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On p-nilpotent groups with extremal p-blocks

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Throughout the present paper, G will represent a finite group, and p a fixed prime number. It is well known that

(I) if G is p-closed, then every p-block of G has full defect, and

(II) if G has the pTI-property, then every p-block of G has either full defect or defect zero.

Here, "G is p-closed" means that a Sylow p-subgroup of G is normal, and "G has the pTI-property" means that the intersection of two distinct Sylow p-subgroups of G is the identity. It is interesting to consider each converse of (I) and (II). In general, neither the converse of (I) nor of (II) is true. In fact, if H is a p-solvable group of p-length greater than 1, then $G=H/O_{p'}(H)$ has only one p-block, but G is neither p-closed nor has the pTI-property. In case $p=2$, several authors studied this problem ([1], [4], [5]). In this paper, we shall show that each converse of (I) and (II) is true if G is a p-nilpotent group. We shall use the following notations: $Z(G)$ is the center of G. Given $g\in G$, we put $x^{g}=gxg^{-1}$ for any $x\in G$, and $S^{g}=\{s^{g}|s\in S\}$ for any subset S of G.

For convenience' sake, we introduce the following definition.

DEFINITION. A group G is a *pFD-group* if every p-block of G has full defect. A group G is a *$pFZD$-group* if every p-block of G has either full defect or defect zero.

The following proposition is an immediate consequence of [7, Theorem 4], and plays an important role in our subsequent study.

PROPOSITION 1. *Let G be a p-nilpotent group with a normal p-complement N. Then G is a pFD-group if and only if, for every $x\in N$, $C_{G}(x)$ contains a Sylow p-subgroup of G.*

By making use of Proposition 1, we can easily obtain the following, which contains [1, Theorem 1].

THEOREM 1. *Let G be a p-nilpotent group. Then G is p-closed if and only if it is a pFD-group.*

PROOF. It suffices to prove the if part. We put $N=O_{p'}(G)$. If $x\in N$, then $x \in C_G(P) \cap N = C_N(P)$ by Proposition 1, where P is a Sylow p-subgroup of G. Hence we have $N = \bigcup_{g \in N} C_N(P)^g$. As is well known, this can happen only if $C_N(P) = N$. Hence G is p-closed.

Let H be a normal subgroup of G such that $|G/H|$ is relatively prime to p. Then by [6, Proposition 4.2], we see that if G is a pFD-group then H is also a pFD-group. Hence, we get the following, which contains [4, Lemma 1].

COROLLARY 1. *Let G be a p-solvable group. Then G is p-closed if and only if it is a pFD-group and has p-length* 1.

By making use of Theorem 1, we can prove the following

THEOREM 2. *Let G be a p-nilpotent group, and $P \cap Q$ an intersection of maximal order of two distinct Sylow p-subgroups of G. Then there exists a p-block of G with defect group $P \cap Q$.*

In advance of proving the theorem, we shall prove the following lemmas.

LEMMA 1. *Let G be a p-nilpotent group, and P a Sylow p-subgroup of G. If D is a subgroup of P, then $N_P(D)$ is a Sylow p-subgroup of $N_G(D)$.*

PROOF. Let R be a Sylow p-subgroup of $N_G(D)$ with $R \supset N_P(D)$, and let S be a Sylow p-subgroup of G with $S \supset R$. Then $S^x = P$ for some x in $O_{p'}(G)$. Let d be an arbitrary element of D. Then, noting that $D^x \subset R^x \subset S^x = P$, we have $d^{-1}d^x \in P \cap O_{p'}(G) = \{1\}$. Hence $d = d^x$. Thus we see that $x \in N_G(D)$. This implies that $R^x \subset P \cap N_G(D) = N_P(D)$, and so $|R| = |R^x| \leqq |N_P(D)|$, proving that $R = N_P(D)$.

LEMMA 2. *Let G be a p-nilpotent group, and $P \cap Q$ an intersection of maximal order of two distinct Sylow p-subgroups of G. Then $N_G(P \cap Q)/P \cap Q$ has the pTI-property.*

PROOF. We put $D = P \cap Q$. Let X and Y be Sylow p-subgroups of $N_G(D)$, and let T and U be Sylow p-subgroups of G with $T \supset X$ and $U \supset Y$. Then $T \cap U \supset X \cap Y \supset D$. Suppose that $X \cap Y \neq D$! Then by the maximality of D, we have $T = U$, and hence $X = N_G(D) \cap T = N_G(D) \cap U = Y$. Hence $N_G(D)/D$ has the pTI-property.

PROOF OF THEOREM 2. By [2, Theorem 58.3], it sufficies to prove that there exists a p-block of $N_G(P \cap Q)$ with defect group $P \cap Q$. So suppose that there exist no p-blocks of $N_G(P \cap Q)$ with defect group $P \cap Q$. Then by Lemma 2, $N_G(P \cap Q)$ is a pFD-group. Hence by Theorem 1, $N_G(P \cap Q)$

is p-cloded, and so $N_P(P\cap Q)=N_Q(P\cap Q)$ (Lemma 1). This implies that $P\cap Q\supset N_P(P\cap Q)\supsetneqq P\cap Q$, a contradiction.

As a corollary to Theorem 2, we get the following, which contains [1, Theorem 2].

THEOREM 3. *Let G be a p-nilpotent group. Then G has the pTI-property if and only if it is a $pFZD$-group.*

The proof of [9, Lemma 1] also holds for general groups which have the pTI-property with $p\neq 2$. Therefore, we see that if a p-solvable group G has the pTI-property, then G has p-length 1. Thus, theorem 3 together with [6, Proposition 4. 2] implies the following

COROLLARY 2. *Let G be a p-solvable group. Then G has the pTI-property if and only if it is a $pFZD$-group and has p-length* 1.

Now, by making use of the same argument as in [1], we shall prove the following results.

LEMMA 3 (cf. [1, Lemma 1. 1]). *Let G be a p-nilpotent group, and P a Sylow p-subgroup of G. If $u\in Z(P)$, then $u\in Z(Q)$ for every Sylow p-subgroup Q of G containing u.*

PROOF. Chose $x\in O_{p'}(G)$ with $Q=P^x$. Noting that u^x and u are elements of Q, we have $u^x u^{-1}\in Q\cap O_{p'}(G)=\{1\}$. Hence $u=u^x\in Z(Q)$.

PROPOSITION 2 (cf. [1, Lemma 1. 2]). *Let G be a p-nilpotent group, P a Sylow p-subgroup of G, and $u\in Z(P)$. If D is a defect group of a p-block of $H=C_G(u)$, then there exists a p-block of G with defect group D.*

PROOF. Let Q be a Sylow p-subgroup of G with $Q\supset D$. Since $u\in D$, we have $u\in Z(Q)$ by Lemma 3. If $x\in N_G(D)$, then $u^x\in D^x=D\subset Q$, and hence $u=u^x$ by [3, Theorem 21. 3]. Thus we have $N_G(D)\subset H$. Hence by [2, Theorem 58.3], we can conclude that there exists a p-block of G with defect group D.

Appendix. Let K be an algebraically closed field of characteristic p. Recently, in [8], Schwarz proved the following:

THEOREM 4 ([8, Satz 6. 3]). *Let B be an arbitrary block ideal of the group algebra KG, and $C=(c_{ij})$ the Cartan matrix of B. If $G/O_{p'p}(G)$ is abelian, then there holds the following*:

(1) $\sum_j c_{ij}$ *is equal to the order of a defect group of B for all i.*

(2) $c_{ii}=c_{jj}$ *for all i, j.*

(3) *The number of non-isomorphic irreducible B-modules is a divisor of* $|G/O_{p'p}(G)|$.

In general, as was claimed in [8], the converse of this theorem need not be true. However, we prove the following, which gives an affirmative answer to a question posed in [8].

THEOREM 5. *Let G be a group of order $p^a m$ $(p\nmid m)$. If the projective cover of the trivial irreducible KG-module has K-dimension p^a, then the following statements are equivalent*:

(1) *$G/O_{p'p}(G)$ is abelian.*

(2) *If B is an arbitrary block ideal of KG and $C=(c_{ij})$ the Cartan matrix of B, then $\sum_j c_{ij}$ is equal to the order of a defect group of B for all i.*

(3) *If $C_0=(c_{kl}^0)$ is the Cartan matrix of the principal block ideal of KG, then $\sum_l c_{kl}^0=p^a$ for all k.*

PROOF. By Theorem 4, it remains only to prove that (3) implies (1). Let B_0 be the principal block ideal of KG, and let $F_1, F_2, \cdots, F_s$ be a full set of non-siomorphic irreducible B_0-modules, where F_1 is the trivial B_0-module and $F_2, \cdots, F_t$ $(t\leqq s)$ have K-dimension 1. We denote by U_i the projective cover of F_i $(1\leqq i\leqq s)$. It is well known that U_i is isomorphic to a direct summand of $U_1\otimes_K F_i$. Hence by our assumption, we have $\dim_K U_k=p^a$ for $1\leqq k\leqq t$. Therefore, we have $p^a=\dim_K U_k=\sum_{l=1}^s c_{kl}^0 \dim_K F_l\geqq \sum_{l=1}^s c_{kl}^0=p^a$. This implies that if $c_{kl}^0\neq 0$ then $\dim_K F_l=1$. Thus we see that if $\dim_K F_k=1$, then all composition factors of U_k have K-dimension 1. Since C_0 is indecomposable, this shows that $t=s$, that is, all irreducible B_0-modules have K-dimension 1. Hence, the commutator subgroup of G is contained in $\bigcap_{i=1}^s \operatorname{Ker} F_i=O_{p'p}(G)$ ([2, Theorem 65. 2]), and so $G/O_{p'p}(G)$ is abelian.

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