Hokkaido Mathematical Journal Vol. 15 (1986), p. 397~403

# Notes on the greatest harmonic minorant

Yukio NAGASAKA
(Received October 25, 1985)

On the unit disk $U$ : $|z|<1$ consider the line segments $I_n=[a_n, b_n]$ ($n=1, 2, \ldots$) on the real line such that $0<b_{n+1}<a_n<b_n$ and $a_n \downarrow 0$. Let $D=U-\bigcup_{n=1}^{\infty} I_n-\{0\}$ and $D_n=U-I_n$. Join $D$ with $D_n$ crosswise along each slit $I_n$ ($n=1, 2, \ldots$). Denote by $R=R_{\{I_n\}}$ this infinite sheeted covering surface over $|z|<1$. Consider $u(z)=\log\frac{1}{|z|}$ on $R$. Then $u$ is superharmonic on $R$. Denote by $H$ the greatest harmonic minorant of $u$, that is $H(z)=\max\{h(z) \mid h\in HP(R)$ and $h\leqq u$ on $R\}$, where $HP(R)$ is the set of nonnegative harmonic functions on $R$. It is an open question whether $H>0$ or $H=0$ on $R$. Our result is the following.

THEOREM A. (i) *If $z=0$ is an irregular boundary point of $D$, then $H>0$ on $R$.*

(ii) *There exists a sequence $\{I_n\}$ such that $z=0$ is a regular boundary point of $D$ and $H=0$ on $R_{\{I_n\}}$.*

Fix any sequences $\{a_n\}$ and $\{b_n\}$ such that $0<b_{n+1}<a_n<b_n<1$ ($n=1, 2, \ldots$) and $a_n \downarrow 0$. Set $I_n=[a_n, b_n]$. Let $\{k_n\}$ be any sequence of positive integers. For each $I_n$, consider

$$I_{n,m}=e^{im\theta_n}I_n \quad \left(\theta_n=\frac{2\pi}{2^{k_n}} : m=1, 2, \ldots, 2^{k_n}\right)$$

$$D=U-\bigcup_{n,m} I_{n,m}-\{0\} \text{ and } D_{n,m}=U-I_{n,m}.$$

Join $D$ with $D_{n,m}$ crosswise along each slit $I_{n,m}$ ($m=1, 2, \ldots, 2^{k_n}$ : $n=1, 2, \ldots$). Denote by $R_{\{k_n\}}$ this covering surface over $|z|<1$. The relative boundary $\partial R_{\{k_n\}}$ of $R_{\{k_n\}}$ consists of $D(|z|=1)=\bar{D}\cap(|z|=1)$ and $D_{n,m}(|z|=1)=\bar{D}_{n,m}\cap(|z|=1)$ ($m=1, 2, \ldots, 2^{k_n}$; $n=1, 2, \ldots$). Let $\phi(r)$ be a non negative continuous function on $(0, 1]$ such that $\lim_{r\to 0}\phi(r)=\infty$ and $\phi(1)=0$.

THEOREM B. *For any $\phi$, there exists $\{k_n\}$ such that every nonnegative harmonic function $v$ on $R_{\{k_n\}}$ with $v(z)\leqq\phi(|z|)$ on $D$ reduces to constant zero.*

REMARK. *For any* $\{k_n\}$, *consider Martin compactification* $R^*$ *of* $R=R_{\{k_n\}}$. *Then there is always a point* $p\in R^*\text{-}R$ *such that the Martin kernel* $K(\cdot, p)$ *is positive harmonic with boundary values* 0 *on* $\partial R$.

Consider $u$ on $U$. Denote by $u_E=u_E^U$ the regularized reduced function of $u$ relative to $E=\bigcup_{n=1}^{\infty} I_n\cup\{0\}$ in $u$ ([2]). Then $u_E$ is superharmonic on $U$, $u_E\leqq u$ on $U$, $u_E=u$ on $\bigcup_{n=1}^{\infty} I_n$ and harmonic on $U\text{-}E$ with boundary values $u$ on $(\bigcup_{n=1}^{\infty} I_n)\cup(|z|=1)$.

LEMMA 1. *$z=0$ is a regular boundary point of $D$ if and only if $u=u_E$ on $U$.*

PROOF. Let $G(z, z_o)$ and $g(z, z_o)$ be the Green functions of $U$ and $D$ with pole $z_o\in D$ respectively. Then $g(z, z_o)=G(z, z_o)-G(\cdot, z_o)_E(z)$ on $D$. Since $g(z, z_o)=0$ on $\bigcup_{n=1}^{\infty} I_n$,

$$\begin{aligned}\overline{\lim_{\substack{z\to 0\\ z\in D}}}\ g(z, z_o)&=\overline{\lim_{\substack{z\to 0\\ z\in U}}}\ g(z, z_o)=G(0, z_o)-\underline{\lim_{\substack{z\to 0\\ z\in U}}}\ G(\cdot, z_o)_E(z)\\ &=G(z_o, 0)-G(\cdot, z_o)_E(0).\end{aligned}$$

Now let show $G(\cdot, z_o)_E(0)=G(\cdot, 0)_E(z_o)$. Set $E_n=\bigcup_{i=1}^{n} I_i$. Then $G(\cdot, z_o)_{E_n}$ and $G(\cdot, 0)_{E_n}$ are Green potentials, that is $G(\cdot, z_o)_{E_n}(z)=\int G(z, w)\, d\mu_{z_o}(w)$ and $G(\cdot, 0)_{E_n}(z)=\int G(z, w) d\mu_o(w)$, where $\mu_{z_o}$ and $\mu_o$ are measures on $E_n$. Since every point of $E_n$ is a regular boundary point of $D$, $G(\cdot, z_o)_{E_n}=G(\cdot, z_o)$ on $E_n$ and $G(\cdot, 0)_{E_n}=G(\cdot, 0)$ on $E_n$. Hence

$$\begin{aligned}G(\cdot, z_o)_{E_n}(0)&=\int G(0, w)\, d\mu_{z_o}(w)=\int G(w, 0)\ d\mu_{z_o}(w)\\ &=\int G(\cdot, 0)_{E_n}(w)\, d\mu_{z_o}(w)=\int\int G(w,z)\, d\mu_0(z) d\mu_{z_o}(w)\end{aligned}$$

and so $G(\cdot, z_o)_{E_n}(0)=G(\cdot, 0)_{E_n}(z_o)$. Since $\{E_n\}$ is an increasing sequence and $E-\bigcup_{n=1}^{\infty} I_n=\{0\}$ is a polar set, it follows that $G(\cdot, z_o)_E(0)=G(\cdot, 0)_E(z_o)=u_E(z_o)$. Hence $\overline{\lim}_{\substack{z\to 0\\ z\in D}} g(z, z_o)=u(z_o)-u_E(z_o)$. Since $z=0$ is a regular boundary point of $D$ if and only if $\overline{\lim}_{\substack{z\to 0\\ z\in D}} g(z, z_o)=0$ for any $z_o\in D$, this lemma follows.

LEMMA 2. *Let $z=0$ be a regular boundary point of $D$ and let $v$ be a positive harmonic function of $D$ such that $v\leqq u$ on $D$. If there exists a real number $0<c<1$ such that $\overline{\lim_{z\to\zeta}}\, v(z)\leqq cu(\zeta)$ for each $\zeta\in\bigcup_{n=1}^{\infty} I_n$, then $v\leqq cu$ on $D$.*

PROOF. Let $v_n$ be harmonic on $G_n=(b_{n+1}<|z|<1)-\bigcup_{i=1}^{n} I_i$ and continuous on $(b_{n+1}\leqq|z|\leqq 1)$ such that $v_n=u$ on $(|z|=b_{n+1})$, $v_n=cu$ on $E_n=\bigcup_{i=1}^{n} I_i$ and $v_n=0$ on $(|z|=1)$ and let $u_{n,m}$ $(n\leqq m)$ be harmonic $(b_{m+1}<|z|<1)$-$E_n$ and continuos on $(b_{m+1}\leqq|z|\leqq 1)$ such that $u_{n,m}=u$ on $E_n$ and $u_{n,m}=0$ on $(|z|=b_{m+1})\cup(|z|=1)$. Then $v\leqq v_m\leqq cu+(u-u_{m,m})$ on $G_m$. Since $u_{n,m}\leqq u_{m,m}\leqq u_E$, $u_{n,m}\uparrow u_{E_n}$ $(m\to\infty)$ and $u_{E_n}\uparrow u_E(n\to\infty)$, it follows that $\lim_{m\to\infty} u_{m,m}=u_E$. By Lemma 1, $u=u_E$. Hence $\lim_{m\to\infty}(u-u_{m,m})=0$ and so $v\leqq cu$ on $D$.

LEMMA 3. *If $\{I_n\}$ satisfies $\sum_{n=1}^{\infty}\log\frac{b_n}{a_n}=\infty$, then $z=0$ is a regular boundary point of $D$.*

PROOF. Fix a point $z_o$ with $b_1<|z_o|<1$. Let $g(z)$ be the Green function on $D$ with pole at $z_o$. Then the Dirichlet integral $D_\Omega(g)$ of $g$ on $\Omega=D\cap(|z|<b_1)$. Set $M(r)=\sup\{g(z)\,|\,|z|=r\}$, $M_n=\inf\{M(r)\,|\,r\in I_n\}$ and $\Omega_n=D\cap(a_n<|z|<b_n)$. By Schwart's inequality,

$$D_{\Omega_n}(g)\leqq\int_0^{2\pi}\int_{a_n}^{b_n}\frac{1}{r^2}\left(\frac{\partial g}{\partial\theta}\right)^2 r\,drd\theta\leqq\int_{a_n}^{b_n}\frac{1}{r}\,\frac{1}{2\pi}\left(\int_0^{2\pi}\left|\frac{\partial g}{\partial\theta}\right|d\theta\right)^2 dr.$$

*Since* $g(r)=0$ for every $r\in I_n$,

$$\int_0^{2\pi}\left|\frac{\partial g}{\partial\theta}\,(re^{i\theta})\right|\,d\theta\geqq\int_0^{\theta}\frac{\partial g}{\partial\theta}(re^{i\theta})\,d\theta=g(re^{i\theta})$$

for each $\theta$ and so $\int_0^{2\pi}\left|\frac{\partial g}{\partial\theta}(re^{i\theta})\right|d\theta\geqq M(r)\geqq M_n$. Hence $D_{\Omega_n}(g)\geqq\frac{1}{2\pi}M_n^2\log\frac{b_n}{a_n}$ and so

$$D_\Omega(g)\geqq\sum_{n=1}^{\infty}D_{\Omega_n}(g)\geqq\frac{1}{2\pi}\sum_{n=1}^{\infty}M_n^2\log\frac{b_n}{a_n}.$$

Since $\sum_{n=1}^{\infty}\log\frac{b_n}{a_n}=\infty$, this shows $\underline{\lim}_{n\to\infty} M_n=0$. Hence $\lim_{z\to 0} g(z)=0$

and so $z=0$ is regular.

Let $0<r<a<b<s<1$ and $I=[a, b]$. Join $S=(r<|z|<s)-I$ and $S'=(|z|<1)-I$ crosswise along the slit $I$, and denote by $\Omega$ this covering surface over $(|z|<1)$. Then $\partial\Omega$ consists of $(|z|=r)\cup(|z|=s)$ on $S$ and $(|z|=1)$ on $S'$. Let $h$ be a harmonic function on $\Omega$ with boundary values $\log\frac{1}{|z|}$ on $\partial\Omega$. Then $h(z)=h(\bar{z})$ on each sheet.

LEMMA 4. *If* $r\leqq a^{\frac{3}{2}}$ *and* $s\geqq b^{\frac{2}{3}}$, *then* $h(z)\leqq\frac{17}{18}\log\frac{1}{|z|}$ *for all* $z\in I$.

PROOF. Since $h(z)\leqq\log\frac{1}{|z|}$ on $\Omega$, $h\leqq\log\frac{1}{a}$ on $S'$ by the maximum theorem. For each $z$ in $A=(r<|z|<s)$ let $z_1$ and $z_2$ be the points in $S$ and $S'$ over $z$ respectively. Then $\phi(z)=h(z_1)+h(z_2)$ is harmonic on $A$. Since $\phi(z)\leqq\log\frac{1}{r}+\log\frac{1}{a}(=\alpha)$ on $(|z|=\mathrm{r})$ and $\phi(z)\leqq 2\log\frac{1}{s}(=\beta)$ on $(|z|=s)$, it follows that

$$\phi(z)\leqq(\alpha-\beta)\frac{\log\frac{s}{|z|}}{\log\frac{s}{r}}+\beta=\left(2-\frac{\log\frac{a}{r}}{\log\frac{s}{r}}\right)\log\frac{1}{|z|}+\frac{\log\frac{a}{r}}{\log\frac{s}{r}}\log\frac{1}{s}$$

$(=\phi_1(z))$ on $A$. Since $h(z)=\frac{1}{2}\phi(z)\leqq\frac{1}{2}\phi_1(z)$ on $I$ and $\log\frac{1}{|z|}\geqq\log\frac{1}{b}$ on $I$,

$$\frac{h(z)}{\log\frac{1}{|z|}}\leqq\frac{\phi_1(z)}{2\log\frac{1}{|z|}}=1-\frac{1}{2}\frac{\log\frac{a}{r}}{\log\frac{s}{r}}+\frac{1}{2}\frac{\log\frac{a}{r}}{\log\frac{s}{r}}\frac{\log\frac{1}{s}}{\log\frac{1}{|z|}}$$

$$\leqq 1-\frac{1}{2}\frac{\log\frac{a}{r}}{\log\frac{s}{r}}\frac{\log\frac{s}{b}}{\log\frac{1}{b}}$$

on $I$. Since $r^{\frac{2}{3}}\leqq a$ and $b^{\frac{2}{3}}\leqq s<1$, $\frac{a}{r}\geqq\left(\frac{1}{r}\right)^{\frac{1}{3}}>\left(\frac{s}{r}\right)^{\frac{1}{3}}$ and $\frac{s}{b}\geqq\left(\frac{1}{b}\right)^{\frac{1}{3}}$.

Hence $\dfrac{h(z)}{\log\frac{1}{|z|}}\leqq 1-\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{3}=\frac{17}{18}$ on $I$.

THE PROOF OF THEOREM A.

(i) Take $\{I_n\}$ such that $z=0$ is an irregular boundary point of $D$ and consider $R=R_{\{I_n\}}$. For $A\subset(|z|\leqq 1)$, denote by $D(A)$ (resp. $D_n(A)$) the part of $D^*=(|z|\leqq 1)-\bigcup_{n=1}^{\infty} I_n$(resp. $D_n^*=(|z|\leqq 1)-I_n$) over $A$. Let $R_n$ be the subregion of R bounded by $\partial R_n=D(|z|=b_{n+1})\cup D(|z|=1)\cup(\bigcup_{i=1}^{n} D_i(|z|=1))$. Let $H_n$ be harmonic on $R_n$ with boundary values $u=\log\frac{1}{|z|}$ on $\partial R_n$. Then $H_n\geqq u-u_E$ on $D$and $H_n\downarrow H$ on $R$ as $n\to\infty$. Since $u\neq u_E$ on $D$ by Lemma 1, it follows that $H>0$ on $R$.

(ii) Let $a_1=0.3$, $b_n=e^{\frac{1}{n}}a_n$ and $b_{n+1}=a_n^{\frac{2}{3}}$ $(n=1,2,\dots)$ and consider $R=R_{\{I_n\}}: I_n=[a_n, b_n]$, $n=1, 2,\dots$. For each $n$, let $\Omega_n$ be the subregion of $R$ bounded by $\partial\Omega_n=D(|z|=b_{n+1})\cup D(|z|=a_{n-1})\cup D_n(|z|=1)$. The $H\leqq\log\frac{1}{|z|}$on $\Omega_n$. Since $b_{n+1}=a_n^{\frac{3}{2}}$ and $b_n^{\frac{2}{3}}=a_{n-1}$, it follows from Lemma 4 that $H(z)\leqq\frac{17}{18}\log\frac{1}{|z|}$ on $I_n$ for each $n$. Since $\log\frac{b_n}{a_n}=\frac{1}{n}$, $\sum\log\frac{b_n}{a_n}=\infty$and so $z=0$ is a regular boundary point of $D$ by Lemma 3. Hence $H(z)\leqq\frac{17}{18}\log\frac{1}{|z|}$on $D$ by Lemma 2. This shows that $H(z)\leqq\left(\frac{17}{18}\right)^k\log\frac{1}{|z|}$on $D$ for every positive integer $k$. Therefore $H=0$ on $R$.

By Schwarz's inequality, we have Lemma 5 and Lemma 6.

LEMMA 5. *Let $f$ be a $C^1$-function on $\Omega=(|z|\leqq 1, \operatorname{Im} z\geqq 0, a\leqq\operatorname{Re} z\leqq b)$ $(0<a<b<1)$ with $f=0$ on $\partial\Omega\cap(|z|=1)$. Then*

$$D\Omega(f)\geqq\int_a^b\int_0^{\sqrt{1-x^2}}\left(\frac{\partial f}{\partial y}\right)^2 dy\,dx\geqq\frac{1}{\sqrt{1-a^2}}\int_a^b[f(x,0)]^2\,dx.$$

LEMMA 6. *Let $f$ be a $C^1$-function on $\Omega=(a<|z|<b, \theta_1<\arg z<\theta_2)$ $(0\leqq\theta_1<\theta_2<2\pi)$. For any $a<r<b$, denote by $Os_\Omega(f:r)$ the oscilation of $f$ on $\{z\in\Omega \mid |z|=r\}$, that is $Os_\Omega(f:r)=\sup\{|f(re^{i\theta})-f(re^{i\theta'})| \mid \theta_1<\theta, \theta'<\theta_2\}$. Then*

$$D_\Omega(f)\geqq\int_{\theta_1}^{\theta_2}\int_a^b\left(\frac{1}{r}\frac{\partial f}{\partial\theta}\right)^2 r\,dr\,d\theta\geqq\frac{1}{(\theta_2-\theta_1)b}\int_a^b[Os_\Omega(f:r)]^2\,dr.$$

LEMMA 7. *Let $0<r<a<b<s<1$ and $I=[a,b]$. For any positive* *integer* $k$ *consider* $I_m = e^{im\theta} I$ $(\theta = \frac{2\pi}{2^k},\ m = 1, 2, \dots, 2^k)$, $S = (r < |z| < s) - \bigcup_{m=1}^{2^k} I_m$ *and* $S_m = (|z| < 1) - I_m$. *Join* $S$ *with* $S_m$ *crosswise along every slit* $I_m$ *and denote* $\Omega_k$ *this covering surface. Then* $\partial\Omega_k = S(|z| = r) \cup S(|z| = s) \cup \left(\bigcup_{m=1}^{2^k} S_m(|z| = 1)\right)$. *Let* $v_k$ *be a non negative harmonic function on* $\Omega_k$ *with boundary values on* $\partial\Omega_k$, $v_k = 1$ *on* $S(|z| = r) \cup S(|z| = s)$ *and* $v_k = 0$ *on* $\bigcup_{m=1}^{2^k} S_m(|z| = 1)$. *Then* $\lim_{k\to\infty} \inf_{a<t<b} \max\{v_k : \Omega_k(|z| = t)\} = 0$ *where* $\Omega_k(|z| = t)$ *is the part of* $\Omega_k$ *over* $(|z| = t)$.

PROOF. Let $\omega$ be harmonic on $S(r < |z| < a) \cup S(b < |z| < s)$ and continuous on $\Omega_k \cup \partial\Omega_k$ such that $\omega = 1$ on $S(|z| = r) \cup S(|z| = s)$ and $\omega = 0$ on $\Omega'_k$, where $\Omega'_k$ is a subregion on $\Omega_k$ bounded by $S(|z| = a) \cup S(|z| = b) \cup \left(\bigcup_{m=1}^{2^k} S_m(|z| = 1)\right)$. Then $D_{\Omega_k}(v_k) \leqq D_{\Omega_k}(\omega)$ $(= A)$ by Dirichlet principle, and $A$ is not dependent on $k$. Let $\Omega_{k,m}$ be the subregion of $\Omega_k$ bounded by $S(|z| = r$ or $s$, $(m - \frac{1}{2})\theta \leqq \arg z \leqq (m + \frac{1}{2})\theta) \cup S(a \leqq |z| \leqq b,\ \arg z = (m \pm \frac{1}{2})\theta) \cup S_m(|z| = 1)$. Since $v_k(z) = v_k(z^*)$ on each sheet $S$ and $S_m$, where $|z| = |z^*|$, $\arg z^* = 2m\theta - \arg z$ for each $m$,

$$D_{\Omega_{k,m}}(v_k) = \frac{1}{2^k} D_{\Omega_k}(v_k) \leqq \frac{A}{2^k} \to 0 \quad (k \to \infty).$$

Fix any $0 < \varepsilon < 1$. Applying Lemma 6 to $e^{im\theta}\Omega$ on $S_m$ there exists a positive integer $k_1$ with following property: for any $k \geqq k_1$ there exists a subset $I'$ of $I$ such that $|I'|$ (= the length of $I'$) $> \frac{3}{4}|I|$ and $v_k(z) < \varepsilon$ for every $z \in e^{im\theta} I'$ $(\theta = \frac{2\pi}{2^k};\ m = 1, 2 \dots, 2^k)$. Applying Lemma 7 to $S'_m = S(a < |z| < b,\ (m-1)\theta < \arg z < m\theta)$ and $S_m$, there exists a positive integer $k_2$ with the following property: for any $k \geqq k_2$ there exists a subset $I''$ of $I$ such that $|I''| > \frac{3}{4}|I|$, $Os_{S'_m}(v_k : r) < \varepsilon$ and $Os_{S_m}(v_k : r) < \varepsilon$. Hence for any $k \geqq \max(k_1, k_2)$ there exists a subset $I^*$ of $I$ such that $|I^*| > \frac{1}{2}|I|$ and $\max\{v_k : \Omega_k(|z| = t)\} < 2\varepsilon$ for any $t \in I^*$

THE PROOF OF THEOREM B.

Let $M_n = \max\{\phi(r) \mid r \in I_n\}$. Fix $0 < \varepsilon < 1$. For each $n$, take a positive integer $k_n$ such that $\max\{M_n v_{k_n} : \Omega_{k_n}(|z| = t_n)\} < \varepsilon$ for some $t_n \in I_n$, where $v_{k_n}$ is the $HP$ function on $\Omega_{k_n}$ stated in Lemma 7 ($r = b_{n+1}$, $a = a_n$, $b = b_n$ and $s = a_{n-1}$). The covering surface $R_{\{k_n\}}$ over $(|z| < 1)$ is a required example. Let $v$ be an $HP$ function on $R_{\{k_n\}}$ such that $v(z) \leqq \phi(|z|)$ on $R_{\{k_n\}}$. Since $v \leqq M_n v_{k_n}$ on $\Omega_{k_n}$ for each $n$, $v < \varepsilon$ on $R_{\{k_n\}}$ and so $v < \varepsilon^m$ for any positive integer $m$. This shows $v = 0$ on $R_{\{k_n\}}$.

## References


[1] C. CONSTANTINESCU and A. CORNEA: Ideale Ränder Riemannscher Flachen, Springer (1963).

[2] L. HELMS: Introduction to potential theory, Wiley-Interscience, (1969).

[3] M. HEINS: On the Lindelöf principle, Ann. of Math. 61 (1955), 440-473.



College of Medical Technology
Hokkaido University