+)$. Then*

(1) *$V_a f = f$ if and only if there exists an even function $g \in L^2_a(\mathbb{C}_+)$ such that*

$$f = (M' \circ t_{m_a})l_{m_a}(g \circ M \circ t_{m_a}).$$

(2) *$V_a f = -f$ if and only if there exists an odd function $g \in L^2_a(\mathbb{C}_+)$ such that*

$$f = (M' \circ t_{m_a}) l_{m_a} (g \circ M \circ t_{m_a}).$$

Proof. We shall only prove (1). The proof of (2) is similar. Suppose that $g(s) = g(-s)$ and that $f = (M' \circ t_{m_a}) l_{m_a} (g \circ M \circ t_{m_a})$. Then

$$V_a f = l_a (f \circ t_a) = l_a (M' \circ t_{m_a} \circ t_a)(l_{m_a} \circ t_a)(g \circ M \circ t_{m_a} \circ t_a).$$

Since by Proposition 3.1,

$$M \circ t_{m_a} \circ t_a = -M \circ t_{m_a} \text{ and } l_a (M' \circ t_{m_a} \circ t_a)(l_{m_a} \circ t_a) = (M' \circ t_{m_a}) l_{m_a},$$

we obtain

$$V_a f = (M' \circ t_{m_a}) l_{m_a} g(-M \circ t_{m_a}) = (M' \circ t_{m_a}) l_{m_a} g(M \circ t_{m_a}) = f.$$

Conversely, suppose that $V_a f = f$; we seek to find an even function g such that

$$f = (M' \circ t_{m_a}) l_{m_a} (g \circ M \circ t_{m_a}).$$

Let $g(Ms) = (M' \circ M)(s) l_{m_a}(s) f(t_{m_a}(s))$. Since $l_{m_a}(s) l_{m_a}(t_{m_a}(s)) \equiv 1$, we have

$$g(Ms) l_{m_a}(t_{m_a}(s)) = M'(Ms) f(t_{m_a}(s)). \tag{3.3}$$

Replacing s by $t_{m_a}(s)$ in (3.3), we get $f(s) = (M' \circ t_{m_a})(s) l_{m_a}(s)(g \circ M \circ t_{m_a})(s)$. Now it remains to show that g is even. It follows from Proposition 3.1 that, for any $s \in \mathbb{C}_+$,

$$\begin{aligned}
(g \circ M \circ t_{m_a})(s) &= (M' \circ M \circ t_{m_a})(s) l_{m_a}(t_{m_a}(s)) f(s) \\
&= (M' \circ M \circ t_{m_a})(s) l_{m_a}(t_{m_a}(s)) l_a(s) f(t_a(s)) \\
&= (M' \circ M \circ t_{m_a})(s) \\
&\quad l_{m_a}(t_{m_a}(s)) l_a(s) (M' \circ t_{m_a} \circ t_a)(s) l_{m_a}(t_a(s)) (g \circ M \circ t_{m_a} \circ t_a)(s) \\
&= (M' \circ M \circ t_{m_a})(s) \\
&\quad l_{m_a}(t_{m_a}(s)) (M' \circ t_{m_a})(s) l_{m_a}(s) g(-(M \circ t_{m_a})(s)) \\
&= g(-M \circ t_{m_a}(s)).
\end{aligned} \tag{3.4}$$

The last identity follows from the fact that $l_{m_a}(t_{m_a}(s)) l_{m_a}(s) = 1$, for all $s \in \mathbb{C}_+$, and

$$\begin{aligned}
(M' \circ M \circ t_{m_a})(s)(M' \circ t_{m_a})(s) &= ([(M' \circ M) M'] \circ t_{m_a})(s) \\
&= [(1 \circ t_{m_a})](s) = 1.
\end{aligned}$$

Replacing s by $(t_{m_a} \circ M)(s)$ in (3.4), we obtain $g(s) = g(-s)$ for all $s \in \mathbb{C}_+$. This proves our claim. ☐

Corollary 3.4. *Suppose that $a \in \mathbb{D}$ and that $f \in L^2_a(\mathbb{C}_+)$. Then $V_a f = f$ if and only if*

$$f = (M' \circ t_{m_a})(g_1 \circ M \circ t_{m_a}) l_{m_a},$$

where

$$(g_1 \circ M)(s) = \frac{1}{2}[(M' \circ M)(s)(f \circ t_{m_a})(s)l_{m_a}(s) + (M' \circ M)(-s)(f \circ t_{m_a})(-s)l_{m_a}(-s)], \quad s \in \mathbb{C}_+,$$

and $V_a f = -f$ *if and only if* $f = (M' \circ t_{m_a})(g_2 \circ t_{m_a})l_{m_a}$, *where*

$$(g_2 \circ M)(s) = \frac{1}{2}[(M' \circ M)(s)(f \circ t_{m_a})(s)l_{m_a}(s) - (M' \circ M)(-s)(f \circ t_{m_a})(-s)l_{m_a}(-s)], \quad s \in \mathbb{C}_+.$$

Proof. Let $V_a = P_a - P_a^{\perp}$ be the spectral decomposition of V_a. Then $V_a f = f$ if and only if $P_a f = f$ for any $f \in L^2(\mathbb{C}_+, d\widetilde{A})$(or $L_a^2(\mathbb{C}_+)$). Thus if M_a is the range space of P_a, then we have $M_a = \{(M' \circ t_{m_a})(g \circ M \circ t_{m_a})l_{m_a} : g \text{ even}\}$. Suppose that $f \in L^2(\mathbb{C}_+, d\widetilde{A})$(or $L_a^2(\mathbb{C}_+)$); then the even function g_1 satisfying

$$P_a f = (M' \circ t_{m_a})(g_1 \circ M \circ t_{m_a})l_{m_a} = f$$

is given by the formula

$$(g_1 \circ M)(s) = \frac{1}{2}[(M' \circ M)(s)(f \circ t_{m_a})(s)l_{m_a}(s) + (M' \circ M)(-s)(f \circ t_{m_a})(-s)l_{m_a}(-s)]$$

and the odd function g_2 with $P_a^{\perp} f = (M' \circ t_{m_a})(g_2 \circ M \circ t_{m_a})l_{m_a} = f$ is given by the formula

$$(g_2 \circ M)(s) = \frac{1}{2}[(M' \circ M)(s)(f \circ t_{m_a})(s)l_{m_a}(s) - (M' \circ M)(-s)(f \circ t_{m_a})(-s)l_{m_a}(-s)].$$

They are obtained by using the formula $P_a = \frac{1}{2}(I + V_a)$ and Theorem 3.3. □

4. Intertwining properties of the unitary operator V_a

In this section, we discuss certain intertwining proposition of the operators $V_a,\ a \in \mathbb{D}$.

Theorem 4.1. *Suppose that* $\phi \in L^\infty(\mathbb{C}_+, d\widetilde{A}), \phi \geq 0, A \geq 0, A \in \mathcal{L}(L_a^2(\mathbb{C}_+))$ *and that* $\mathcal{T}_\phi \leq A \leq V_a \mathcal{T}_\phi V_a$ *for some* $a \in \mathbb{D}$. *Then* $A = \mathcal{T}_\phi = \mathcal{T}_{\phi \circ t_a}$.

Proof. Suppose that $\mathcal{T}_\phi \leq A \leq V_a \mathcal{T}_\phi V_a = \mathcal{T}_{\phi \circ t_a}$. Since $\phi \geq 0$, hence $\langle \mathcal{T}_\phi f, f\rangle = \langle P_+(\phi f), f\rangle = \langle \phi f, f\rangle = \int_{\mathbb{C}_+} \phi |f|^2 d\widetilde{A} \geq 0$ for every $f \in L_a^2(\mathbb{C}_+)$. Hence $\mathcal{T}_\phi \geq 0$. Choose $\lambda > 0$ such that $\mathcal{T}_\phi + \lambda > 0$. Put $S = (\mathcal{T}_\phi + \lambda)^{\frac{1}{2}} V_a$. Thus

$$SS^* = \mathcal{T}_\phi + \lambda \leq A + \lambda \leq V_a(\mathcal{T}_\phi + \lambda)V_a = S^*S. \tag{4.1}$$

Hence S is a hyponormal operator. Further $|S| = V_a(\mathcal{T}_\phi + \lambda)^{\frac{1}{2}} V_a = V_a S$. Let $S = V|S|$ be the polar decomposition of S. Hence $S = V V_a S$. It follows from the invertibility of S that $I = V V_a$; that is, $V = V_a$. By [9], the operator S is normal, and from (4.1) it follows that $A = \mathcal{T}_\phi = \mathcal{T}_{\phi \circ t_a}$. □

Theorem 4.2. *Let* $T \in \mathcal{L}(L_a^2(\mathbb{C}_+))$. *The following conditions hold:*

(1) *If* $TV_a = V_aT$, *for all* $a \in \mathbb{D}$, *then* $T = \alpha I_{\mathcal{L}(L_a^2(\mathbb{C}_+))}$ *for some* $\alpha \in \mathbb{C}$.
(2) *If* $TV_{\phi_a} = V_{\phi_a}T$, *for all* $a \in \mathbb{D}$, *then* $T = \beta I_{\mathcal{L}(L_a^2(\mathbb{C}_+))}$ *for some* $\beta \in \mathbb{C}$.
(3) *If* $TV_a = V_aT$, *for some* $a \in \mathbb{D}$, *then* $M_a = \{(M' \circ t_{m_a})l_{m_a}(g \circ M \circ t_{m_a}) : g \text{ even}\}$ *is a reducing subspace of* T.

Proof. To prove (1), suppose $TV_a = V_aT$ for all $a \in \mathbb{D}$. Let $WSW^{-1}WU_aW^{-1} = WU_aW^{-1}WSW^{-1}$ for some $S \in \mathcal{L}(L_a^2(\mathbb{D}))$. This implies that $SU_a = U_aS$, for all $a \in \mathbb{D}$ where $S = W^{-1}TW$. Hence, by [2], $S = \alpha I_{\mathcal{L}(L_a^2(\mathbb{D}))}$ for some $\alpha \in \mathbb{C}$. Hence $T = \alpha I_{\mathcal{L}(L_a^2(\mathbb{C}_+))}$.

For proof of (2), assume that $TV_{\phi_a} = V_{\phi_a}T$ for all $a \in \mathbb{D}$. Let $T = WSW^{-1}$, $S \in \mathcal{L}(L_a^2(\mathbb{D}))$. Then $(WSW^{-1})(WU_{\phi_a}W^{-1}) = (WU_{\phi_a}W^{-1})(WSW^{-1})$ for all $a \in \mathbb{D}$. Therefore $SU_{\phi_a} = U_{\phi_a}S$ for all $a \in \mathbb{D}$. Hence $S = \beta I_{\mathcal{L}(L_a^2(\mathbb{D}))}$ for some $\beta \in \mathbb{C}$. This follows from a well-known fact from representation theory [6] of the Lie group $\mathrm{Aut}(\mathbb{D}) = SU(1,1) = SL_2(\mathbb{R})$. Thus $T = \beta I_{\mathcal{L}(L_a^2(\mathbb{C}_+))}$.

To prove (3), let $TV_a = V_aT$ for some $a \in \mathbb{D}$. Let $V_a = P_a - P_a^{\perp}$ be the spectral decomposition of V_a. Then $V_af = f$ if and only if $P_af = f$ for any $f \in L_a^2(\mathbb{C}_+)$. It follows from Theorem 3.3 that $V_af = f$ if and only if there exists an even function $g \in L_a^2(\mathbb{C}_+)$ such that $f = (M' \circ t_{m_a})l_{m_a}(g \circ M \circ t_{m_a})$. Thus if M_a is the range space of P_a, then we have $M_a = \{(M' \circ t_{m_a})l_{m_a}(g \circ M \circ t_{m_a}) : g \text{ even}\}$. Now $TV_a = V_aT$, for some $a \in \mathbb{D}$, if and only if $TP_a = P_aT$. This is true if and only if M_a is a reducing subspace of T. $\square$

For $T \in \mathcal{L}(L_a^2(\mathbb{C}_+))$, let $\sigma(T)$ denote the spectrum of T.

Theorem 4.3. *Let* $T \in \mathcal{L}(L_a^2(\mathbb{C}_+))$. *Suppose there exists* $a \in \mathbb{D}$ *such that* $TV_a = -V_aT$. *Then*

(1) *there exist* $T_1, T_2 \in \mathcal{L}(L_a^2(\mathbb{C}_+))$ *such that* $T_1^2 = T_2^2 = 0$ *and* $\sigma(T) = \sigma(-T)$;
(2) *if* $T^2 = -I$, *then* $T = \frac{1}{2}(T + V_a) + \frac{1}{2}(T - V_a)$, *and*
(3) *if* T *is invertible, then* T *is similar to* $S \oplus (-S)$ *for some invertible* $S \in \mathcal{L}(L_a^2(\mathbb{C}_+))$.

Proof. Let $T \in \mathcal{L}(L_a^2(\mathbb{C}_+))$, and let $TV_a = -V_aT$ for some $a \in \mathbb{D}$. Let $T_1 = \frac{1}{2}T(I - V_a)$, and let $T_2 = \frac{1}{2}T(I + V_a)$. Then it is easy to verify that $T_1^2 = T_2^2 = 0$ and $T = T_1 + T_2$. Further, if $S = T_1 - T_2$ and $\lambda \in \mathbb{C}$, then

$$\begin{aligned}(T - \lambda I)(S - T - \lambda I) &= TS - \lambda S - T^2 + \lambda T - \lambda T + \lambda^2 I\\ &= (T_1 + T_2)(T_1 - T_2) - \lambda S - T^2 + \lambda^2 I\\ &= -ST - \lambda S - T^2 + \lambda^2 I\\ &= (S + T - \lambda I)(-T - \lambda I).\end{aligned}$$

It is not difficult to see that $(S - T)^2 = (S + T)^2 = 0$. Now, if $\lambda \neq 0$, then both $S - T - \lambda I$ and $S + T - \lambda I$ are invertible. We deduce from the above that $(T - \lambda I)$ is invertible if and only if $(-T - \lambda I)$ is too. Thus $\sigma(T) \setminus \{0\} = \sigma(-T) \setminus \{0\}$. Therefore, $\sigma(T) = \sigma(-T)$. This proves (1).

To prove (2), assume that $T^2 = -I$. Then as $TV_a = -V_a$, we have $T = \frac{1}{2}(T + V_a) + \frac{1}{2}(T - V_a) = T_1 + T_2$ and $T_1^2 = 0 = T_2^2$. To establish (3), assume in addition that T is invertible and that $TV_a = -V_aT$ for some $a \in \mathbb{D}$. Since $V_a^2 = I$ and $\sigma(V_a) = \{+1, -1\}$, V_a is similar to an operator of the form $I_1 \oplus (-I_2)$, where I_1 and

I_2 are the identity operators on some Hilbert spaces H_1 and H_2, respectively. Let X be an invertible operator implementing this similarity, $XV_a = (I_1 \oplus (-I_2))X$. We have $XTX^{-1}(I_1 \oplus (-I_2)) = -(I_1 \oplus (-I_2))XTX^{-1}$. If $XTX^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ on the decomposition $H_1 \oplus H_2$, then carrying out the above matrix multiplication yields that

$$\begin{pmatrix} A & -B \\ C & -D \end{pmatrix} = \begin{pmatrix} -A & -B \\ C & D \end{pmatrix}.$$

Therefore, $A = 0$ and $D = 0$. In other words, T is similar to $\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$ on $H_1 \oplus H_2$. Since

$$T^2 \approx \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}^2 = \begin{pmatrix} BC & 0 \\ 0 & CB \end{pmatrix},$$

BC and CB are both invertible. Thus B and C are invertible. Hence we may assume, for simplicity, that $H_1 = H_2$. We have $\sigma(BC) = \sigma(CB) = \sigma(T^2)$ (for details, see [5]). By our assumption, $S \equiv (CB)^{\frac{1}{2}}$ exists. If

$$X = \begin{pmatrix} C^{-1}S & -C^{-1}S \\ I & I \end{pmatrix},$$

then X is invertible [5] and

$$\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} X = X \begin{pmatrix} S & 0 \\ 0 & -S \end{pmatrix}.$$

This implies that T is similar to $S \oplus (-S)$.

$\square$

Remark 4.4. Notice that $\begin{pmatrix} S & 0 \\ 0 & -S \end{pmatrix} = \frac{1}{2}\begin{pmatrix} S & -S \\ S & -S \end{pmatrix} + \frac{1}{2}\begin{pmatrix} S & S \\ -S & S \end{pmatrix}$ is the sum of two operators whose squares are zero.

Corollary 4.5. *Let $T \in \mathcal{L}(L^2_a(\mathbb{C}_+))$ be an invertible operator. Then there exist $T_1, T_2 \in \mathcal{L}(L^2_a(\mathbb{C}_+))$ such that $T = T_1 + T_2$ where $T_1^2 = T_2^2 = 0$ if and only if there exists an involution V such that $TV = -VT$.*

Proof. Assume that $T = T_1 + T_2$ where $T_1^2 = T_2^2 = 0$. Let $V = (T_1 - T_2)T^{-1}$. Since

$$\begin{aligned}(T_1 - T_2)T &= (T_1 - T_2)(T_1 + T_2) \\ &= T_1T_2 - T_2T_1 \\ &= -(T_1 + T_2)(T_1 - T_2) \\ &= -T(T_1 - T_2)\end{aligned}$$

and

$$\begin{aligned}(T_1 - T_2)^2 &= -T_1T_2 - T_2T_1 \\ &= -(T_1 + T_2)^2 \\ &= -T^2,\end{aligned}$$

we have

$$\begin{aligned} V^2 &= (T_1 - T_2)T^{-1}(T_1 - T_2)T^{-1} \\ &= (T_1 - T_2)^2 T^{-2} = I. \end{aligned}$$

Moreover, $TV = T(T_1 - T_2)T^{-1} = -(T_1 - T_2)TT^{-1} = -(T_1 - T_2)T^{-1}T = -VT$. Now suppose there exists an involution V such that $TV = -VT$. Let $T_1 = \frac{1}{2}T(I - V)$ and $T_2 = \frac{1}{2}T(I + V)$. Then $T = T_1 + T_2$ where $T_1^2 = T_2^2 = 0$. The result follows. □

Theorem 4.6. *Let $A \in \mathcal{L}(L_a^2(\mathbb{C}_+))$. Suppose that $ImA = \frac{A-A^*}{2i} > k > 0$. Then $AV_a \neq V_a A^*$ for all $a \in \mathbb{D}$.*

Proof. Suppose that $A \in \mathcal{L}(L_a^2(\mathbb{C}_+))$ and that $ImA > k > 0$. Let $a \in \mathbb{D}$, and let $V_a = C + iD$ be the Cartesian decomposition of V_a. We shall show that $\|AC - CA^*\| > 2k\|C\|$ and that $\|AD - DA^*\| > 2k\|D\|$. Let $|c_0| = \|C\|$; then there is a sequence $\{f_n\}$ of unit vectors in $L_a^2(\mathbb{C}_+)$ such that $\|(C - c_0)f_n\| \to 0$ as $n \to \infty$. Therefore,

$$\begin{aligned} &\|AC - CA^*\| \\ &\quad > |\langle (AC - CA^*)f_n, f_n\rangle| \\ &\quad = |\langle A(C - c_0)f_n, f_n\rangle - \langle (C - c_0)A^* f_n, f_n\rangle + c_0\langle Af_n, f_n\rangle - c_0\langle A^* f_n, f_n\rangle| \\ &\quad > |c_0|\; |\langle (A - A^*)f_n, f_n\rangle| - |\langle A(C - c_0)f_n, f_n\rangle| - |\langle (C - c_0)A^* f_n, f_n\rangle| \\ &\quad > 2|c_0|k - \text{ term which goes to zero as } n \to \infty. \end{aligned}$$

Thus $\|AC - CA^*\| > 2k\|C\|$. Similarly we get $\|AD - DA^*\| > 2k\|D\|$. Since $AC - CA^* = Re(AV_a - V_a A^*)$ and $AD - DA^* = Im(AV_a - V_a A^*)$, it follows that

$$\begin{aligned} 2\|AV_a - V_a A^*\| &> \|AC - CA^*\| + \|AD - DA^*\| \\ &> 2k(\|C\| + \|D\|) \\ &> 2k\|V_a\| = 2k. \end{aligned}$$

Hence $\|AV_a - V_a A^*\| > k$. The result follows. □

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