Hokkaido Mathematical Journal Vol. 43 (2014) p. 21–42

Examples of certain kind of minimal orbits of Hermann actions

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(Received March 2, 2012; Revised January 28, 2013)

Abstract. We give examples of certain kind of minimal orbits of Hermann actions and discuss whether each of the examples is austere.

Key words: Hermann action, minimal submanifold, austere submanifold.

1. Introduction

Let $N = G/K$ be a symmetrc space of compact type equipped with the G-invariant metric induced from the Killing form of the Lie algebra of G. Let H be a symmetric subgroup of G (i.e., $(\mathrm{Fix}\,\tau)_0 \subset H \subset \mathrm{Fix}\,\tau$ for some involution τ of G), where $\mathrm{Fix}\,\tau$ is the fixed point group of τ and $(\mathrm{Fix}\,\tau)_0$ is the identity component of $\mathrm{Fix}\,\tau$. The natural action of H on N is called a *Hermann action* (see [HPTT], [Kol]). Let θ be an involution of G with $(\mathrm{Fix}\,\theta)_0 \subset K \subset \mathrm{Fix}\,\theta$. According to [Co], when G is simple, we may assume that $\theta \circ \tau = \tau \circ \theta$ by replacing H to a suitable conjugate group of H if necessary except for the following three Hermann action:

(i) $Sp(p+q) \curvearrowright SU(2p+2q)/S(U(2p-1) \times U(2q+1)) \quad (p \geq q+2)$,
(ii) $U(p+q+1) \curvearrowright Spin(2p+2q+2)/Spin(2p+1) \times_{\mathbf{Z}_2} Spin(2q+1) \quad (p \geq q+1)$,
(iii) $Spin(3) \times_{\mathbf{Z}_2} Spin(5) \curvearrowright Spin(8)/\mu(Spin(3) \times_{\mathbf{Z}_2} Spin(5))$,

where μ is the triality automorphism of $Spin(8)$. Here we note that we remove transitive Hermann actions.

Assumption In the sequel, we assume that $\theta \circ \tau = \tau \circ \theta$. Then the Hermann action $H \curvearrowright G/K$ is said to be *commutative*.

Let $\mathfrak{g}, \mathfrak{k}$ and $\mathfrak{h}$ be the Lie algebras of G, K and H, respectively. Denote the involutions of $\mathfrak{g}$ induced form θ and τ by the same symbols θ and τ, respectively. Set $\mathfrak{p} := \mathrm{Ker}(\theta + \mathrm{id})$ and $\mathfrak{q} := \mathrm{Ker}(\tau + \mathrm{id})$. The vector space $\mathfrak{p}$

2010 Mathematics Subject Classification : 53C40, 53C35.

is identified with $T_{eK}(G/K)$, where e is the identity element of G. Denote by $B_{\mathfrak g}$ the Killing form of $\mathfrak g$. Give G/K the G-invariant metric arising from $B_{\mathfrak g}|_{\mathfrak p\times\mathfrak p}$. Take a maximal abelian subspace $\mathfrak b$ of $\mathfrak p\cap\mathfrak q$. For each $\beta\in\mathfrak b^*$, we set $\mathfrak p_\beta:=\{X\in\mathfrak p\mid \mathrm{ad}(b)^2(X)=-\beta(b)^2X\ (\forall\, b\in\mathfrak b)\}$ and $\triangle':=\{\beta\in\mathfrak b^*\setminus\{0\}\mid\mathfrak p_\beta\neq\{0\}\}$. This set $\triangle'$ is a root system. Let $\Pi'=\{\beta_1,\dots,\beta_r\}$ be the simple root system of the positive root system $\triangle'_+$ of $\triangle'$ under a lexicographic ordering of $\mathfrak b^*$. Set ${\triangle'}^V_+:=\{\beta\in\triangle'_+\mid\mathfrak p_\beta\cap\mathfrak q\neq\{0\}\}$ and ${\triangle'}^H_+:=\{\beta\in\triangle'_+\mid\mathfrak p_\beta\cap\mathfrak h\neq\{0\}\}$. Define a subset $\widetilde C$ of $\mathfrak b$ by

$$\widetilde C:=\left\{b\in\mathfrak b\mid 0<\beta(b)<\pi\left(\forall\,\beta\in{\triangle'}^V_+\right),\ -\frac{\pi}{2}<\beta(b)<\frac{\pi}{2}\left(\forall\,\beta\in{\triangle'}^H_+\right)\right\}.$$

The closure $\overline{\widetilde C}$ of $\widetilde C$ is a simplicial complex. Set $C:=\mathrm{Exp}(\widetilde C)$, where Exp is the exponential map of G/K at eK. Each principal H-orbit passes through only one point of C and each singular H-orbit passes through only one point of $\mathrm{Exp}(\partial\widetilde C)$. For each simplex σ of $\overline{\widetilde C}$, only one minimal H-orbit through $\mathrm{Exp}(\sigma)$ exists. See proofs of Theorems A and B in [K2] (also [I]) about this fact. For $\beta\in\triangle'_+$, we set $\beta=\sum_{i=1}^r n_i^\beta\beta_i$, $m_\beta:=\dim\mathfrak p_\beta$, $m_\beta^V:=\dim(\mathfrak p_\beta\cap\mathfrak q)$ and $m_\beta^H:=\dim(\mathfrak p_\beta\cap\mathfrak h)$. Let Z_0 be a point of $\mathfrak b$. We consider the following two conditions for Z_0:

$$\text{(I)}\quad\left\{\begin{aligned}&\beta(Z_0)\equiv 0,\ \frac{\pi}{6},\ \frac{\pi}{3},\ \frac{\pi}{2},\ \frac{2\pi}{3},\ \frac{5\pi}{6}\ (\mathrm{mod}\,\pi)\ (\forall\,\beta\in\triangle'_+)\ \&\\ &\sum_{\substack{\beta\in{\triangle'}^V_+\text{ s.t. }\beta(Z_0)\\ \equiv\pi/6\ (\mathrm{mod}\,\pi)}}3n_i^\beta m_\beta^V+\sum_{\substack{\beta\in{\triangle'}^V_+\text{ s.t. }\beta(Z_0)\\ \equiv\pi/3\ (\mathrm{mod}\,\pi)}}n_i^\beta m_\beta^V\\ &\quad+\sum_{\substack{\beta\in{\triangle'}^H_+\text{ s.t. }\beta(Z_0)\\ \equiv2\pi/3\ (\mathrm{mod}\,\pi)}}3n_i^\beta m_\beta^H+\sum_{\substack{\beta\in{\triangle'}^H_+\text{ s.t. }\beta(Z_0)\\ \equiv5\pi/6\ (\mathrm{mod}\,\pi)}}n_i^\beta m_\beta^H\\ &=\sum_{\substack{\beta\in{\triangle'}^V_+\text{ s.t. }\beta(Z_0)\\ \equiv2\pi/3\ (\mathrm{mod}\,\pi)}}n_i^\beta m_\beta^V+\sum_{\substack{\beta\in{\triangle'}^V_+\text{ s.t. }\beta(Z_0)\\ \equiv5\pi/6\ (\mathrm{mod}\,\pi)}}3n_i^\beta m_\beta^V\\ &\quad+\sum_{\substack{\beta\in{\triangle'}^H_+\text{ s.t. }\beta(Z_0)\\ \equiv\frac{\pi}{6}\ (\mathrm{mod}\,\pi)}}n_i^\beta m_\beta^H+\sum_{\substack{\beta\in{\triangle'}^H_+\text{ s.t. }\beta(Z_0)\\ \equiv\pi/3\ (\mathrm{mod}\,\pi)}}3n_i^\beta m_\beta^H\\ &\hspace{20em}(i=1,\dots,r).\end{aligned}\right.$$

and

$$
\text{(II)} \quad \begin{cases}
\beta(Z_0) \equiv 0,\, \dfrac{\pi}{4},\, \dfrac{\pi}{2},\, \dfrac{3\pi}{4} \ (\mathrm{mod}\, \pi)\ (\forall\, \beta \in \triangle'_+)\ \& \\
\displaystyle\sum_{\substack{\beta\in\triangle'^V_+ \text{ s.t. } \beta(Z_0) \\ \equiv \pi/4 \ (\mathrm{mod}\,\pi)}} n_i^\beta m_\beta^V + \sum_{\substack{\beta\in\triangle'^H_+ \text{ s.t. } \beta(Z_0) \\ \equiv 3\pi/4 \ (\mathrm{mod}\,\pi)}} n_i^\beta m_\beta^H \\
\quad = \displaystyle\sum_{\substack{\beta\in\triangle'^V_+ \text{ s.t. } \beta(Z_0) \\ \equiv 3\pi/4 \ (\mathrm{mod}\,\pi)}} n_i^\beta m_\beta^V + \sum_{\substack{\beta\in\triangle'^H_+ \text{ s.t. } \beta(Z_0) \\ \equiv \pi/4 \ (\mathrm{mod}\,\pi)}} n_i^\beta m_\beta^H \\
\hfill (i = 1, \ldots, r).
\end{cases}
$$

Denote by L the isotropy group of H at $\mathrm{Exp}\, Z_0$. Denote by $\mathfrak{h}$ (resp. $\mathfrak{l}$) the Lie algebra of H (resp. L) and $B_{\mathfrak{g}}$ the Killing form of $\mathfrak{g}$. Also, denote by g_I the induced metric on the submanifold M in G/K and $\nabla^\perp$ the normal connection of the submanifold M. In the case where $(\mathfrak{h}, \mathfrak{l})$ admits a reductive decomposition $\mathfrak{h} = \mathfrak{l} + \mathfrak{m}$, we denote the canonical connection of the principal L-bundle $\pi : H \to H/L (= M)$ with respect to this reductive decomposition by $\omega_{\mathfrak{m}}$. Let $F^\perp(M)$ be the normal frame bundle of M. Define a map $\eta : H \to F^\perp(M)$ by $\eta(h) = h_* u_0$ $(h \in H)$, where u_0 is an arbitrary fixed element of $F^\perp(M)_{\mathrm{Exp}\, Z_0}$, where $F^\perp(M)_{\mathrm{Exp}\, Z_0}$ is the fibre of $F^\perp(M)$ over $\mathrm{Exp}\, Z_0$. This map η is an embedding. By identifying H with $\eta(H)$, we regard $\pi : H \to H/L (= M)$ as a subbundle of $F^\perp(M)$. Denote by the same symbol $\omega_{\mathfrak{m}}$ the connection of $F^\perp(M)$ induced from $\omega_{\mathfrak{m}}$ and $\nabla^{\omega_{\mathfrak{m}}}$ the linear connection on $T^\perp M$ associated with $\omega_{\mathfrak{m}}$.

In this paper, we prove the following results for the orbit $M = H(\mathrm{Exp}\, Z_0)$ of the Hermann action $H \curvearrowright G/K$.

Theorem A *If Z_0 satisfies the condition* (I) *or* (II)*, then the orbit M is a minimal submanifold satisfying the following conditions*:

(i) $(\mathfrak{h}, \mathfrak{l})$ *admits a reductive decomposition* $\mathfrak{h} = \mathfrak{l} + \mathfrak{m}$ *such that* $B_{\mathfrak{g}}(\mathfrak{l}, \mathfrak{m}) = 0$,

(ii) $\nabla^\perp = \nabla^{\omega_{\mathfrak{m}}}$ *holds.*

Also, $\bigcap_{v \in T_x^\perp M} \mathrm{Ker}\, A_v$ *is equal to*

$$g_{0*}(\mathfrak{z}_{\mathfrak{p}\cap\mathfrak{h}}(\mathfrak{b}))+\sum_{\beta\in\triangle'^V_+\ \text{s.t.}\ \beta(Z_0)\equiv\pi/2\pmod{\pi}}g_{0*}(\mathfrak{p}_\beta\cap\mathfrak{q})$$
$$+\sum_{\beta\in\triangle'^H_+\ \text{s.t.}\ \beta(Z_0)\equiv 0\pmod{\pi}}g_{0*}(\mathfrak{p}_\beta\cap\mathfrak{h}),$$

where $\mathfrak{z}_{\mathfrak{p}\cap\mathfrak{h}}(\mathfrak{b})$ *is the centralizer of* $\mathfrak{b}$ *in* $\mathfrak{p}\cap\mathfrak{h}$.

Let M be a submanifold in a Riemannian manifold N. If, for any unit normal vector v, the spectrum of the shape operator A_v is invariant with respect to the (-1)-multiple (with considering the multiplicities), then M is called an *austere submanifold*. By using Theorem A, we can show the following fact.

Theorem B *Assume that* Z_0 *satisfies the condition* (I) *or* (II). *If* $m_\beta^V = m_\beta^H$ *for all* $\beta\in\triangle'_+$ *and if* Z_0 *satisfies* $\beta(Z_0)\equiv 0,\ \pi/4,\ \pi/2, 3\pi/4 \pmod{\pi}$ *for all* $\beta\in\triangle'_+$, *then the orbit* M *is an austere submanifold satisfying the conditions* (i) *and* (ii) *in Theorem A.*

Remark 1.1 The austere orbits of the commutative Hermann actions were classified in [I].

Also, we can show the following facts.

Theorem C *Assume that* Z_0 *satisfies the condition* (I). *In particular, if* $\triangle'^V_+\cap\triangle'^H_+=\emptyset$, *if* $\beta(Z_0)\equiv 0,\ \pi/3,\ 2\pi/3 \pmod{\pi}$ *for all* $\beta\in\triangle'^V_+$ *and if* $\beta(Z_0)\equiv\pi/6,\ \pi/2,\ 5\pi/6\pmod{\pi}$ *for all* $\beta\in\triangle'^H_+$, *then* M *is a minimal submanifold satisfying the conditions* (i), (ii) *in Theorem A. Furthermore, if the cohomogeneity of the* H*-action is equal to the rank of* G/K, *then* $(g_I)_{eL}=(3/4)B_{\mathfrak{g}}|_{\mathfrak{m}\times\mathfrak{m}}$ *and* $\bigcap_{v\in T^\perp_xM}\mathrm{Ker}\,A_v=\{0\}$ *hold.*

Theorem D *Assume that* Z_0 *satisfies the condition* (I). *In particular, if* $\triangle'^V_+\cap\triangle'^H_+=\emptyset$, *if* $\beta(Z_0)\equiv 0,\ \pi/6,\ 5\pi/6 \pmod{\pi}$ *for all* $\beta\in\triangle'^V_+$ *and if* $\beta(Z_0)\equiv\pi/3,\ \pi/2,\ 2\pi/3\pmod{\pi}$ *for all* $\beta\in\triangle'^H_+$, *then* M *is a minimal submanifold satisfying the conditions* (i), (ii) *in Theorem A. Furthermore, if the cohomogeneity of the* H*-action is equal to the rank of* G/K, *then* $(g_I)_{eL}=(1/4)B_{\mathfrak{g}}|_{\mathfrak{m}\times\mathfrak{m}}$ *and* $\bigcap_{v\in T^\perp_xM}\mathrm{Ker}\,A_v=\{0\}$ *hold.*

Theorem E *Assume that* Z_0 *satisfies the condition* (II). *In particular, if* $\triangle'^V_+\cap\triangle'^H_+=\emptyset$, *if* $\beta(Z_0)\equiv 0,\ \pi/4,\ 3\pi/4 \pmod{\pi}$ *for all* $\beta\in\triangle'^V_+$ *and if*

$\beta(Z_0) \equiv \pi/4,\ \pi/2,\ 3\pi/4 \pmod{\pi}$ *for all* $\beta \in {\triangle'}^H_+$, *then* M *is a minimal submanifold satisfying the conditions* (i), (ii) *in Theorem A. Furthermore, if the cohomogeneity of the* H*-action is equal to the rank of* G/K, *then* $(g_I)_{eL} = (1/2)B_{\mathfrak{g}}|_{\mathfrak{m}\times\mathfrak{m}}$ *and* $\bigcap_{v\in T^\perp_x M} \mathrm{Ker}\, A_v = \{0\}$ *hold.*

Theorem F *If* ${\triangle'}^V_+ \cap {\triangle'}^H_+ = \emptyset$, *if* $\beta(Z_0) \equiv 0,\ \pi/2 \pmod{\pi}$ *for all* $\beta \in {\triangle'}_+$, *then* M *is a totally geodesic submanifold satisfying the conditions* (i), (ii) *in Theorem A. Furthermore, if the cohomogeneity of the* H*-action is equal to the rank of* G/K, *then* $(g_I)_{eL} = B_{\mathfrak{g}}|_{\mathfrak{m}\times\mathfrak{m}}$ *holds.*

Remark 1.2

(i) If $H = K$ then we have ${\triangle'}^H_+ = \emptyset$ and hence ${\triangle'}^V_+ \cap {\triangle'}^H_+ = \emptyset$.

(ii) In Theorems C$\sim$F, when G is simple, there exists an inner automorphism ρ of G with $\rho(K) = H$ by Proposition 4.39 of [I].

In the final section, we give examples of Hermann actions $H \curvearrowright G/K$ and $Z_0 \in \mathfrak{b}$ as in Theorems B, C and F.

2. Basic notions and facts

In this section, we recall some basic notions and facts.

Shape operators of orbits of Hermann actions

Let $H \curvearrowright G/K$ be a Hermann action and θ (resp. τ) an involution of G with $(\mathrm{Fix}\,\theta)_0 \subset K \subset \mathrm{Fix}\,\theta$ (resp. $(\mathrm{Fix}\,\tau)_0 \subset H \subset \mathrm{Fix}\,\tau$). Assume that $\theta\circ\tau = \tau\circ\theta$. Let $\mathfrak{k}$, $\mathfrak{p}$, $\mathfrak{h}$, $\mathfrak{q}$, $\mathfrak{b}$, $\mathfrak{p}_\beta$, $\triangle'$, ${\triangle'}^V_+$ and ${\triangle'}^H_+$ be as in Introduction. Fix $Z_0 \in \mathfrak{b}$. Set $M := H(\mathrm{Exp}\, Z_0)$ and $g_0 := \exp Z_0$, where Exp is the exponential map of G/K at eK and exp is the exponential map of G. Set

$$
{\triangle'}^V_{Z_0} := \left\{\beta \in {\triangle'}^V_+ \mid \beta(Z_0) \equiv 0 \pmod{\pi}\right\}
$$

and

$$
{\triangle'}^H_{Z_0} := \left\{\beta \in {\triangle'}^H_+ \mid \beta(Z_0) \equiv \frac{\pi}{2} \pmod{\pi}\right\}.
$$

Denote by A the shape tensor of M. The tangent space $T_{\mathrm{Exp}\, Z_0} M$ of M at $\mathrm{Exp}\, Z_0$ is given by

$$T_{\mathrm{Exp}\,Z_0}M = g_{0*}\left(\mathfrak{z}_{\mathfrak{p}\cap\mathfrak{h}}(\mathfrak{b}) + \sum_{\beta\in\triangle'^V_+\setminus\triangle'^V_{Z_0}}(\mathfrak{p}_\beta\cap\mathfrak{q}) + \sum_{\beta\in\triangle'^H_+\setminus\triangle'^H_{Z_0}}(\mathfrak{p}_\beta\cap\mathfrak{h})\right) \quad (2.1)$$

and hence

$$T^\perp_{\mathrm{Exp}\,Z_0}M = g_{0*}\left(\mathfrak{b} + \sum_{\beta\in\triangle'^V_{Z_0}}(\mathfrak{p}_\beta\cap\mathfrak{q}) + \sum_{\beta\in\triangle'^H_{Z_0}}(\mathfrak{p}_\beta\cap\mathfrak{h})\right). \quad (2.2)$$

Denote by L the isotropy group of the H-action at $\mathrm{Exp}\,Z_0$. The slice representation $\rho^S_{Z_0} : L \to GL(T^\perp_{\mathrm{Exp}\,Z_0}M)$ of the H-action at $\mathrm{Exp}\,Z_0$ is given by $\rho^S_{Z_0}(h) = h_{*\,\mathrm{Exp}\,Z_0}|_{T^\perp_{\mathrm{Exp}\,Z_0}M}$ $(h \in H_{Z_0})$. Then we have $\bigcup_{h\in H_{Z_0}}\rho^S_{Z_0}(h)(g_{0*}\mathfrak{b}) = T^\perp_{\mathrm{Exp}\,Z_0}M$ and

$$\begin{aligned} A_{\rho^S_{Z_0}(h)(g_{0*}v)}\big|_{\rho^S_{Z_0}(h)(g_{0*}(\mathfrak{z}_{\mathfrak{p}\cap\mathfrak{h}}(\mathfrak{b})))} &= 0, \\ A_{\rho^S_{Z_0}(h)(g_{0*}v)}\big|_{\rho^S_{Z_0}(h)(g_{0*}(\mathfrak{p}_\beta\cap\mathfrak{q}))} &= -\frac{\beta(v)}{\tan\beta(Z_0)}\,\mathrm{id}\ \left(\beta\in\triangle'^V_+\setminus\triangle'^V_{Z_0}\right), \quad (2.3) \\ A_{\rho^S_{Z_0}(h)(g_{0*}v)}\big|_{\rho^S_{Z_0}(h)(g_{0*}(\mathfrak{p}_\beta\cap\mathfrak{h}))} &= \beta(v)\tan\beta(Z_0)\,\mathrm{id}\ \left(\beta\in\triangle'^H_+\setminus\triangle'^H_{Z_0}\right), \end{aligned}$$

where $h \in L$ and $v \in \mathfrak{b}$.

The canonical connection

Let H/L be a reductive homogeneous space and $\mathfrak{h} = \mathfrak{l}+\mathfrak{m}$ be a reductive decomposition (i.e., $[\mathfrak{l},\mathfrak{m}] \subset \mathfrak{m}$), where $\mathfrak{h}$ (resp. $\mathfrak{l}$) is the Lie algebra of H (resp. L). Also, let $\pi : P \to H/L$ be a principal G-bundle, where G is a Lie group. Assume that H acts on P as $\pi(h\cdot u) = h\cdot\pi(u)$ for any $u \in P$ and any $h \in H$. Then there uniquely exists a connection ω of P such that, for any $X \in \mathfrak{m}$ and any $u \in P$, $t \mapsto (\exp tX)(u)$ is a horizontal curve with respect to ω, where exp is the exponential map of H. This connection ω is called the *canonical connection* of P associated with the reductive decomposition $\mathfrak{h} = \mathfrak{l} + \mathfrak{m}$.

3. Proof of Theorems A∼F

In this section, we shall first prove Theorems A∼F. We use the notations in Introduction. Let $H \curvearrowright G/K$ be a Hermann action and Z_0 be an element of $\mathfrak{b}$. Set $M := H(\mathrm{Exp}\,Z_0)$.

Proof of Theorem A. Denote by $\mathcal{H}$ the mean curvature vector of M. From (2.1) and (2.3), we have

$$\begin{aligned}&\left\langle \mathcal{H}_{\mathrm{Exp}\,Z_0}, \rho^S_{Z_0}(h)(g_{0*}v)\right\rangle \\ &\quad = -\sum_{i=1}^r \sum_{\beta\in\triangle'^V_+\setminus\triangle'^V_{Z_0}} \frac{n_i^\beta m_\beta^V}{\tan\beta(Z_0)}\beta_i(v) + \sum_{i=1}^r \sum_{\beta\in\triangle'^H_+\setminus\triangle'^H_{Z_0}} n_i^\beta m_\beta^H \tan\beta(Z_0)\beta_i(v)\end{aligned}$$

for any $v\in\mathfrak{b}$ and any $h\in L$. Hence, $\mathcal{H}_{\mathrm{Exp}\,Z_0}$ vanishes if and only if the following relations hold:

$$\sum_{\beta\in\triangle'^V_+\setminus\triangle'^V_{Z_0}} \frac{n_i^\beta m_\beta^V}{\tan\beta(Z_0)} = \sum_{\beta\in\triangle'^H_+\setminus\triangle'^H_{Z_0}} n_i^\beta m_\beta^H \tan\beta(Z_0) \qquad (i=1,\ldots,r). \tag{3.1}$$

Since Z_0 satisfies the condition (I) or (II) in Theorem A, (3.1) holds, that is, $\mathcal{H}_{\mathrm{Exp}\,Z_0}$ vanishes. Therefore M is minimal.

Next we shall show that there exists a reductive decomposition $\mathfrak{h}=\mathfrak{l}+\mathfrak{m}$ with $B_{\mathfrak{g}}(\mathfrak{l},\mathfrak{m})=0$. Easily we have

$$\mathfrak{l} = \mathfrak{z}_{\mathfrak{k}\cap\mathfrak{h}}(\mathfrak{b}) + \sum_{\beta\in\triangle'^V_{Z_0}} (\mathfrak{k}_\beta\cap\mathfrak{h}) + \sum_{\beta\in\triangle'^H_{Z_0}} (\mathfrak{p}_\beta\cap\mathfrak{h}). \tag{3.2}$$

Define a subspace $\mathfrak{m}$ of $\mathfrak{h}$ by

$$\mathfrak{m} := \mathfrak{z}_{\mathfrak{p}\cap\mathfrak{h}}(\mathfrak{b}) + \sum_{\beta\in\triangle'^V_+\setminus\triangle'^V_{Z_0}} (\mathfrak{k}_\beta\cap\mathfrak{h}) + \sum_{\beta\in\triangle'^H_+\setminus\triangle'^H_{Z_0}} (\mathfrak{p}_\beta\cap\mathfrak{h}). \tag{3.3}$$

Easily we can show that $\mathfrak{h}=\mathfrak{l}+\mathfrak{m}$ is a reductive decomposition and that $B_{\mathfrak{g}}(\mathfrak{l},\mathfrak{m})=0$.

Next we shall show that $\nabla^{\omega_{\mathfrak{m}}}=\nabla^\perp$. Take $v\in\mathfrak{b}$ $(\subset g_{0*}^{-1}T^\perp_{\mathrm{Exp}\,Z_0}M)$. Set $g_s := \exp(1-s)Z_0$. Let $Z:[0,1]\to\mathfrak{b}$ be a C^∞-curve such that $Z(0)=Z_0$ and that $Z((0,1])$ is contained in a fundamental domain of the Coxeter group associated with the principal H-orbit at an intersection point of the orbit and $\mathfrak{b}$. Set $M_s := H(\mathrm{Exp}\,Z(1-s))$ $(0\le s\le 1)$. Denote by A^s the shape tensor of M_s and $\widetilde{\nabla}$ the Levi-Civita connection of G/K. Let $\widetilde{v}^s$ be the H-equivariant normal vector field of M_s $(0\le s<1)$ arising from $g_{s*}v$.

Since M_s $(0 \leq s < 1)$ is a principal orbit of a Hermann (hence hyperpolar) action, $\widetilde{v}^s$ is well-defined and it is a parallel normal vector field with respect to $\nabla^{\perp}$. Take $X \in \mathfrak{k}_\beta \cap \mathfrak{h}$ $(\subset \mathfrak{m})$ $(\beta \in \triangle'^V_+ \setminus \triangle'^V_{Z_0})$. Then, by using (2.3), we have

$$\widetilde{\nabla}_{X^*_{\mathrm{Exp}\,Z(1-s)}}\widetilde{v}^s = -A^s_v X^*_{\mathrm{Exp}\,Z(1-s)} = \frac{\beta(v)}{\tan\beta(Z_0)}X^*_{\mathrm{Exp}\,Z(1-s)}.$$

and hence

$$\begin{aligned}\widetilde{\nabla}_{X^*_{\mathrm{Exp}\,Z_0}}(\exp tX)_{*\,\mathrm{Exp}(Z_0)}(v) &= \lim_{s\to 1-0}\widetilde{\nabla}_{X^*_{\mathrm{Exp}\,Z(1-s)}}\widetilde{v}^s \\ &= \frac{\beta(v)}{\tan\beta(Z_0)}X^*_{\mathrm{Exp}\,Z_0} \in T_{\mathrm{Exp}\,Z_0}M.\end{aligned}$$

Hence we obtain $\nabla^{\perp}_{X^*_{\mathrm{Exp}\,Z_0}}(\exp tX)_{*\,\mathrm{Exp}(Z_0)}(v) = 0$. Take $Y \in \mathfrak{p}_\beta \cap \mathfrak{h}$ $(\subset \mathfrak{m})$ $(\beta \in \triangle'^H_+ \setminus \triangle'^H_{Z_0})$. Then, by using (2.3), we have

$$\widetilde{\nabla}_{Y^*_{\mathrm{Exp}\,Z(1-s)}}\widetilde{v}^s = -A^s_v Y^*_{\mathrm{Exp}\,Z(1-s)} = -\beta(v)\tan\beta(Z_0)Y^*_{\mathrm{Exp}\,Z(1-s)}.$$

and hence

$$\begin{aligned}\widetilde{\nabla}_{Y^*_{\mathrm{Exp}\,Z_0}}(\exp tY)_{*\,\mathrm{Exp}\,Z_0}(v) &= \lim_{s\to 1-0}\widetilde{\nabla}_{Y^*_{\mathrm{Exp}\,Z(1-s)}}\widetilde{v}^s \\ &= -\beta(v)\tan\beta(Z_0)Y^*_{\mathrm{Exp}\,Z_0} \in T_{\mathrm{Exp}\,Z_0}M.\end{aligned}$$

Hence we obtain $\nabla^{\perp}_{Y^*_{\mathrm{Exp}\,Z_0}}(\exp tY)_{*\,\mathrm{Exp}(Z_0)}(v) = 0$. Therefore, it follows from the arbitrariness of X, Y and β that $t \mapsto (\exp t\hat{X})_{*\,\mathrm{Exp}\,Z_0}(v)$ is $\nabla^{\perp}$-parallel along $t \mapsto (\exp t\hat{X})(\mathrm{Exp}\,Z_0)$ for any $\hat{X} \in \mathfrak{m}$. Take any $h \in L$. Similarly we can show that $t \mapsto (\exp t\hat{X})_{*\,\mathrm{Exp}\,Z_0}(\rho^S_{Z_0}(h))(g_{0*}v))$ is $\nabla^{\perp}$-parallel along $t \mapsto (\exp t\hat{X})(\mathrm{Exp}\,Z_0)$ for any $\hat{X} \in \mathfrak{m}$. Note that this fact has been showed in [IST] in different method. On the other hand, it follows from the definition of ω that $t \mapsto (\exp t\hat{X})_{*\,\mathrm{Exp}\,Z_0}(\rho^S_{Z_0}(h)(g_{0*}v))$ is $\nabla^{\omega_{\mathfrak{m}}}$-parallel along $t \mapsto (\exp t\hat{X})(\mathrm{Exp}\,Z_0)$ for any $\hat{X} \in \mathfrak{m}$. Therefore we obtain $\nabla^{\perp} = \nabla^{\omega_{\mathfrak{m}}}$. The statement for $\bigcap_{v \in T^{\perp}_x M}\mathrm{Ker}\,A_v$ follows from (2.3) directly. □

Next we prove Theorem B.

Proof of Theorem B. This statement of this theorem follows from (2.3) directly. □

Next we prove Theorems C∼F.

Proof of Theorems C∼F. Define a diffeomorphism $\psi : H/L \to M$ by $\psi(hL) := h\cdot \mathrm{Exp}\, Z_0$ ($h \in H$). Next we shall show that $(\psi^* g_I)_{eL} = cB_{\mathfrak{g}}|_{\mathfrak{m}\times\mathfrak{m}}$, where

$$c = \begin{cases} \dfrac{3}{4} & \text{(in case of Theorems C)} \\ \dfrac{1}{4} & \text{(in case of Theorem D)} \\ \dfrac{1}{2} & \text{(in case of Theorem E)} \\ 1 & \text{(in case of Theorem F).} \end{cases}$$

In the sequel, we omit the notation ψ^*. For each $X \in \mathfrak{m}(= T_{eL}(H/L) = T_{\mathrm{Exp}\, Z_0} M)$, denote by X^* the Killing field on M associated with X, that is, $X^*_p := d/dt|_{t=0}(\exp tX)(p)$ ($p \in M$). From the definition of ψ, we have $\psi_{*eL} X = X^*_{\mathrm{Exp}\, Z_0}$. Take $S_{\beta_1} \in \mathfrak{k}_{\beta_1} \cap \mathfrak{h}$ ($\beta_1 \in {\triangle'_+}^H \setminus {\triangle'_{Z_0}}^H$) and $\hat{S}_{\beta_2} \in \mathfrak{p}_{\beta_2} \cap \mathfrak{h}$ ($\beta_2 \in {\triangle'_+}^V \setminus {\triangle'_{Z_0}}^V$). Let T_{β_1} be the element of $\mathfrak{p}_{\beta_1} \cap \mathfrak{q}$ such that $\mathrm{ad}(b)(S_{\beta_1}) = \beta_1(b) T_{\beta_1}$ for any $b \in \mathfrak{b}$. Then we have

$$\psi_{*eL}(S_{\beta_1}) = (S^*_{\beta_1})_{\mathrm{Exp}\, Z_0} = -\sin \beta_1(Z_0)(\exp Z_0)_*(T_{\beta_1}) \tag{3.4}$$

and

$$\psi_{*eL}(\hat{S}_{\beta_2}) = (\hat{S}^*_{\beta_2})_{\mathrm{Exp}\, Z_0} = \cos \beta_2(Z_0)(\exp Z_0)_*(\hat{S}_{\beta_2}). \tag{3.5}$$

Hence, since H and Z_0 is as in Theorems C∼F, we have $(g_I)_{eL}(S_{\beta_1}, S_{\beta_1}) = cB_{\mathfrak{g}}(S_{\beta_1}, S_{\beta_1})$ and $(g_I)_{eL}(\hat{S}_{\beta_2}, \hat{S}_{\beta_2}) = cB_{\mathfrak{g}}(\hat{S}_{\beta_2}, \hat{S}_{\beta_2})$. If the cohomogeneity of the H-action is equal to the rank of G/K, then we have $\mathfrak{z}_{\mathfrak{p}\cap\mathfrak{h}}(\mathfrak{b}) = 0$. Therefore we obtain $(g_I)_{eL} = cB_{\mathfrak{g}}|_{\mathfrak{m}\times\mathfrak{m}}$. Also, in Theorems C∼E, $\bigcap_{v\in T^\perp_x M} \mathrm{Ker}\, A_v = \{0\}$ follows from the statement for $\bigcap_{v\in T^\perp_x M} \mathrm{Ker}\, A_v$ in Theorem A directly. □

4. Examples

In this section, we give examples of a Hermann action $H \curvearrowright G/K$ and $Z_0 \in \widetilde{C}$ as in Theorems B, C and F. We use the notations in Introduction.

Example 1 We consider the isotropy action of $SU(3n+3)/SO(3n+3)$. Then we have $\triangle_+ = \triangle'_+ = {\triangle'}^V_+$ (which is of $(\mathfrak{a}_{3n+2})$-type) and ${\triangle'}^H_+ = \emptyset$. Let $\Pi = \{\beta_1, \ldots, \beta_{3n+2}\}$ be a simple root system of $\triangle'_+$, where we order $\beta_1, \ldots, \beta_{3n+2}$ as the Dynkin diagram of $\triangle'_+$ is as in Figure 1, $\triangle'_+ = \{\beta_i + \cdots + \beta_j \mid 1 \leq i, j \leq 3n+2\}$. For any $\beta \in \triangle'_+$, we have $m_\beta = 1$. Let Z_0 be the point of $\mathfrak{b}$ defined by $\beta_{n+1}(Z_0) = \beta_{2n+2}(Z_0) = \pi/3$ and $\beta_i(Z_0) = 0$ $(i \in \{1, \ldots, 3n+2\} \setminus \{n+1, 2n+2\})$. Clearly we have $m^V_\beta = 1$, $m^H_\beta = 0$ and $\beta(Z_0) \equiv 0$, $\pi/3$ or $2\pi/3 \pmod{\pi}$ for any $\beta \in \triangle'_+$. For simplicity, set $\beta_{ij} := \beta_i + \cdots + \beta_j$ $(1 \leq i \leq j \leq 3n+2)$. Easily we can show

$$\left\{\beta \in {\triangle'}^V_+ \mid \beta(Z_0) \equiv \frac{\pi}{3} \pmod{\pi}\right\}$$
$$= \{\beta_{ij} \mid 1 \leq i \leq n+1 \leq j < 2n+2, \text{ or } n+1 < i \leq 2n+2 \leq j \leq 3n+2\}$$

and

$$\left\{\beta \in {\triangle'}^V_+ \mid \beta(Z_0) \equiv \frac{2\pi}{3} \pmod{\pi}\right\}$$
$$= \{\beta_{ij} \mid 1 \leq i \leq n+1, \ 2n+2 \leq j \leq 3n+2\}.$$

From these facts, it follows that the condition (I) holds. Thus Z_0 is as in the statement of Theorem C. Also, it is easy to show that M is not austere.

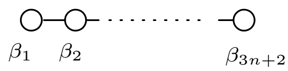

Figure 1.

Example 2 We consider the isotropy action of $SU(6n+6)/Sp(3n+3)$. Then we have $\triangle_+ = \triangle'_+ = {\triangle'}^V_+$ (which is of $(\mathfrak{a}_{3n+2})$-type) and ${\triangle'}^H_+ = \emptyset$. Let $\Pi = \{\beta_1, \ldots, \beta_{3n+2}\}$ be a simple root system of $\triangle'_+$, where we order $\beta_1, \ldots, \beta_{3n+2}$ as above. We have $m_\beta = 4$ for any $\beta \in \triangle'_+$. Let Z_0 be the point of the closure of $\mathfrak{b}$ defined by $\beta_{n+1}(Z_0) = \beta_{2n+2}(Z_0) = \pi/3$ and $\beta_i(Z_0) = 0$ $(i \in \{1, \ldots, 3n+2\} \setminus \{n+1, 2n+2\})$. Clearly we have $m^V_\beta = 4$,

$m^H_\beta = 0$ and $\beta(Z_0) \equiv 0, \pi/3$ or $2\pi/3 \pmod{\pi}$ for any $\beta \in \triangle'_+$. For simplicity, set $\beta_{ij} := \beta_i + \cdots + \beta_j$ $(1 \le i \le j \le 3n+2)$. Easily we can show

$$\left\{\beta \in {\triangle'}^V_+ \mid \beta(Z_0) \equiv \frac{\pi}{3} \pmod{\pi}\right\}$$
$$= \{\beta_{ij} \mid 1 \le i \le n+1 \le j < 2n+2, \text{ or } n+1 < i \le 2n+2 \le j \le 3n+2\}$$

and

$$\left\{\beta \in {\triangle'}^V_+ \mid \beta(Z_0) \equiv \frac{2\pi}{3} \pmod{\pi}\right\}$$
$$= \{\beta_{ij} \mid 1 \le i \le n+1, 2n+2 \le j \le 3n+2\}.$$

From these facts, it follows that the condition (I) holds. Thus Z_0 is as in the statement of Theorem C. Also, it is easy to show that M is not austere.

Example 3 We consider the isotropy action of $SU(3)/S(U(1)\times U(2))$ (2-dimensional complex projective space). Then we have $\triangle_+ = \triangle'_+ = {\triangle'}^V_+ = \{\beta, 2\beta\}$ and ${\triangle'}^H_+ = \emptyset$, $m_\beta = 2$ and $m_{2\beta} = 1$. Let Z_0 be the point of $\mathfrak{b}$ defined by $\beta(Z_0) = \pi/3$. Clearly Z_0 satisfies the condition (I). Thus Z_0 is as in the statement of Theorem C. Also, it is easy to show that M is not austere.

Example 4 We consider the isotropy action of $Sp(3n+2)/U(3n+2)$. Then we have $\triangle_+ = \triangle'_+ = {\triangle'}^V_+$ (which is of $(\mathfrak{c}_{3n+2})$-type) and ${\triangle'}^H_+ = \emptyset$. Let $\Pi = \{\beta_1, \ldots, \beta_{3n+2}\}$ be a simple root system of $\triangle'_+$, where we we order $\beta_1, \ldots, \beta_{3n+2}$ as the Dynkin diagram of $\triangle'_+$ is as in Fig. 2. We have $m_\beta = 1$ for any $\beta \in \triangle'_+$. Let Z_0 be the point of $\mathfrak{b}$ defined by $\beta_{n+1}(Z_0) = \beta_{2n+2}(Z_0) = \beta_{3n+2}(Z_0) = \pi/3$ and $\beta_i(Z_0) = 0$ $(i \in \{1, \ldots, 3n+2\} \setminus \{n+1, 2n+2, 3n+2\})$. Clearly we have $m^V_\beta = 1$, $m^H_\beta = 0$ and $\beta(Z_0) \equiv 0$, $\pi/3$ or $2\pi/3 \pmod{\pi}$ for any $\beta \in \triangle'_+$. For simplicity, set $\beta_{ij} := \beta_i + \cdots + \beta_j$ $(1 \le i \le j \le 3n+2)$, $\widehat{\beta}_i := 2(\beta_i + \cdots + \beta_{3n+1}) + \beta_{3n+2}$ and $\widehat{\beta}_{ij} := \beta_i + \cdots + \beta_{j-1} + 2(\beta_j + \cdots + \beta_{3n+1}) + \beta_{3n+2}$ $(1 \le i < j \le 3n+1)$. Easily we can show

$$\left\{\beta \in {\triangle'}_+^V \mid \beta(Z_0) \equiv \frac{\pi}{3} \ (\mathrm{mod}\, \pi)\right\}$$
$$= \{\beta_{ij} \mid 1 \leq i \leq n+1 \leq j < 2n+2 \text{ or } n+1 < i \leq 2n+2 \leq j < 3n+2$$
$$\text{or } 2n+3 \leq i \leq j = 3n+2\}$$
$$\cup \left\{\widehat{\beta}_i \mid 2n+3 \leq i \leq 3n+1\right\}$$
$$\cup \left\{\widehat{\beta}_{ij} \mid 2n+3 \leq i < j \leq 3n+1 \text{ or } 1 \leq i \leq n+1 < j \leq 2n+2\right\}$$

and

$$\left\{\beta \in {\triangle'}_+^V \mid \beta(Z_0) \equiv \frac{2\pi}{3} \ (\mathrm{mod}\, \pi)\right\}$$
$$= \{\beta_{ij} \mid \text{“}1 \leq i \leq n+1 \ \&\ 2n+2 \leq j \leq 3n+1\text{” or}$$
$$\text{“}n+2 \leq i \leq 2n+2 \ \&\ j = 3n+2\text{”}\}$$
$$\cup \left\{\widehat{\beta}_i \mid 1 \leq i \leq n+1\right\}$$
$$\cup \left\{\widehat{\beta}_{ij} \mid 1 \leq i < j \leq n+1 \text{ or } n+2 \leq i \leq 2n+2 < j \leq 3n+1\right\}.$$

From these facts, it follows that the condition (I) holds. Thus Z_0 is as in the statement of Theorem C. Also, it is easy to show that M is not austere.

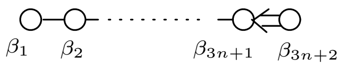

Figure 2.

By refering Tables 1 and 2 in [K2], we shall list up Hermann actions of cohomogeneity two on irreducible symmetric spaces of compact type and rank two satisfying

$$\text{(i) } m_\beta^V = m_\beta^H \ (\forall\, \beta \in \triangle'_+) \quad \text{or} \quad \text{(ii) } {\triangle'}_+^V \cap {\triangle'}_+^H = \emptyset.$$

All of such Hermann actions satisfying (i) are as in Table 1. In Table 1, $\underset{(m)}{\beta}$ means $m_\beta^V = m_\beta^H = m$. All of such Hermann actions satisfying (ii) are the dual actions (see Table 3) of Hermann actions on symmetric spaces of non-compact type as in Table 2. In Table 3, $\underset{(m)}{\beta}$ means m_β^V or m_β^H is equal to m. Since the Hermann actions in Table 2 are commutative, so are also the Hermann actions in Table 3. Also, since ${\triangle'}^V_+ \cap {\triangle'}^H_+ = \emptyset$ as in Table 3 and G/K is irreducible, there exists an inner automorphism ρ of G with $\rho(K) = H$ by Proposition 4.39 in [I]. According to the proof of the proposition, ρ is given explicitly by $\rho = \mathrm{Ad}_G(\exp b)$, where Ad_G is the adjoint representation of G and b is the element of $\mathfrak{b}$ satisfying

$$(\beta_1(b), \beta_2(b)) = \begin{cases} \left(0, \dfrac{\pi}{2}\right) & \text{(in case of (1),(2),(3),(4),(6),(9),(10),(11))} \\ \left(\dfrac{\pi}{2}, 0\right) & \text{(in case of (5),(7))} \\ \left(\dfrac{\pi}{2}, \dfrac{\pi}{2}\right) & \text{(in case of (8)).} \end{cases}$$

Table 1.

$H \curvearrowright G/K$	${\triangle'}^V_+ = {\triangle'}^H_+$
$SO(6) \curvearrowright SU(6)/Sp(3)$	$\{\underset{(2)}{\beta_1}, \underset{(2)}{\beta_2}, \underset{(2)}{\beta_1+\beta_2}\}$
$SO(2)^2 \times SO(3)^2 \curvearrowright (SO(5)\times SO(5))/SO(5)$	$\{\underset{(1)}{\beta_1}, \underset{(1)}{\beta_2}, \underset{(1)}{\beta_1+\beta_2}, \underset{(1)}{2\beta_1+\beta_2}\}$
$SU(2)^2 \cdot SO(2)^2 \curvearrowright (Sp(2)\times Sp(2))/Sp(2)$	$\{\underset{(1)}{\beta_1}, \underset{(1)}{\beta_2}, \underset{(1)}{\beta_1+\beta_2}, \underset{(1)}{2\beta_1+\beta_2}\}$
$Sp(4) \curvearrowright E_6/F_4$	$\{\underset{(4)}{\beta_1}, \underset{(4)}{\beta_2}, \underset{(4)}{\beta_1+\beta_2}\}$
$SU(2)^4 \curvearrowright (G_2\times G_2)/G_2$	$\{\underset{(1)}{\beta_1}, \underset{(1)}{\beta_2}, \underset{(1)}{\beta_1+\beta_2}, \underset{(1)}{2\beta_1+\beta_2}, \underset{(1)}{3\beta_1+\beta_2}, \underset{(1)}{3\beta_1+2\beta_2}\}$

Table 2.

(1)	$SO_0(1,2) \curvearrowright SL(3,\mathbb{R})/SO(3)$
(2)	$Sp(1,2) \curvearrowright SU^*(6)/Sp(3)$
(3)	$U(2,3) \curvearrowright SO^*(10)/U(5)$
(4)	$SO_0(2,3) \curvearrowright SO(5,\mathbb{C})/SO(5)$
(5)	$U(1,1) \curvearrowright Sp(2,\mathbb{R})/U(2)$
(6)	$Sp(2,\mathbb{R}) \curvearrowright Sp(2,\mathbb{C})/Sp(2)$
(7)	$Sp(1,1) \curvearrowright Sp(2,\mathbb{C})/Sp(2)$
(8)	$SO^*(10)\cdot U(1) \curvearrowright E_6^{-14}/Spin(10)\cdot U(1)$
(9)	$F_4^{-20} \curvearrowright E_6^{-26}/F_4$
(10)	$SL(2,\mathbb{R})\times SL(2,\mathbb{R}) \curvearrowright G_2^2/SO(4)$
(11)	$G_2^2 \curvearrowright G_2^{\mathbf{C}}/G_2$

Table 3.

	$H \curvearrowright G/K$	$\triangle'^V_+$	$\triangle'^H_+$
(1)	$SO_0(1,2)^* \curvearrowright SU(3)/SO(3)$	$\{\underset{(1)}{\beta_1}\}$	$\{\underset{(1)}{\beta_2}, \underset{(1)}{\beta_1+\beta_2}\}$
(2)	$Sp(1,2)^* \curvearrowright SU(6)/Sp(3)$	$\{\underset{(4)}{\beta_1}\}$	$\{\underset{(4)}{\beta_2}, \underset{(4)}{\beta_1+\beta_2}\}$
(3)	$U(2,3)^* \curvearrowright SO(10)/U(5)$	$\{\underset{(4)}{\beta_1}, \underset{(1)}{2\beta_1}, \underset{(1)}{2\beta_1+2\beta_2}\}$	$\{\underset{(4)}{\beta_2}, \underset{(4)}{\beta_1+\beta_2}, \underset{(4)}{2\beta_1+\beta_2}\}$
(4)	$SO_0(2,3)^* \curvearrowright$ $(SO(5)\times SO(5))/SO(5)$	$\{\underset{(2)}{\beta_1}\}$	$\{\underset{(2)}{\beta_2}, \underset{(2)}{\beta_1+\beta_2}, \underset{(2)}{2\beta_1+\beta_2}\}$
(5)	$U(1,1)^* \curvearrowright Sp(2)/U(2)$	$\{\underset{(1)}{\beta_2}, \underset{(1)}{2\beta_1+\beta_2}\}$	$\{\underset{(1)}{\beta_1}, \underset{(1)}{\beta_1+\beta_2}\}$
(6)	$Sp(2,\mathbb{R})^* \curvearrowright$ $(Sp(2)\times Sp(2))/Sp(2)$	$\{\underset{(2)}{\beta_1}\}$	$\{\underset{(2)}{\beta_2}, \underset{(2)}{\beta_1+\beta_2}, \underset{(2)}{2\beta_1+\beta_2}\}$
(7)	$Sp(1,1)^* \curvearrowright$ $(Sp(2)\times Sp(2))/Sp(2)$	$\{\underset{(2)}{\beta_2}, \underset{(2)}{2\beta_1+\beta_2}\}$	$\{\underset{(2)}{\beta_1}, \underset{(2)}{\beta_1+\beta_2}\}$
(8)	$(SO^*(10)\cdot U(1))^* \curvearrowright$ $E_6/Spin(10)\cdot U(1)$	$\{\underset{(8)}{\beta_1}, \underset{(1)}{2\beta_1}, \underset{(1)}{2\beta_1+2\beta_2}\}$	$\{\underset{(6)}{\beta_2}, \underset{(9)}{\beta_1+\beta_2}, \underset{(5)}{2\beta_1+\beta_2}\}$
(9)	$(F_4^{-20})^* \curvearrowright E_6/F_4$	$\{\underset{(8)}{\beta_1}\}$	$\{\underset{(8)}{\beta_2}, \underset{(8)}{\beta_1+\beta_2}\}$
(10)	$(SL(2,\mathbb{R})\times SL(2,\mathbb{R}))^* \curvearrowright$ $G_2/SO(4)$	$\{\underset{(1)}{\beta_1}, \underset{(1)}{3\beta_1+2\beta_2}\}$	$\{\underset{(1)}{\beta_2}, \underset{(1)}{\beta_1+\beta_2}, \underset{(1)}{2\beta_1+\beta_2}, \underset{(1)}{3\beta_1+\beta_2}\}$
(11)	$(G_2^2)^* \curvearrowright (G_2\times G_2)/G_2$	$\{\underset{(2)}{\beta_1}, \underset{(2)}{3\beta_1+2\beta_2}\}$	$\{\underset{(2)}{\beta_2}, \underset{(2)}{\beta_1+\beta_2}, \underset{(2)}{2\beta_1+\beta_2}, \underset{(2)}{3\beta_1+\beta_2}\}$

According to Theorem B, we obtain the following fact.

Proposition 4.1 *Let $H \curvearrowright G/K$ be a Hermann action in Table 1 and Z_0 an element of $\mathfrak{b}$ satisfying $(\beta_1(Z_0), \beta_2(Z_0)) = (0, \pi/4)$, $(\pi/4, 0)$ or $(\pi/4, \pi/4)$. Then $M = H(\mathrm{Exp}\, Z_0)$ is a (non-totally geodesic) austere submanifold.*

Denote by $Z_{(a,b)}$ the element Z of $\mathfrak{b}$ satisfying $(\beta_1(Z), \beta_2(Z)) = (a, b)$. In the case where $\triangle'$ is of type $(\mathfrak{a}_2)$, three points of $\mathfrak{b}$ as in Proposition 4.1 are as in Figure 3.

Proposition 4.2 *Let $H \curvearrowright G/K$ be a Hermann action in Table 3 and Z_0 an element of the closure of $\widetilde{C}(\subset \mathfrak{b})$ such that $H(\mathrm{Exp}\, Z_0)$ is minimal. Then, as in Tables $4 \sim 13$, Z_0 satisfies the condition in Theorem C or F, or it does not satisfy the conditions in Theorems C$\sim$F.*

Remark 4.1 There exist exactly seven elements Z_0 of the closure of $\widetilde{C}(\subset \mathfrak{b})$ such that $H(\mathrm{Exp}\, Z_0)$ is minimal.

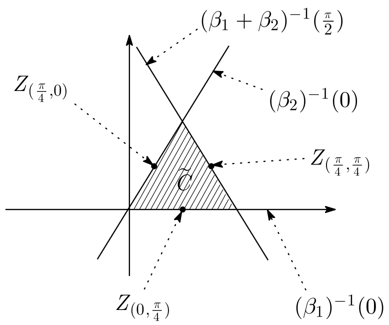

Figure 3.

Table 4.

(a,b)	$Z_{(a,b)}$	$M = SO_0(1,2)^*(\mathrm{Exp}\, Z_{(a,b)})$	$\dim M$
$(0,-\frac{\pi}{2})$	as in Theorem F	one-point set	0
$(0,\frac{\pi}{2})$	as in Theorem F	one-point set	0
$(\pi,-\frac{\pi}{2})$	as in Theorem F	one-point set	0
$(0,0)$	as in Theorem F	totally geodesic	2
$(\frac{\pi}{2},0)$	as in Theorem F	totally geodesic	2
$(\frac{\pi}{2},-\frac{\pi}{2})$	as in Theorem F	totally geodesic	2
$(\frac{\pi}{3},-\frac{\pi}{6})$	as in Theorem C	not austere	3

$$SO_0(1,2)^* \curvearrowright SU(3)/SO(3)$$
$$(\dim SU(3)/SO(3) = 5)$$

The positions of Z_0's in Table 4 are as in Figure 4.

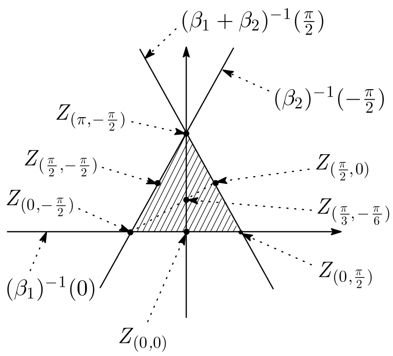

Figure 4.

Table 5.

(a,b)	$Z_{(a,b)}$	$M = Sp(1,2)^*(\operatorname{Exp} Z_{(a,b)})$	$\dim M$
$(0,-\frac{\pi}{2})$	as in Theorem F	one-point set	0
$(0,\frac{\pi}{2})$	as in Theorem F	one-point set	0
$(\pi,-\frac{\pi}{2})$	as in Theorem F	one-point set	0
$(0,0)$	as in Theorem F	totally geodesic	8
$(\frac{\pi}{2},0)$	as in Theorem F	totally geodesic	8
$(\frac{\pi}{2},-\frac{\pi}{2})$	as in Theorem F	totally geodesic	8
$(\frac{\pi}{3},-\frac{\pi}{6})$	as in Theorem C	not austere	12

$$Sp(1,2)^* \curvearrowright SU(6)/Sp(3)$$
$$(\dim SU(6)/Sp(3) = 14)$$

The positions of Z_0's in Table 5 are as in Figure 4.

Table 6.

(a,b)	$Z_{(a,b)}$	$M = U(2,3)^*(\operatorname{Exp} Z_{(a,b)})$	$\dim M$
$(0,\frac{\pi}{2})$	as in Theorem F	one-point set	0
$(0,0)$	as in Theorem F	totally geodesic	12
$(\frac{\pi}{2},-\frac{\pi}{2})$	as in Theorem F	totally geodesic	8
$(\arctan\sqrt{\frac{7}{3}},\frac{\pi}{2}-\arctan\sqrt{\frac{7}{3}})$	not as in Theorems C$\sim$F	not austere	14
$(0,\arctan\frac{1}{\sqrt{13}})$	not as in Theorems C$\sim$F	not austere	13
$(\arctan\frac{\sqrt{5}}{3},-\arctan\frac{\sqrt{5}}{3})$	not as in Theorems C$\sim$F	not austere	17
(a_0,b_0)	not as in Theorems C$\sim$F	not austere	18

$$U(2,3)^* \curvearrowright SO(10)/U(5)$$
$$(\dim SO(10)/U(5) = 20)$$

The positions of Z_0's in Table 6 are as in Figure 5. Also, the numbers a_0 and b_0 in Table 6 are real numbers such that $a_0, b_0 \not\equiv \pi/6,\ \pi/3,\ \pi/4,\ 3\pi/4$ $(\operatorname{mod}\pi)$.

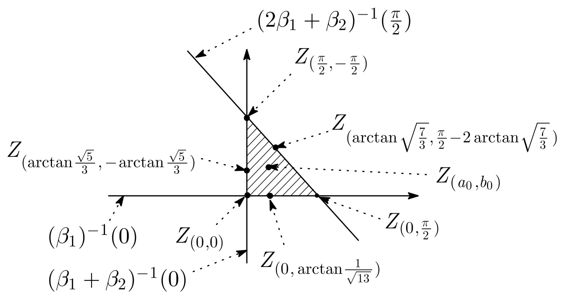

Figure 5.

Table 7.

(a,b)	$Z_{(a,b)}$	$M=SO_0(2,3)^*(\mathrm{Exp}\,Z_{(a,b)})$	$\dim M$
$(0,-\frac{\pi}{2})$	as in Theorem F	one-point set	0
$(0,\frac{\pi}{2})$	as in Theorem F	one-point set	0
$(\frac{\pi}{2},-\frac{\pi}{2})$	as in Theorem F	totally geodesic	4
$(0,0)$	as in Theorem F	totally geodesic	6
$(\arctan\sqrt{3},-\frac{\pi}{2})$	not as in Theorems C$\sim$F	not austere	6
$(\arctan\sqrt{3},\frac{\pi}{2}-2\arctan\sqrt{3})$	not as in Theorems C$\sim$F	not austere	6
$(\arctan\frac{1}{\sqrt{2}},-\arctan\frac{1}{\sqrt{2}})$	not as in Theorems C$\sim$F	not austere	8

$$SO_0(2,3)^* \curvearrowright (SO(5)\times SO(5))/SO(5)$$
$$(\dim(SO(5)\times SO(5))/SO(5)=10)$$

The positions of Z_0's in Table 7 are as in Figure 6.

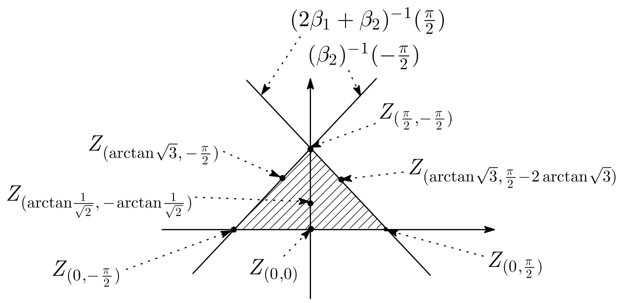

Figure 6.

Table 8.

(a,b)	$Z_{(a,b)}$	$M=U(1,1)^*(\mathrm{Exp}\,Z_{(a,b)})$	$\dim M$
$(\frac{\pi}{2},0)$	as in Theorem F	one-point set	0
$(-\frac{\pi}{2},\pi)$	as in Theorem F	one-point set	0
$(0,0)$	as in Theorem F	totally geodesic	2
$(\frac{\pi}{6},0)$	as in Theorem C	not austere	3
$(-\frac{\pi}{6},\frac{\pi}{3})$	as in Theorem C	not austere	3
$(0,\frac{\pi}{2})$	as in Theorem F	totally geodesic	3
$(0,\arctan\sqrt{2})$	not as in Theorems C$\sim$F	not austere	4

$$U(1,1)^* \curvearrowright Sp(2)/U(2)$$
$$(\dim Sp(2)/U(2)=6)$$

The positions of Z_0's in Table 8 are as in Figure 7.

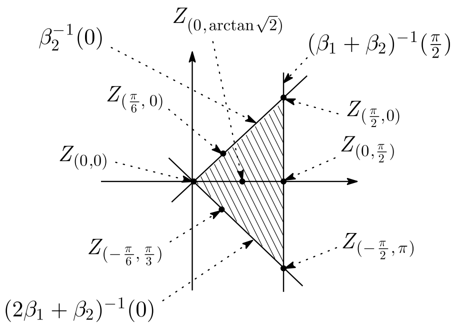

Figure 7.

Table 9.

(a,b)	$Z_{(a,b)}$	$M=Sp(2,\mathbb{R})^*(\mathrm{Exp}\,Z_{(a,b)})$	$\dim M$
$(0,-\frac{\pi}{2})$	as in Theorem F	one-point set	0
$(0,\frac{\pi}{2})$	as in Theorem F	one-point set	0
$(\frac{\pi}{2},-\frac{\pi}{2})$	as in Theorem F	totally geodesic	4
$(0,0)$	as in Theorem F	totally geodesic	6
$(\arctan\sqrt{3},-\frac{\pi}{2})$	not as in Theorems C$\sim$F	not austere	6
$(\arctan\sqrt{3},\frac{\pi}{2}-2\arctan\sqrt{3})$	not as in Theorems C$\sim$F	not austere	6
$(\arctan\frac{1}{\sqrt{2}},-\arctan\frac{1}{\sqrt{2}})$	not as in Theorems C$\sim$F	not austere	8

$$Sp(2,\mathbb{R})^* \curvearrowright (Sp(2)\times Sp(2))/Sp(2)$$
$$(\dim(Sp(2)\times Sp(2))/Sp(2)=10)$$

The positions of Z_0's in Table 9 are as in Figure 6.

Table 10.

(a,b)	$Z_{(a,b)}$	$M = Sp(1,1)^*(\mathrm{Exp}\, Z_{(a,b)})$	$\dim M$
$(\frac{\pi}{2}, 0)$	as in Theorem F	one-point set	0
$(-\frac{\pi}{2}, \pi)$	as in Theorem F	one-point set	0
$(0,0)$	as in Theorem F	totally geodesic	4
$(\frac{\pi}{6}, 0)$	as in Theorem C	not austere	6
$(-\frac{\pi}{6}, \frac{\pi}{3})$	as in Theorem C	not austere	6
$(0, \frac{\pi}{2})$	as in Theorem F	totally geodesic	6
$(0, \arctan\sqrt{2})$	not as in Theorems C$\sim$F	not austere	8

$$Sp(1,1)^* \curvearrowright (Sp(2)\times Sp(2))/Sp(2)$$
$$(\dim(Sp(2)\times Sp(2))/Sp(2) = 10)$$

The positions of Z_0's in Table 10 are as in Figure 7.

Table 11.

(a,b)	$Z_{(a,b)}$	$M = (SO^*(10)\cdot U(1))^*(\mathrm{Exp}\, Z_{(a,b)})$	$\dim M$
$(0,0)$	as in Theorem F	totally geodesic	20
$(0, \frac{\pi}{2})$	as in Theorem F	one-point set	0
$(\frac{\pi}{2}, -\frac{\pi}{2})$	as in Theorem F	totally geodesic	17
$(0, a_1)$	not as in Theorems C$\sim$F	not austere	21
$(a_2, -a_2)$	not as in Theorems C$\sim$F	not austere	29
$(a_3, \frac{\pi}{2} - 2a_3)$	not as in Theorems C$\sim$F	not austere	25
(a_4, b)	not as in Theorems C$\sim$F	not austere	30

$$(SO^*(10)\cdot U(1))^* \curvearrowright E_6/Spin(10)\cdot U(1)$$
$$(\dim E_6/Spin(10)\cdot U(1) = 32)$$

The positions of Z_0's in Table 11 are as in Figure 8. The numbers a_i $(i = 1,2,3,4)$ and b in Table 11 are real numbers such that $a_i, b \not\equiv \pi/6$, $\pi/3$, $\pi/4$, $3\pi/4 \pmod \pi$.

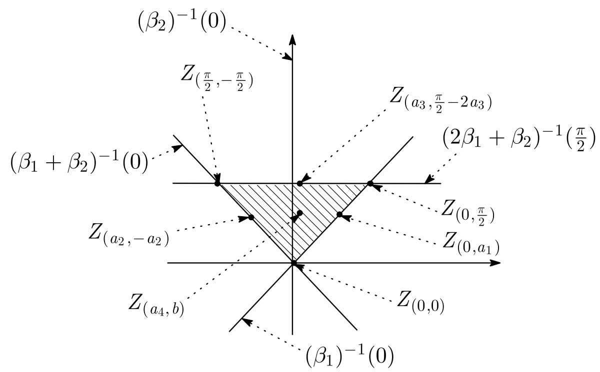

Figure 8.

Table 12.

(a,b)	$Z_{(a,b)}$	$M=(F_4^{-20})^*(\mathrm{Exp}\,Z_{(a,b)})$	$\dim M$
$(0,-\frac{\pi}{2})$	as in Theorem F	one-point set	0
$(0,\frac{\pi}{2})$	as in Theorem F	one-point set	0
$(\pi,-\frac{\pi}{2})$	as in Theorem F	one-point set	0
$(0,0)$	as in Theorem F	totally geodesic	16
$(\frac{\pi}{2},0)$	as in Theorem F	totally geodesic	16
$(\frac{\pi}{2},-\frac{\pi}{2})$	as in Theorem F	totally geodesic	16
$(\frac{\pi}{3},-\frac{\pi}{6})$	as in Theorem C	not austere	24

$$(F_4^{-20})^* \curvearrowright E_6/F_4$$
$$(\dim E_6/F_4=26)$$

The positions of Z_0's in Table 12 are as in Figure 4.

Table 13.

(a,b)	$Z_{(a,b)}$	$M=(SL(2,\mathbb{R})\times SL(2,\mathbb{R}))^*(\mathrm{Exp}\,Z_{(a,b)})$	$\dim M$
$(0,-\frac{\pi}{2})$	as in Theorem F	one-point set	0
$(0,\frac{\pi}{2})$	as in Theorem F	one-point set	0
$(\frac{\pi}{2},-\frac{\pi}{2})$	as in Theorem F	totally geodesic	4
$(\frac{\pi}{3},-\frac{\pi}{2})$	as in Theorem C	not austere	3
$(\arctan\sqrt{5},\frac{\pi}{2}-2\arctan\sqrt{5})$	not as in Theorems C$\sim$F	not austere	5
(a_4,b_2)	not as in Theorems C$\sim$F	not austere	6

$$(SL(2,\mathbb{R})\times SL(2,\mathbb{R}))^* \curvearrowright G_2/SO(4)$$
$$(\dim G_2/SO(4)=8)$$

The positions of Z_0's in Table 13 are as in Figure 9. The numbers a_4 and b_2 in Table 13 are real numbers such that $a_4, b_2 \not\equiv \pi/6,\ \pi/3,\ \pi/4,\ 3\pi/4 \pmod{\pi}$.

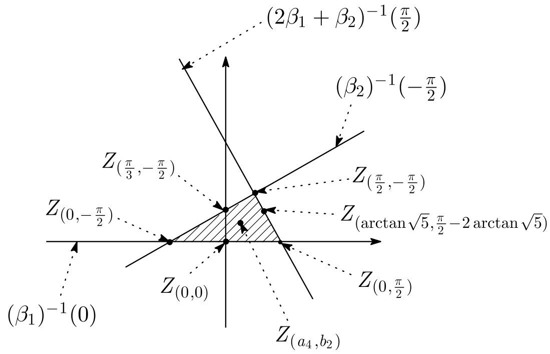

Figure 9.

Table 14.

(a,b)	$Z_{(a,b)}$	$M=(G_2^2)^*(\mathrm{Exp}\,Z_{(a,b)})$	$\dim M$
$(0,-\frac{\pi}{2})$	as in Theorem F	one-point set	0
$(0,\frac{\pi}{2})$	as in Theorem F	one-point set	0
$(\frac{\pi}{2},-\frac{\pi}{2})$	as in Theorem F	totally geodesic	8
$(\frac{\pi}{3},-\frac{\pi}{2})$	as in Theorem C	not austere	6
$(\arctan\sqrt{5},\frac{\pi}{2}-2\arctan\sqrt{5})$	not as in Theorems C$\sim$F	not austere	10
(a_5,b_3)	not as in Theorems C$\sim$F	not austere	12

$$(G_2^2)^* \curvearrowright (G_2\times G_2)/G_2$$
$$(\dim(G_2\times G_2)/G_2 = 14)$$

The positions of Z_0's in Table 14 are as in Figure 9. The numbers a_5 and b_3 in Table 14 are real numbers such that $a_4, b_2 \not\equiv \pi/6,\ \pi/3,\ \pi/4,\ 3\pi/4 \pmod{\pi}$.

Acknowledgement. The author wishes to thank the referee for his valuable comments.

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