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DISAPPEARING SOLUTIONS FOR DISSIPATIVE HYPERBOLIC SYSTEMS OF CONSTANT MULTIPLICITY

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1. Introduction

Let $n\geq 3$ and Ω be an open domain in $\boldsymbol{R}^n$ with a bounded complement and boundary $\partial\Omega$ assumed real analytic and connected. Consider the mixed problem

$$
(1.1)\qquad \begin{cases} (\partial_t-\sum_{j=1}^{n} A_j\partial_{x_j})u=0 & \text{on } (0,\infty)\times\Omega, \\ \Lambda(x)u=0 & \text{on } (0,\infty)\times\partial\Omega, \\ u(0,x)=f(x). \end{cases}
$$

where A_j, $\Lambda(x)$ are $(r\times r)$ matrices, $\Lambda(x)$ is real analytic and $f(x)\in L^2(\Omega;\boldsymbol{C}^r)$. We shall assume the following conditions fulfilled

(H_1) A_j are constant Hermittian matrices,

$$
(\mathrm{H}_2)\begin{cases}\text{the eigenvalues of the matrix } A(\xi)=\sum_{j=1}^{n}A_j\xi_j \\ \text{have constant multiplicity for } \xi\in\boldsymbol{R}^n\backslash\{0\}.\end{cases}
$$

The above conditions show that the dimension q of the positive eigenspace of the matrix $A(\xi)$ is equal to the dimension of the negative eigenspace. The boundary condition will be assumed maximal dissipative one, i. e.

$$
(\mathrm{H}_3)\begin{cases}\text{a) } \langle A(\nu(x))u, u\rangle\leq 0 \text{ for } u\in\mathrm{Ker}\Lambda(x),\ x\in\partial\Omega, \\ \text{b) } \mathrm{Ker}\Lambda(x) \text{ is the maximal subspace in } \boldsymbol{C}^r, \\ \qquad \text{satisfying the condition a).}\end{cases}
$$

Here $\nu(x)$ is the unit normal at $x\in\partial\Omega$ pointed into $K=\boldsymbol{R}^n\backslash\Omega$, $\langle\ ,\ \rangle$ is the inner product in $\boldsymbol{C}^r$. Moreover, we shall assume the boundary condition coercive (see [5]-[7], [18] for the precise definition). It is well known (see [12], [15], [18]) that the above conditions are valid for a wide class important physical problems such as the Maxwell's equations, accoustic wave equation, Pauli, Dirac's equations etc.

In this work we study the disappearing solutions (D. S.) to the problem

(1. 1). A solution $u(t, x)$ to (1. 1) is called disappearing if $f \neq 0$ and there exists $T_0>0$ such that $u(t, x)=0$ for $t \geq T_0$.

There are at least three important reasons to study the disappearing solutions for dissipative boundary value problems.

First, the disappearing solutions are closely connected with the outgoing and incoming spaces D_+^ρ, D_-^ρ playing a central role in the abstract approach to scattering theory developed by P. Lax and R. Phillips [14], [16]. More precisely, the solution to the problem (1. 1) can be represented in the form $u(t, x)=V(t)f$, where $\{V(t); t \geq 0\}$ is a semigroup of contraction operators acting in the Hilbert space $\mathscr{H}=L^2(\Omega, C^r)$. The inner product in $\mathscr{H}$ is defined by

$$(f, g)_{\mathscr{H}}=\int_\Omega <f(x), g(x)> dx.$$

To state our first result, we denote by $\mathscr{H}_b^\perp$ the orthogonal complement of the linear space $\mathscr{H}_b$ spaned by the eigenvectors of the generator G of $V(t)$ with eigenvalues on the imaginary axis.

THEOREM 1. *Let $f \in \mathscr{H}_b^\perp$ and n be odd. Then the following conditions are equivalent*:

a) *there exists $\rho>0$, such that $V(t)f \perp (D_+^\rho+D_-^\rho)$ for $t \geq 0$,*

b) *there exists $\rho>0$, such that $f \perp D_-^\rho$ and $\lim_{t\to\infty} V(t)f=0$,*

c) *$V(t)f$ is a disappearing solution to* (1. 1).

The above theorem enables one to prove the existence of solutions $V(t)f$ such that $V(t)f \perp (D_+^\rho+D_-^\rho)$ for any $t \geq 0$. The solutions satisfying the condition a) allow one to introduce the notion of the controllability of the scattering operator [17].

The second reason to deal with D. S. is connected with the images of the wave operators $W_\pm$, defined as follows

$$W_- g=\lim_{t\to\infty} V(t)J_0U_0(-t)g,$$

$$W_+ g=\lim_{t\to\infty} V^*(t)J_0U_0(t)g$$

for $g \in \mathscr{H}_{ac}(G_0)$. Here $U_0(t)$ is the unperturbed group connected with the Cauchy problem

$$\begin{cases} (\partial_t - \sum_{j=1}^{n} A_j \partial_{x_j}) u = 0 & \text{in } \boldsymbol{R}^{n+1}, \\ u(0, x) = g, \end{cases}$$

where $g \in \mathscr{H}_0 = L^2(\boldsymbol{R}^n; \boldsymbol{C}^r)$. The operator G_0 is the generator of $U_0(t)$ and J_0 is the orthogonal projection from $\mathscr{H}_0$ onto $\mathscr{H} \subset \mathscr{H}_0$. The problem for description of the images of the wave operators for dissipative systems has been suggested by B. Simon [24], who proved the inclusions Im $W_\pm \subset \mathscr{H}_6$. The complete characterization of the closures $\overline{\text{Im } W_\pm}$ was obtained in [5]-[7].

$$(1.2) \qquad \overline{\text{Im } W_-} = \mathscr{H}_6^\perp \ominus \mathscr{H}_\infty^-, \ \overline{\text{Im } W_+} = \mathscr{H}_6^\perp \ominus \mathscr{H}_\infty^+,$$

where

$$\mathscr{H}_\infty^- = \{ f \in \mathscr{H}_6^\perp ; \lim_{t \to \infty} V(t) f = 0 \},$$

$$\mathscr{H}_\infty^+ = \{ f \in \mathscr{H}_6^\perp ; \lim_{t \to \infty} V^*(t) f = 0 \}.$$

The relations (1. 2) arise the question when the spaces $\mathscr{H}_\infty^\pm$ are nontrivial ones. The answer to this question is closely connected with the existence of disappearing solutions in view of the following

THEOREM 2. *Suppose that n is odd. Then the following conditions are equivalent*

i) $f \perp Im\ W_-$ *and* $f \in \mathscr{H}_6^\perp \ominus D_-^\rho$,

ii) $V(t)f$ *is a disappearing solution to* (1. 1).

This theorem shows that the appearence of D. S. could change the images of the wave operators for dissipative hyperbolic systems.

The third reason to study the D. S. is connected with some inverse scattering problems. More precisely, let us consider the problem to recover the convex hull of the obstacle $K = \boldsymbol{R}^n \backslash \Omega$ from the leading singularity of the kernel of the scattering operator. Recall that the kernel of the scattering operator is a matrix-valued distribution $\{S^{jk}(s, \theta, \omega)\}_{j, k=1}^{q}$, *where* $(s, \theta, \omega) \in \boldsymbol{R} \times S^{n-1} \times S^{n-1}$. For back-scattering data, i. e. $\theta = -\omega$, the leading singularity of the kernel of the scattering operator was investigated by Petkov [22]. He proved that the convex hull of the obstacle can be recovered from the leading singularity of the back-scattering kernel provided the condition

$$(1.3) \qquad N(x) \not\subset \text{Ker}\Lambda(x) \ \text{ for any } x \in \partial\Omega$$

holds. Here $N(x)$ is the negative eigenspace of the matrix $A(\nu(x))$ for $x \in \partial\Omega$. The crucual role of the condition (1. 3) is connected with the fact that the leading singularity of the back-scattering kernel is nonzero if and only if (1.3) holds (see [22]). On the other hand, the scattering operator is a composition of the wave operators. Theorem 2 says that the existence of D. S. can perturbe the image of the wave operators. Consequently, the existence of D. S. changes the kernel of the scattering operator and may influence the solvability of the inverse scattering problem. We shall discuss more completely the relation between the existence of D. S. and the inverse scattering problem in a forthcoming paper.

Those were the reasons that led us to study disappearing solutions for dissipative hyperbolic systems.

The existence of D. S. depends essentially on that which of the following three cases appears

(A) $N(x)\cap\operatorname{Ker}\Lambda(x)=\{0\}$ for any $x\in\partial\Omega$,

(B) $N(x)\cap\operatorname{Ker}\Lambda(x)\neq\{0\}$ for some $x\in\partial\Omega$ and (1.3) holds,

(C) $N(x)\subset\operatorname{Ker}\Lambda(x)$ for at least one $x\in\partial\Omega$.

In this work we shall treat only the cases (A) and (C), while the case (B) will be analyzed in a forthcoming paper.

Our first goal is to study the existence of D. S. in the case (A).

THEOREM 3. *Suppose the condition (A) holds. Then there is no disappearing solution to* (1.1).

Our second goal is to find sufficient condition for the existence of D. S. in the case (C). We consider only Maxwell's equations, which are an important example of mathematical physics. More precisely, assuming the obstacle $K=\boldsymbol{R}^n\backslash\Omega$ to be a strictly convex neighbourhood of the origin 0, introduce the condition

$$N(-x/|x|)\subset\operatorname{Ker}\Lambda(x) \text{ for any } x\in\partial\Omega. \tag{1.4}$$

If the boundary $\partial\Omega$ is a sphere then $\nu(x)=-x/|x|$ and it is obvious that (1.4) implies the case (C) is valid. For general strictly convex obstacles K the property (1.4) is also a stronger version of the condition (C) (see lemma 4.1 below). Then we have the following

THEOREM 4. *Suppose the condition* (1.4) *holds. Then there exists a D. S. to the mixed problem* (1.1) *associated with Maxwell's equations.*

The theorem shows that the boundary conditions, satisfying (1.4), form an important class of boundary conditions, which enable one to construct D. S. A similar construction of D. S. can be used for other physical problems such as the wave equation, Dirac's and Pauli equations etc., provided the same property (1.4) fulfilled.

Our third goal is to obtain some information about the first instant when the D. S. becomes zero, i. e. we wish to estimate the quantity $T(f)=\inf\{t;\ u(t,x)=0\}$ provided $u(t,x)=V(t)f$ is a D. S. Denote by $M(R)$ the maximal length in Ω between the points on the boundary $\partial\Omega$ and the sphere $\{x;\ |x|=R\}$, where $R=\max\ \{|x|;\ x\in\mathrm{supp}\, f\}$ (see section 7 for the precise definition of the maximal length in Ω). In the case when the obstacle $K=\boldsymbol{R}^n\backslash\Omega$ is a strictly convex neighbourhood of the origin 0 the maximal length in Ω between the points on $\partial\Omega$ and the sphere $\{x;\ |x|=R\}$ is

$$M(R)=\max\ R-|x|,\ x\in\partial\Omega$$

Let $c_{\min}$ be the infimum of the nonzero positive eigenvalues of the matrix $A(\xi)$ for $\xi\in S^{n-1}$.

Then we have the following estimate

THEOREM 5. $T(f)\leq M(R)/c_{\min}$.

REMARK. The example from [22] and our construction described in section 4 show that $T(f)=M(R)=M(R)/c_{\min}$ for wide class disappearing solutions $V(t)f$.

Theorem 1 is an extention of the result obtained in [11], [17]. The novelty in the proof of theorem 1, compared with [11], [17] is connected with the application of the wave operator

$$W=s-\lim_{t\to\infty} U_0(-t)JV(t)$$

The existence of W is proved in [5]-[7] by the use of Enss method. Moreover, constructing suitable approximation of any element $f\in\mathscr{H}_6^\perp$ by elements in $\mathscr{H}_6^\perp\cap\bigcap_{N=1}^{\infty}\mathscr{D}(G^N)$ we succeed to weaken the assumptions introduced in [11]. The result of theorem 2 is new even for the wave equation and enables one to find sufficient conditions connected only with the initial data f, which guarantee that $V(t)f$ is a D. S.

The proof of theorem 3 follows the idea introduced in [17] and the construction of Duff [2], [3]. This construction will be used in another work, where disappearing solutions with jumps will be discussed. Following the construction of Duff it is natural to expect that one can find a D. S. in the case (C). Working on this problem we met some essential difficulties which forced us to consider a stronger version of the condition (C) that is sufficient condition for the existence of D. S. The condition (1. 4) is such a sufficient condition.

Finally, theorem 5 to our knowledge is the first result concerning the first instant $T(f)$ when D. S. $V(t)f$ becomes zero. The main tool for the proof of this theorem is the Holmgren's uniqueness theorem in the form obtained in [8]. The author is gratefull to the referee for his suggestion to study the quantity $T(f)$ and obtain some information about it.

We shall sketch the plan of the work. In section 2 we consider the link among D. S., outgoing (incoming) spaces and the images of the wave operators. Some preliminary lemmas are proved in section 3. The construction of D. S. for the Maxwell's equation is given in section 4. The proof of theorem 3 is discussed in sections 5, 6. An important part in the proof of this theorem is the establishment of the convergence of the constructed series. Section 6 is devoted to this problem. Finally, in section 7 we consider the first instant when the disappearing solution becomes zero and obtain an estimate of this quantity.

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2. Link among disappearing solutions, outgoing (incoming) spaces and the image of the wave operators

The solution to the mixed problem (1.1) can be represented by a semigroup $\{V(t),\ t\geq 0\}$ provided the assumptions (H_1)-(H_3) fulfilled (see [15], [16]). The semigroup $V(t)$ acts in the Hilbert space $\mathscr{H}=L^2(\Omega\,;\boldsymbol{C}^r)$ and represents the solution to (1.1) by the equality $u(t,x)=V(t)f$. In order to simplify the proofs in this section we shall assume that

$$(H_4)\begin{cases}\text{the matrix } A(\xi)=\sum A_j\xi_j \text{ is an invertible}\\ \text{one for } \xi\in\boldsymbol{R}^n\backslash\{0\}.\end{cases}$$

In order to introduce the wave and scattering operators one compares the actions of the perturbed group $V(t)$ and the unperturbed group $U_0(t)$. The latter acts in the Hilbert space $\mathscr{H}_0=L^2(\boldsymbol{R}^n\,;\boldsymbol{C}^r)$. An important role in the scattering theory is played by the spaces D_+ and D_- having the properties

$$(2.1)\qquad\begin{cases}\text{i)}\ \ U_0(t)D_+\subset D_+\subset\mathscr{H}_0,\ U_0(-t)D_-\subset D_-\subset\mathscr{H}_0 \text{ for } t\geq 0,\\ \text{ii)}\ \ \bigcap_t U_0(t)D_+=\bigcap_t U_0(t)D_-=0,\\ \text{iii)}\ \ \lim_{t\to+\infty} P_+U_0(t)f=0 \text{ for } f\in\mathscr{H}_0.\end{cases}$$

The precise definition of the outgoing D_+ and incoming D_- spaces is given in [14], [16], [22]. P_+ is the orthogonal projection on the orthogonal complement of D_+. The outgoing and incoming spaces for the perturbed system can be defined by the equalities $D_+^\rho = U_0(\rho)D_+$, $D_-^\rho = U_0(-\rho)D_-$, where the number $\rho > 0$ is choosen so large that

$$K = \boldsymbol{R}^n \backslash \Omega \subset \{x\,;\,|x| < \rho\}.$$

After this preparation work we can turn to the

Proof of theorem 1: *a*)$\Rightarrow$*b*): According to the results in [5]-[7] given any $f \in \mathscr{H}_b^\perp$, there exists $g \in \mathscr{H}_0$, so that

$$(2.2) \qquad \lim_{t\to\infty} \|JV(t)f - U_0(t)g\|_{\mathscr{H}_0} = 0.$$

Recall that J is the inclusion map $J: \mathscr{H} \to \mathscr{H}_0$. It is easy to check the property

$$(2.3) \qquad g \perp D_+^\rho.$$

Indeed, given any $h \in D_+^\rho$ we have $U_0(t)h \in D_+^\rho$ according to (2.1). The equalities

$$(g,\, h)_{\mathscr{H}_0} = \lim_{t\to\infty} (U_0(-t)JV(t)f,\, h)_{\mathscr{H}_0} = \lim_{t\to\infty} (JV(t)f,\, U_0(t)h)_{\mathscr{H}_0}$$

together with the condition a) yield the equality $(g,\, h)_{\mathscr{H}_0} = 0$ and show that (2.3) is valid. In the same manner we get $U_0(t)g \perp D_+^\rho$ for $t \geq 0$. Then we use the property (2.1) iii) and obtain $\lim_{t\to\infty} U_0(t)g = 0$. From this equality and (2.2) we find $\lim_{t\to\infty} V(t)f = 0$. This proves b).

b)$\Rightarrow$*c*): Let $\varepsilon > 0$ be fixed and the condition b) holds. We shall find an element

$$\varphi_\varepsilon \in \mathscr{H}_b^\perp \cap \bigcap_{N=1}^{\infty} \mathscr{D}(G^N),$$

such that $\|\varphi_\varepsilon - f\|_{\mathscr{H}} \leq \varepsilon$, $\varphi_\varepsilon \perp D_-^\rho$ and $\lim_{t\to\infty} V(t)\varphi_\varepsilon = 0$. For the purpose define inductively a sequence $f_0, f_1, f_2, \ldots$. More precisely, set $f_0 = f$. Suppose that $f_0, f_1, \ldots, f_\nu$ $(\nu \geq 0)$ are defined so that $f_j \in \mathscr{D}(G^j)$ for $j = 1, \ldots, \nu$. Set $f_{\nu+1} = (1/\varepsilon_{\nu+1}) \int_0^{\varepsilon_{\nu+1}} V(\tau) f_\nu d\tau$. Then $f_{\nu+1} \in \mathscr{H}(G^{\nu+1})$ and

$$Gf_{\nu+1} = (1/\varepsilon_{\nu+1}) \Big[\, V(\varepsilon_{\nu+1}) f_\nu - f_\nu \,\Big].$$

Choosing $\varepsilon_{\nu+1} > 0$ sufficiently small we can arrange the properties

$$\begin{cases} \text{a)} & \varepsilon_{\nu+1} < \varepsilon/2, \\ \text{b)} & \|G^k f_{\nu+1} - G^k f_\nu\|_{\mathscr{H}} < \varepsilon/2^{\nu+1} \text{ for } k=0,1,\dots,\ \nu. \end{cases}$$

Utilyzing the above estimate we obtain the property

$$\begin{cases} \text{given any integers } N \geq 0,\ \nu \geq N \text{ and } \mu \geq 0 \text{ we have} \\ \|G^N f_{\nu+\mu} - G^N f_\nu\|_{\mathscr{H}} \leq \varepsilon/2^\nu. \end{cases}$$

Since G^N are closed operators, on can find an element

$$\varphi = \varphi_\varepsilon \in \bigcap_{N=1}^{\infty} \mathscr{D}(G^N),$$

such that $\lim_{\nu \to \infty} G^N f_\nu = G^N \varphi$ for each $N \geq 0$. Moreover our choice of the sequence $f_0,\ f_1, \dots$ guarantees that $\varphi \perp D_{-}^{\rho},\ \varphi \in \mathscr{H}_b^{\perp}$.

On the other hand, it is easy to obtain the equality

$$\lim_{t \to \infty} V(t)\varphi = 0. \tag{2.4}$$

Indeed, given any number $\delta > 0$, there exists $\nu = \nu(\delta)$ such that $\|\varphi - f_\nu\|_{\mathscr{H}} \leq \delta/2$. Then we have the inequalities

$$\begin{cases} \|V(t)\varphi\|_{\mathscr{H}} \leq \|V(t)\varphi - V(t)f_\nu\|_{\mathscr{H}} + \|V(t)f_\nu\|_{\mathscr{H}} \\ \leq \delta/2 + \|V(t)f_\nu\|_{\mathscr{H}}. \end{cases} \tag{2.5}$$

To complete the proof of (2.4) it is sufficient to check that (2.6) $\lim_{t \to \infty} V(t) f_\nu = 0$ for any integer $\nu \geq 0$. This property follows from the condition b), when $\nu = 0$. Moreover, we have

$$\|V(t)f\|_{\mathscr{H}} \leq \max_{t \leq \tau \leq t+\varepsilon_{\nu+1}} \|V(\tau)f\|_{\mathscr{H}}.$$

Utilyzing this estimate we obtain inductively (2.6). From (2.5) and (2.6) we derive (2.4).

The choice of the element φ_ε enables us to apply theorem 1 from [11] and conclude that $V(t)\varphi_\varepsilon$ is a D. S., that is $V(t)\varphi_\varepsilon = 0$ for $t \geq T_0$. According to remark 2 in [11] the number $T_0 > 0$ depends on ρ and the matrices $A_1, \dots, A_n$, but T_0 is independent of φ_ε. Taking $\varepsilon \to 0$, we finish the proof of c).

c)⇒a): One can directly apply the result from [11]. This completes the proof of theorem 1.

Proof of theorem 2: Suppose that $f \in \mathscr{H} \ominus (\mathrm{Im}\, W_- + D^{\rho}_-)$. Then the equalities (1.2) lead to the inclusion $f \in \mathscr{H}^-_{\infty}$. This inclusion and theorem 1 show that $V(t)f$ is a D. S. to (1.1).

Suppose that $V(t)f$ is a D. S. Denote by $\overline{\mathscr{H}_6}$ the closure of the linear space $\mathscr{H}_6$, described in the introduction. First, we shall verify the property

$$(2.7) \qquad \| V(t)g\|_{\mathscr{H}} = \|g\|_{\mathscr{H}} \text{ for any } g \in \overline{\mathscr{H}_6}.$$

The above equality is fulfilled when g is an eigenvector of the generator G with eigenvalue on the imaginary axis. On the other hand, the linear space $\mathrm{Ker}(G-\lambda)$ coincides with $\mathrm{Ker}(G^*-\bar{\lambda})$ according to lemma 9.1 from [24]. This fact leads to the equality $(g, h)_{\mathscr{H}} = 0$ provided $Gg = i\lambda g$, $Gh = i\mu h$, $\lambda \neq \mu$, $\lambda, \mu \in \boldsymbol{R}$. Using this property, we can verify (2.7) when g is represented by the linear combination $g = \sum_{k=1}^{\nu} a_k g_k$, $Gg_k = i\lambda_k g_k$, $\lambda_k \in \boldsymbol{R}$ and $\lambda_k \neq \lambda_s$ for $k \neq s$. Consequently, (2.7) is valid for $g \in \mathscr{H}_6^{\perp}$. Since the linear space $\mathscr{H}_6$ is dense in $\overline{\mathscr{H}_6^{\perp}}$ and $V(t)$ are contraction operators, we obtain (2.7) for any $g \in \overline{\mathscr{H}_6^{\perp}}$.

Let $f = f_1 + f_2$, where $f_1 \perp \mathscr{H}_6$ and $f_2 \in \overline{\mathscr{H}_6}$. According to lemma 9.1 from [24] the linear space $\mathscr{H}_6^{\perp}$ is invariant by $V(t)$. The same is valid for $\overline{\mathscr{H}_6}$. The fact that $V(t)f$ is D. S. implies that $\| V(t)f \|^2_{\mathscr{H}} = \|v(t)f_1\|^2_{\mathscr{H}} + \| V(t)f_2 \|^2_{\mathscr{H}}$ vanishes when $t \to \infty$. The property (2.7) shows that $f_2 = 0$ and $f \in \mathscr{H}^-_{\infty}$. Applying the equality (1.2) again, we are going to the property $f \perp \mathrm{Im}\, W_-$. Finally, from the fact that $V(t)f$ is a D. S. and theorem 1 we derive that $f \perp D^{\rho}_-$.

This proves the theorem.

3. Some preliminary lemmas.

Consider the equation

$$(3.1) \qquad \det\,[\tau I + A(\xi)] = 0,$$

where $A(\xi) = \sum A_j \xi_j$. The assumption (H_1) implies that

$$(3.2) \qquad \det\,[\tau I + A(\xi)] = \tau^{q_0} \prod_{j=1}^{m} (\tau - \tau_j(\xi))^{q_j},$$

where $\tau_j(\xi) \neq 0$ for $\xi \in \boldsymbol{R}^n \backslash \{0\}$. The matrix $A(\xi)$ is a Hermittian one and the nonzero rooths of (3.1) can be ordered

$$(3.3) \qquad \tau_1(\xi) > \tau_2(\xi) > \dots > \tau_m(\xi).$$

Moreover, we have the properties (see [14])

$$(3.4)\qquad \begin{cases} \text{a)} & m \text{ is even,} \\ \text{b)} & \tau_j(\xi) \text{ are real analytic functions on } \boldsymbol{R}^n\backslash\{0\} \text{ and} \\ & \text{they are homogeneous of degree 1 with respect to } \xi, \\ \text{c)} & \tau_j(\xi)=\tau_{j'}(-\xi),\ j'=m-j+1, \\ \text{d)} & \mathrm{sgn}(\tau_k(\xi))=\mathrm{sgn}(m/2+\frac{1}{2}-k). \end{cases}$$

Here $\mathrm{sgn}(x)$ denotes the sign of the real number x. The properties (3. 2) and (3. 3) enable one to define the projectors

$$(3.5)\qquad \pi_j(\xi)=(2\pi i)^{-1}\int_{|z-\tau_j(\xi)|=\varepsilon}(zI+A(\xi))^{-1}dz,$$

where $j=0, 1, \ldots, m$, $\tau_0(\xi)=0$ and the number $\varepsilon>0$ is chosen so small that the unique eigenvalue of $A(-\xi)$ in the ball $\{z\in\boldsymbol{C}\,;\,|z-\tau_j(\xi)|\leq\varepsilon\}$ is $\tau_j(\xi)$. We need the following properties of these projectors (see [13])

$$(3.6)\qquad \begin{cases} \text{i)} & \mathrm{Im}\ \pi_j(\xi)=\mathrm{Ker}\ (\tau_j(\xi)I+A(\xi)), \\ \text{ii)} & \sum_{j=0}^{m}\pi_j(\xi)=I, \\ \text{iii)} & \pi_j^*(\xi)=\pi_j(\xi), \\ \text{iv)} & \pi_j(\xi)\pi_k(\xi)=\delta_{jk}\pi_j(\xi), \end{cases}$$

where δ_{jk} is 1 if $j=k$ and 0 for $j\neq k$.

Next, we shall introduce the characteristic surfaces of the operator ∂_t-G. These surfaces are determined by the equality $t=\Psi_k(x)$, where the functions $\Psi_k(x)$ are solutions to the Cauchy problem

$$(3.7)\qquad \begin{cases} \det(I+A(\nabla\Psi_k))=0 & \text{in } U, \\ \nabla\Psi_k(x)=\nu(x)/\tau_k(\nu(x)) & \text{on } \partial\Omega, \\ \Psi_k(x)=0 & \text{on } \partial\Omega. \end{cases}$$

The existence of the function $\Psi_k(x)$ is guaranteed by

LEMMA 3. 1. *Let $1\leq k\leq m$ be fixed. Then there exists a neighbourhood U of the boundary $\partial\Omega$, such that the problem (3. 7) has a unique real analytic solution $\Psi_k(x)$.*

PROOF: We shall discuss only the case $k>m/2$, since the case $k\leq m/2$ can be considered in a similar way. The factorization (3. 2) shows that it is sufficient to solve the problem

$$(3.8)\qquad \begin{cases} \tau_{m-k+1}(\nabla\Psi_k)=1 & \text{in } U, \\ \nabla\Psi_k=\nu/\tau_k(\nu) & \text{on } \partial\Omega, \\ \Psi_k(x)=0 & \text{on } \partial\Omega. \end{cases}$$

Let $x^0 \in \partial\Omega$ be fixed. The boundary $\partial\Omega$ is real analytic and we can change the variables near x^0 so that the boundary is determined by $x_n=0$ near x^0, while the domain Ω is described by $x_n>0$. Then the vector $\nu(x)$ takes the form $(0, 0, \ldots, 0, -1)$ in the new variables. Moreover, the first equation in (3.8) becomes $\tilde{\tau}_{m-k+1}(x, \nabla\Psi_k)=1$, where $\tilde{\tau}_{m+1-k}(x, \xi)$ is a real analytic function homogeneous of degree 1 with respect to ξ and $\tilde{\tau}_{m-k+1}(x, 0, \ldots, 0, -1)=\tau_{m-k+1}(\nu(x))$ for $x \in \partial\Omega$. Consider the equation

$$(3.9) \qquad \tilde{\tau}_{m-k+1}(x, \xi', \xi_n)=1, \ \xi'=(\xi_1, \ldots, \xi_{n-1}).$$

The Euler's equality implies that $\partial_{\xi_n}\tilde{\tau}_{m-k+1}(x, 0, \ldots, 0, -1)=-\tilde{\tau}_{m-k+1}(x, 0, \ldots, 0, -1)=-\tau_{m-k+1}(\nu)\neq 0$ for $x \in \partial\Omega$. Applying the implicit function theorem, we find a real analytic function $\xi_n(x, \xi')$ defined in a small conical neighbourhood of $(x, \xi')=(x^0, 0)$ which satisfies the equation (3.9) and the condition $\xi_n(x, 0)=-1/\tau_k(\nu(x))$ for $x \in \partial\Omega$. Consider the Cauchy problem

$$\begin{cases} \partial_{x_n}\Psi=\xi_n(x, \nabla_{x'}\Psi), \ x'=(x_1, \ldots, x_{n-1}), \\ \Psi(x)=0 \text{ for } x_n=0. \end{cases}$$

The Cauchy-Kovalewska theorem ([20]) yields the existence and uniqueness of the solution to this problem in a small neighbourhood of x^0. Combining this fact with the choice of $\xi_n(x, \xi')$, we obtain $\partial_{x_j}\Psi(x)=0$, $j=1, \ldots, n-1$, $\partial_{x_n}\Psi=\xi_n(x, 0)=-1/\tau_k(\nu)$ when $x_n=0$. Consequently, the problem (3.8) has a solution in a small neighbourhood of the point $x^0 \in \partial\Omega$. Since the boundary $\partial\Omega$ is a compact set, applying Holmgren's uniqueness theorem ([20]) we complete the proof of the lemma.

Let the nonzero eigenvalues of the matrix $A(-\nabla\Psi_k)$ be ordered as follows

$$(3.10) \qquad \lambda_1^k(x)>\lambda_2^k(x)> \ldots >\lambda_m^k(x)$$

and $\lambda_0^k(x)=0$. The corresponding projectors can be determined similarly to (3.5)

$$(3.11) \qquad \phi_j^k(x)=(2\pi i)^{-1}\int_{|z-\lambda_j^k(x)|=\varepsilon}(zI+A(\nabla\Psi_k)^{-1}dz,$$

where $j=0, 1, \ldots, m$ and $\varepsilon>0$ is a sufficiently small number. The functions $\lambda_j^k(x)$ and $\phi_j^k(x)$ are real analytic in U. They have the properties similar to (3.4)

$$(3.12)\quad \begin{cases} \text{i)} & \operatorname{Im} \phi_j^k(x) = \operatorname{Ker}(\lambda_j^k(x) I + A(\nabla \Psi_k)), \\ \text{ii)} & \sum_{j=0}^{m} \phi_j^k(x) = I, \\ \text{iii)} & \phi_j^k(x)^* = \phi_j^k(x), \\ \text{iv)} & \phi_s^k(x)\phi_j^k(x) = \delta_{sj}\phi_j^k(x). \end{cases}$$

LEMMA 3.2. *$\lambda_k^k(x)=1$ for $k \leq m/2$ and $\lambda_{m-k+1}^k(x)=1$ for $k > m/2$.*

PROOF: We shall consider only the case $k > m/2$. The first equation in (3.7) implies that the number 1 is an eigenvalue of $A(-\nabla\Psi_k)$. The number $\tau_k(\nu)$ is negative one and the property (3.3) leads to the following arrangement of the eigenvalues of the matrix $A(-\nu/\tau_k(\nu))$

$$\tau_m(\nu)/\tau_k(\nu) > \ldots > \tau_k(\nu)/\tau_k(\nu) > \ldots > \tau_1(\nu)/\tau_k(\nu).$$

Utilyzing (3.10) we get $\lambda_j^k(x) = \tau_{m-j+1}(\nu)/\tau_k(\nu)$ for $x \in \partial\Omega$ and $\lambda_{m-k+1}^k(x) = 1$ for $x \in U$. This proves the lemma.

Let $N(x)$ be the negative eigenspace of the matrix $A(\nu(x))$ and $P(x)$ be the positive one.

LEMMA 3.3. *Suppose that $N(x) \subset \operatorname{Ker}\Lambda(x)$ for $x \in \partial\Omega$ and $\Lambda(x)$ satisfies the assumption (H_3). Then* $\operatorname{Ker}\Lambda(x) = [P(x)]^\perp$.

PROOF: Let $x \in \partial\Omega$ be fixed. Since the boundary condition $v \in \operatorname{Ker}\Lambda(x)$ is maximally dissipative one, we have $\operatorname{Ker} A(\nu(x)) \subset \operatorname{Ker}\Lambda(x)$. This inclusion and the assumptions of the lemma lead to the property $\operatorname{Ker} A(\nu(x)) + N(x) \subset \operatorname{Ker}\Lambda(x)$. On the other hand, any vector u is orthogonal to the linear space $\operatorname{Ker} A(\nu(x)) + N(x)$ if and only if $u \in P(x)$. Consequently, $[P(x)]^\perp = \operatorname{Ker} A(\nu(x)) + N(x) \subset \operatorname{Ker}\Lambda(x)$. Since the boundary condition is maximal dissipative one, we have the equality $\dim \operatorname{Ker}\Lambda(x) = r - q = \dim P(x)$.

This proves the lemma.

LEMMA 3.4. *Suppose that $N(x) \cap \operatorname{Ker}\Lambda(x) = \{0\}$ for $x \in \partial\Omega$. Given any real analytic vectorvalued function $f(x)$ there exists a unique couple $(f_+(x), f_-(x))$ of real analytic vectorvalued functions, such that*

i) $f(x) = f_+(x) + f_-(x)$,
ii) $f_+ \in P(x)$, $f_- \perp A(\nu)\operatorname{Ker}\Lambda(x)$.

PROOF: Our assumptions imply that

$$(3.13)\quad N(x) + \operatorname{Ker}\Lambda(x) = \boldsymbol{C}^r.$$

It is easy to see that

(3.14) $\quad P(x)\cap[A(\nu)\mathrm{Ker}\Lambda(x)]^{\perp}=\{0\}$.

Indeed, any vector $g\in P(x)\cap[A(\nu)\mathrm{Ker}\Lambda(x)]^{\perp}$ can be represented in the form $g=g_1+g_2$, $g_1\in N(x)$, $g_2\in\mathrm{Ker}\Lambda(x)$ according to (3.13). From $g\in P(x)$ we get $<A(\nu)g, g_1>=0$. Moreover the assumption $g\perp A(\nu)\mathrm{Ker}\Lambda(x)$ gives $<A(\nu)g, g_2>=0$. Hence, $<A(\nu)g, g>=0$ for $g\in P(x)$. This proves the equality $g=0$ and the property (3.14) is verified.

On the other hand, one can find a local basis in $P(x)$ (respectively in $[A(\nu)\mathrm{Ker}\Lambda(x)]^{\perp}$) formed by real analytic vectorvalued functions $e_1^+(x),\dots,e_q^+(x)$ (respectively $e_1^-(x), e_2^-(x),\dots, e_{r-q}^-(x)$). Taking advantage of (3.14) we see that the vectors $e_1^+,\dots,e_q^+, e_1^-,\dots,e_{r-q}^-$ form a basis in $\boldsymbol{C}^r$. Thus any real analytic vector-valued function $f(x)$ on $\partial\Omega$ has the form

$$f(x)=\sum_{k=1}^{q} f_+^k(x)e_k^+(x)+\sum_{k=1}^{r-q} f_1^k(x)e_k^-(x),$$

where f_+^k, f_-^k are real analytic vectorvalued functions. Setting $f_+=\sum_{k=1}^{q} f_+^k e_k^+$, $f_-=\sum_{k=1}^{r-q} f_-^k e_k^-$, we complete the proof of the lemma.

Finally, we turn our attention to the linear spaces

$$\hat{N}(x)=\sum_{k=1}^{m/2}\mathrm{Im}\,\Phi_k^k(x),\quad \hat{P}(x)=\sum_{k=m/2+1}^{m}\mathrm{Im}\,\Phi_{m-k+1}^k(x).$$

LEMMA 3.5. *Suppose the condition* $N(x)\cap\mathrm{Ker}\Lambda(x)=\{0\}$ *holds and* $x^0\in\partial\Omega$ *be fixed. Then there exists a neighbourhood* V *of* x^0 *and a basis* $e_1^+(x),\dots,e_q^+(x), e_1(x),\dots,e_{r-q}(x)$ *in* $\boldsymbol{C}^r$ *formed by real analytic vectorvalued functions in* V, *such that*

i) *given any integer* j, $j\le q$, *there exists an integer* $k_j>m/2$, *such that* $A(\nabla\Psi_{k_j})e_j=-e_j$ *in* V,

ii) *the vectors*

$$e_1^+(x)+e_1(x),\dots, e_q^+(x)+e_q(x),\ e_{q+1}(x),\dots, e_{r-q}(x)$$

form a basis in $\mathrm{Ker}\Lambda(x)$ *for* $x\in V\cap\partial\Omega$.

PROOF: Since $\mathrm{Ker}\,A(\nu)\subset\mathrm{Ker}\Lambda(x)$, we can choose a basis

(3.15) $\quad e_{q+1}(x),\dots, e_{r-q}(x)$

in $\mathrm{Ker}\,A(\nu)$ and complete the above basis with vectors

(3.16) $\quad e_1^+(x)+e_1(x),\dots, e_q^+(x)+e_q(x),\ e_j^+\in\hat{P}(x),\ e_j\in\hat{N}(x)$,

such that the vectors (3.15) and (3.16) form a basis in $\mathrm{Ker}\Lambda(x)$. The assumption $N(x)\cap \mathrm{Ker}\Lambda(x)=\{0\}$ shows that $e_1^+, \dots, e_q^+$ are linear independent vectors, while the assumption (H_3) implies the same for the vectors $e_1, \dots, e_q$. Taking suitable linear combinations of the vectors (3.16) we see that $e_1^+, \dots, e_q^+$ can be chosen as eigenvectors of the projectors ϕ^k_{m-k+1}, i. e. the property i) is fulfilled. This proves the lemma.

4. Sufficient conditions for existence of disappearing solutions to Maxwell's equations.

Maxwell's equations in an exterior domain $\Omega\subset \boldsymbol{R}^3$ have the form

$$(4.1)\qquad \begin{cases} \partial_t E=\mathrm{rot}\ H \text{ in } \Omega, \\ \partial_t H=-\mathrm{rot}\ E \text{ in } \Omega, \end{cases}$$

where $E=(E_1, E_2, E_3)$ and $H=(H_1, H_2, H_3)$ are the vectors of the electric and magnetic fields. The equations (4.1) can be written down as follows $\partial_t u=Gu$, where

$$(4.2)\qquad G=\begin{pmatrix} 0 & \mathrm{rot} \\ -\mathrm{rot} & 0 \end{pmatrix} \text{ and } u=(E, H).$$

Our goal is to construct D. S. to the mixed problem

$$(4.3)\qquad \begin{cases} (\partial_t - G)u=0 & \text{in } (0, \infty)\times\Omega, \\ \Lambda(x)u=0 & \text{on } (0, \infty)\times\partial\Omega, \\ u(0, x)=f(x). \end{cases}$$

Throughout this section we suppose the obstacle $K=\boldsymbol{R}^3\backslash\Omega$ is strictly convex. Without lose of generality we can assume the origin 0 of the coordinate system lies in the interior of the obstacle K. The following assumption concerning the boundary condition in (4.3) will play a crucual role in our considerations

$$(4.4)\qquad N(-x/|x|)\subset \mathrm{Ker}\Lambda(x) \text{ for } x\in\partial\Omega.$$

Here $N(\xi)$, $\xi\in S^2$, is the negative eigenspace of the principle symbol $A(\xi)$ of the operator $i^{-1}G$, which is the matrix

$$(4.5)\qquad A(\xi)=\begin{pmatrix} 0 & D(\xi) \\ -D(\xi) & 0 \end{pmatrix}.$$

Here $D(\xi)$ denotes a (3×3) matrix acting on any vector $v\in\boldsymbol{R}^3$ according to the equality

$$(4.6)\qquad D(\xi)v=\xi\times v.$$

We mentioned in the introduction it is natural to expect that D. S. exist when

(4.7) $\quad N(x) \subset \mathrm{Ker}\Lambda(x)$ for some $x \in \partial\Omega$.

The first step in this section is to prove that the condition (4. 4) does lead to the condition (4. 7), which plays an important role in the investigation of the inverse seattering problem (see [4], [22])

LEMMA 4. 1. *Suppose that* $K = \boldsymbol{R}^3 \backslash \Omega$ *is strictly convex and* (4. 4) *holds. Then the condition* (4. 7) *holds.*

PROOF: Since K is strictly convex and the origin 0 lies in the interior of K, the map $x \in \partial\Omega \longrightarrow x/|x| \in S^2$ is a diffeomorphism of $\partial\Omega$ onto S^2. On the other hand, the Euler's characteristic $\varkappa(S^2)$ of the sphere S^2 is equal to 2 (see &11 in [19]). Applying corollary 9. 7, &9 in [19], we conclude that

$$(4.8) \qquad \begin{cases} \text{for any tangential vector field } v(x) \text{ on } \partial\Omega \text{ one} \\ \text{can find } x^0 \in \partial\Omega, \text{ such that } v(x^0) = 0. \end{cases}$$

We denote by $\pi(x)$ the projection of $x/|x|$ on the tangential plane at $x \in \partial\Omega$. The property (4. 8) guarantees that there exists $x^0 \in \partial\Omega$, such that $\pi(x^0) = 0$. But the equality $\pi(x^0) = 0$ implies that $\nu(x^0) = -x^0/|x^0|$. Hence, we are going to the property $N(-x^0/|x^0|) = N(\nu(x^0))$. Using (4. 4) we get $N(\nu(x^0)) \subset \mathrm{Ker}\Lambda(x^0)$. This proves the lemma.

The second step in this section is the following

Proof of theorem 4: Since the origin 0 lies in the interior of K and K is strictly convex, we can introduce polar coordinates in Ω : $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$, where $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi)$, $r > 0$.

The first equation in (4. 3) takes the form

$$(4.9) \qquad \partial_t u - A(e)\partial_r u - r^{-1}(A(f)\partial_\theta u + A(g)\partial_\varphi u/\sin\theta) = 0,$$

where

$$(4.10) \qquad \begin{cases} e = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta), \\ f = \partial_\theta e, \\ g = \partial_\varphi e/\sin\theta, \end{cases}$$

We shall look for a solution to (4. 3) of the following type

$$(4.11) \qquad u = \Psi(t, r)\ Y(\theta, \varphi),$$

where $\Psi(t, r)$ is a real valued function, while $Y(\theta, \varphi)$ is a vectorvalued function.

Our next goal is to determine the function $Y(\theta, \varphi)$ by using the projectors of the matrix $A(\xi)$, $\xi \in S^2$. Taking into account the equalities (4.5), (4.6), we find $\det(A(\xi)+\tau I)=\tau^2(\tau^2-|\xi|^2)^2$. Consequently, the nonzero eigenvalues of the matrix $A(-\xi)$ are $1=|\xi|>-1=-|\xi|$ for $\xi \in S^2$. Then the projectors on the positive, negative and null eigenspaces of the matrix $A(\xi)$ can be determined as follows

$$\pi_{\pm}(\xi)=\frac{1}{2}\begin{bmatrix} \pi(\xi) & \pm D(\xi) \\ \mp D(\xi) & \pi(\xi) \end{bmatrix}, \quad \pi_0(\xi)=\begin{bmatrix} I-\pi(\xi) & 0 \\ 0 & I-\pi(\xi) \end{bmatrix}. \tag{4.12}$$

Here $D(\xi)$ is a (3×3) matrix defined according to (4.6) and $\pi(\xi)=-D(\xi)D(\xi)$. The fact that $\pi_+(\xi)$, $\pi_-(\xi)$ and $\pi_0(\xi)$ are the projectors on the eigenspaces of the matrix $A(\xi)$ follows directly from the equalities

$$A(\xi)\pi_{\pm}(\xi)=\pm\pi_{\pm}(\xi),\ A(\xi)\pi_0(\xi)=0 \text{ for } \xi \in S^2. \tag{4.13}$$

Next we shall find a function $Y(x/|x|)$ defined on S^2 and satisfying the properties

$$\begin{cases} \text{a)} & Y(x/|x|) \in \operatorname{Im} \pi_+(x/|x|), \\ \text{b)} & A(f)\partial_\theta Y + A(g)\partial_\varphi Y/\sin\theta = Y. \end{cases} \tag{4.14}$$

Since $x/|x|=e$, a direct calculation shows that the image of $\pi_+(x/|x|)=\pi_+(e)$ is spaned by the vectors (g, f) and $(f, -g)$, where $e,$, f, g are defined in (4.10). Set $Y(x/|x|)=\mu(\theta, \varphi)$ (g, f), where $\mu(\theta, \varphi)$ is an unknown function. Substituting the above representation of Y into (4.14) b) and using (4.10) we are going to the following equation for μ

$$(\mathrm{sdn}\theta)\partial_\theta\mu+(\cos\theta)\mu=0.$$

Choosing $\mu=1/\sin\theta$ we obtain that the function

$$Y=(g/\sin\theta, f/\sin\theta) \tag{4.15}$$

is a solution to (4.14). The function (4.15) leads simply to the construction of D. S. of the form (4.11). Substituting the equality (4.11) into (4.9) and exploiting (4.14), we find

$$\partial_t\Psi-\partial_r\Psi-r^{-1}\Psi=0.$$

This equation is satisfied if we choose $\Psi=\phi(t+r)/r$, where $\phi(s)$ is an arbitrary function on $\boldsymbol{R}$. Especially, given any $T>0$ we introduce the function $\phi(s)=\phi_T(s)=s-T$ if $s\leq T$ and $\phi(s)=0$ if $s>T$. The above observation shows that $u=\phi(t+r)$ $(r\sin\theta)^{-1}(g, f)$ is a solution to the first equation in (4.3). On the boundary $\partial\Omega$ the solution $u(t, x)$ belongs to $\operatorname{Im}\pi_+(x/|x|)\subset N(-x/|x|)$. Applying the assumption (4.4), we conclude that $u(t, x)\in \operatorname{Ker}\Lambda(x)$ for $x\in\partial\Omega$ and the boundary condition in (4.3) is fulfilled.

Moreover, the initial data $u(0,x)=\Phi(r)\ (r\ \sin\theta)^{-1}(g,f)$ is nonzero provided $\phi_T(|x|)\neq 0$ for some $x\in\partial\Omega$. The condition $\phi_T(|x|)\not\equiv 0$ is fulfilled when the number T is chosen sufficiently large and $x\in\partial\Omega$.

Finally, our choice of the function $\phi(s)$ guarantees that $u(t,x)=0$ for $t\geq T$ and it is clear that $u(t,x)$ is a D. S. to (4.3). This proves the theorem.

5. Sufficient conditions for nonexistence of disappearing solutions.

In this section we turn again our attention to the general case of first order dissipative hyperbolic systems of type (1.1). Our goal is to prove theorem 3, i. e. there is no D. S. to (1.1) provided

$$\mathrm{Ker}\Lambda(x)\cap N(x)=\{0\}\ \text{ for } x\in\partial\Omega. \tag{5.1}$$

In order to do this consider the mixed problem

$$\begin{cases} (\partial_t-\sum_{j=1}^n A_j\partial_{x_j})u=0 & \text{on } (0,\varepsilon)\times\Omega,\\ (u-h)\perp A(\nu)\mathrm{Ker}\Lambda(x) & \text{on } (0,\varepsilon)\times\partial\Omega,\\ u(0,x)=0, \end{cases} \tag{5.2}$$

where $\varepsilon>0$ and $h(t,x)$ is a real analytic vectorvalued function on $\partial\Omega$ satisfying the compatibility condition $h(0,x)=0$.

First, we shall define more precisely the solutions to (5.2) (see [12], [15], [23])

Definition 5.1. We shall say that the function $v(t,x)\in C([0,\varepsilon];\mathscr{H}^1(\Omega;\boldsymbol{C}^r))\cap C^1((0,\varepsilon);\mathscr{H}^0(\Omega;\boldsymbol{C}^r))$ is a solution to (5.2) if the boundary and initial conditions in (5.2) are fulfilled and the following equality

$$\int_0^\varepsilon\int_\Omega< v(t,x),\ (\partial_t-\sum_{j=1}^n A_j\partial_{x_j})w(t,x)>dx\ dt$$
$$=\int_\Omega< v(\varepsilon,x),\ w(\varepsilon,x)>dx-\int_0^\varepsilon\int_{\partial\Omega}< A(\nu)v,\ w>dS_x\ dt$$

holds for any smooth function $w(t,x)\in C_0^\infty(\boldsymbol{R}^{n+1};\boldsymbol{C}^r)$.

Set

$$D_\varepsilon=C([0,\varepsilon];\mathscr{H}^1(\Omega;\boldsymbol{C}^r))\cap C^1((0,\varepsilon);\mathscr{H}^0(\Omega;\boldsymbol{C}^r)). \tag{5.3}$$

Let us suppose that the mixed problem (5.2) has a solution for $\varepsilon>0$ sufficiently small and any reall analytic vectorvalued function $h(t,x)$ satisfying the compatibility condition

$$h(0,x)=0\ \text{ for } x\in\partial\Omega. \tag{5.4}$$

After this preparation we can start the

Proof of theorem 3: Let $u(t, x)$ be a D. S. to (1. 1) and $T=\inf\{t>0; u(t, x)=0\}$. Given any real analytic vectorvalued function $\varphi(t, x)$ on $\boldsymbol{R}\times\partial\Omega$ we set $h(t, x)=t\varphi(t, x)$. Then the function $h(t, x)$ satisfies the compatibility condition (5. 4). We suppose that (5. 2) has a solution $v(t, x)\in D_\varepsilon$, when $\varepsilon>0$ is sufficiently small. It is easy to see that the equality in definition 5. 1 is fulfilled when $w(t, x)=u(t+T-\varepsilon, x)$. Indeed, the solution $u(t, x)$ can be approximated by functions $u_s(t, x)\in C_0^\infty(\boldsymbol{R}^{n+1}; \boldsymbol{C}^r)$, $s=1, 2, \dots$, such that

$$
(5.5)\qquad \begin{cases} \max\limits_{0\le t\le T} \|u_s(t, x)-u(t, x)\|_{L^2(\Omega)}\to 0, \\ \max\limits_{0\le t\le T} \|A(\nu)(u_s(t, x)-u(t, x))\|_{\mathscr{H}^{-1/2}(\partial\Omega)}\to 0, \\ \max\limits_{0\le t\le T} \|(\partial_t-\sum\limits_{j=1}^{n} A_j\partial_{x_j})u_s(t, x)\|_{L^2(\Omega)}\to 0 \end{cases}
$$

as s tends to $+\infty$ (see &4 in [15], [21]). Setting $w_s(t, x)=u_s(t+T-\varepsilon, x)$, $\tilde{w}_s=(\partial_t-\sum A_j\partial_{x_j})w_s$ and utilyzing the equality in definition 5. 1 we obtain

$$
\int_0^\varepsilon\int_\Omega <v(t, x), \tilde{w}_s(t, x)> dx\ dt
$$

$$
=\int_\Omega <v(\varepsilon, x), w_s(\varepsilon, x)> dx-\int_0^\varepsilon\int_{\partial\Omega} <A(\nu)v, w_s> dS_x\ dt.
$$

Choosing $s\to+\infty$, from (5. 5) we derive

$$
<v(\varepsilon, x), u(T, x)> dx=\int_0^\varepsilon\int_{\partial\Omega} <A(\nu)v, u(t+T-\varepsilon, x)> dS_x\ dt.
$$

Since $u(t, x)$ is a D. S. and $u(T, x)=0$ we are going to

$$
\int_0^\varepsilon\int_{\partial\Omega} <A(\nu)v(t, x), u(t+T-\varepsilon, x)> dS_x\ dt=0.
$$

The boundary conditions in (5. 2), (1. 1) lead to the equality

$$
\int_0^\varepsilon\int_{\partial\Omega} t<\varphi(t, x), u(t+T-\varepsilon, x)> dS_x\ dt=0.
$$

On the other hand, $\varphi(t, x)$ is an arbitrary real analytic function on $\boldsymbol{R}\times\partial\Omega$ and hence $u(t+T-\varepsilon, x)=0$ for $t\in(0, \varepsilon)$, $x\in\partial\Omega$. Using standart integration by parts (see [14]) for the solution to $(\partial_t-\sum A_j\partial_{x_j})u=0$ in the domain $(0, \varepsilon)\times\Omega$ we conclude that $u(t, x)=0$ for $T-\varepsilon<t<T$, $x\in\Omega$. This contradicts to our choice $T=\inf\{t>0\,; u(t, x)=0\}$. The contradiction shows that there is no D. S.

In order to complete the proof of theorem 3 we have to establish

PROPOSITION 5. 1. *Suppose $h(t, x)$ is a real analytic vectorvalued function satisfying the compatibility condition* ((5. 4). *Then there exists $\varepsilon_0>0$, such that the mixed problem* (5. 2) *has a solution $v(t, x)\in D_\varepsilon$ for $0<\varepsilon\leq\varepsilon_0$.*

Proof of proposition 5. 1. Given any integer k, $1+m/2\leq k\leq m$, consider the characteristic surface $t=\Psi_k(x)$, where the function $\Psi_k(x)$ is defined according to lemma 3. 1 in a small neighbourhood U of $\partial\Omega$.

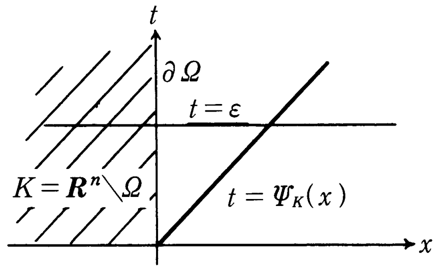

Fig.1

Following the approach of Duff [2], [3], we shall determine the solution to the problem (5. 2) in the form

$$(5.6)\qquad v(t, x)=\sum_{k=1+m/2}^{m} v_k(t, x).$$

The vectorvalued functions $v_k(t, x)$ will be defined in the region $t\geq\Psi_k(x)$ by the series

$$(5.7)\qquad v_k(t, x)=\sum_{p=1}^{\infty} w_p^k(x)(t-\Psi_k(x))^p,$$

where $w_p^k(x)$ are suitably chosen vectorvalued functions. Extending the functions $v_k(t, x)$ as 0 for $t<\Psi_k(x)$ we shall obtain the needed result (see [1]).

We start with the construction of the series (5. 7). The proof of the convergence of this series will be discussed in the next section. The substitution of the series (5. 7) into the first equation in (5. 2) gives the equalities

$$(5.8)\qquad \begin{cases} (I+A(\nabla\Psi_k))w_1^k=0, \\ (p+1)(I+A(\nabla\Psi_k))w_{p+1}^k-G(w_p^k)=0, \end{cases}$$

where $p=1, 2, \ldots,$ $G=\sum_j A_j\partial_{x_j}$. Let the nonzero eigenvalues of the matrix $A(-\nabla\Psi_k)$ be ordered as follows

$$(5.9) \qquad \lambda_1^k(x)>\lambda_2^k(x)> \ldots >\lambda_m^k(x)$$

and $\Phi_1^k(x), \ldots, \Phi_m^k(x)$ be the corresponding orthogonal projectors, defined in (3.7). Lemma 3.2 shows that $\lambda_{m-k+1}^k(x)=1$ and the linear space Ker $(I+A(\nabla\Psi_k))$ coincides with Im $\Phi_{m-k+1}^k(x)$. This fact leads to the following representation of the solutions to (5.8)

$$(5.10) \qquad \begin{cases} w_1^k(x)=\phi_{m-k+1}^k(x)f_k^1(x) \\ w_{p+1}^k(x)=\phi_{m-k+1}^k(x)f_k^{p+1}(x)+(p+1)^{-1}H_kG(w_p^k), \end{cases}$$

where $p=1, 2, \ldots,$ $f_k^p(x)$ are vectorvalued real analytic functions. Moreover, $H_k=H_k(x)$ is a linear operator in $\boldsymbol{C}^r$ defined by

$$(5.11) \qquad H_k(v)=\sum_{\substack{j=0 \\ j\neq m-k+1}}^{m} \Phi_j^k(x)v/(1-\lambda_j^k(x)) \text{ for } v\in\boldsymbol{C}^r.$$

Lemma 3.2 and the inequalities (5.9) guarantees that $\lambda_j^k(x)\neq 1$ for $j\neq m-k+1$, $x\in U$ and the equality (5.11) defines correctly the linear operator H_k. From (5.10) we obtain inductively

$$(5.12) \qquad w_p^k(x)=\sum_{j=1}^{p}\frac{j\,!}{p\,!}(H_kG)^{p-j}\phi_{m-k+1}^k(x)f_k^j(x)$$

for $p=1, 2, \ldots$ Consider the following first order differential operator

$$(5.13) \qquad Q_k=\phi_{m-k+1}^k\, G\phi_{m-k+1}^k.$$

Applying the operator $\Phi_{m-k+1}^k(x)$ to the both sides of (5.8) and using (5.12) we get

$$(5.14) \qquad \begin{cases} Q_k(f_k^1)=0 \text{ in } U, \\ Q_k(f_k^p)=-\sum_{j=1}^{p-1}\frac{j\,!}{p\,!}\phi_{m-k+1}^k G(H_kG)^{p-j}\phi_{m-k+1}^k f_k^j \text{ in } U, \end{cases}$$

for $p=2, 3, \ldots$ Our next goal is to choose the boundary values of $f_k^p(x)$ on $\partial\Omega$ for $p=1, 2, \ldots$ Having in view the form of $v_k(t, x)$, it is natural to exploit the series expansion $h(t, x)=\sum_{p=0}^{\infty} t^p h_p(x)$ for $x\in\partial\Omega$. Notice that the assumption (5.4) implies that $h_0(x)=0$ and

$$(5.15) \qquad h(t, x)=\sum_{p=1}^{\infty} t^p h_p(x).$$

Then the boundary condition in (5.2) will be fulfilled if

$$\left[\sum_{k=1+m/2}^{m} w_p^k(x)-h_p(x)\right]\perp A(\nu)(x))\ \mathrm{Ker}\ \Lambda(x). \tag{5.16}$$

This condition for $p=1$ is equivalent to the property

$$\left[\sum_{k=1+m/2}^{m} \phi^k_{m-k+1}(x)f^1_k(x)-h_1(x)\right]\perp A(\nu(x))\ \mathrm{Ker}\ \Lambda(x)$$

for $x\in\partial\Omega$. Applying lemma 3.4 we conclude that the unknown functions $f^1_k(x)|_{\partial\Omega}$, $k=1+m/2,\dots,m$, can be determined from this requirement. Set $\varphi^1_k(x)=f^1_k(x)|_{\partial\Omega}$. In a similar manner combining (5.12), (5.16) and lemma 3.4 we obtain that the functions $\varphi^p_k(x)=f^p_k(x)|\partial\Omega$ can be determined inductively from (5.16). Thus we are going to the following Cauchy problemr for the functions $f^1_k(x)$, $f^2_k(x)$, ...

$$\begin{cases} Q_k(f^1_k(x))=0 \text{ in } U,\\ f^1_k(x)=\varphi^1_k(x) \text{ on } \partial\Omega,\end{cases} \tag{5.17}$$

$$\begin{cases} Q_k(f^p_k)=-\sum_{j=1}^{p-1}\frac{j\,!}{p\,!}\phi^k_{m-k+1}G(H_kG)^{p-j}\phi^k_{m-k+1}f^j_k \text{ in } U,\\ f^p_k(x)=\varphi^p_k(x) \text{ on } \partial\Omega,\end{cases} \tag{5.17$_p$}$$

where $p=1, 2, \dots$ In order to apply the Cauchy-Kowalevska theorem we must verify the condition

$$\det Q_k(\Psi(x))\neq 0 \text{ for } \Psi(x)=0, \tag{5.18}$$

where $\Psi(x)=0$ represents the boundary $\partial\Omega$ in a small neighbourhood of a fixed point $x^0\in\partial\Omega$. In order to check the property (5.18) we choose a basis $e_1(x),\dots, e_{q_k}(x)$ of the linear space $\mathrm{Im}\ \Phi^k_{m-k+1}(x)$ formed by real analytic vectorvalued functions. Given any point $x\in\partial\Omega$ near x^0 we have $\nabla\Psi=c\nu(x)$, where $c\neq 0$. Using this fact and the representation formula (5.13) of the differential operator Q_k, we obtain that the matrix $Q_k(\Psi(x))$ in the basis $e_1,\dots, e_{q_k}$ has the form

$$\{c<A(\nu)e_i, e_j>\}_{i,j=1}^{q_k}=-c\tau_k(\nu)\{<e_i, e_j>\}_{i,j=1}^{q_k}.$$

This matrix is invertible, since $e_1,\dots, e_{q_k}$ are linear independent vectors. Consequently, the boundary $\partial\Omega$ is noncharacteristic with respect to the operator Q_k for $x\in\partial\Omega$ near x^0. Applying the local Cauchy-Kowalevska theorem [20] we can solve the equations (5.17) in a small neighbourhood of x^0. Since the boundary $\partial\Omega$ is a compact set, we can choose a finite number of open sets, covering the boundary, in which the equations (5.17) can be solved. Holmgren's uniqueness theorem [20] implies that the solutions to these problems can be determined in a small neighbourhood U of the boundary $\partial\Omega$. Thus we can find the functions f_k^1, f_k^2, ... Utilyzing (5.12) we determine the vectorvalued functions $w_p^k(x)$ and the series (5.7).

In the following section we shall discuss the convergence of the series (5.7) and complete the proof of proposition 5.1.

6. Convergence of the series (5.7)

Let $(y^0, t^0)\in\partial\Omega\times\boldsymbol{R}$ be fixed point and $\partial\Omega$ be given near y^0 by the equation $x_n=g(x')$, $x'=(x_1, \ldots, x_{n-1})$, Then we make the change of the variables $y'=x'$, $y_n=x_n-g(x')$. In the new variables the problem (5.2) takes the form

$$(6.1)\qquad \begin{cases} (\partial_t-A_n(y)\partial_{y_n}-\sum_{j=1}^{n-1} A_j\partial_{y_j})v=0, \\ (v-h)\perp A(\nu)\mathrm{Ker}\Lambda(y) \text{ for } y_n=0, \\ v(0, y)=0, \end{cases}$$

where $A_n(y)=A_n-\sum_{j=1}^{n-1} A_j\partial_{y_j}g(y')$. Lemma 3.5 guarantees that there exists an orthogonal basis $e_1^+(y), \ldots, e_q^+(y)$, $e_1(y), \ldots, e_{r-q}(y)$ in $\boldsymbol{C}^r$ formed by real analytic vectorvalued functions, such that

$$(6.2)\qquad \begin{cases} \text{i) given any integer } i\leq q \text{ there exists an integer} \\ \quad k_i>m/2, \text{ such that } A(\nabla\Psi_{k_i})e_i^+=-e_i^+, \\ \text{ii) the vectors } e_{q+1}(y), \ldots, e_{r-q}(y),\ e_1^+(y)+e_1(y), \ldots, \\ \quad e_q^+(y)+e_q(y) \text{ form a basis in } [A(\nu)\mathrm{Ker}\Lambda(y)]^\perp \text{ for } y_n=0. \end{cases}$$

An important role in our proof of the convergence will play the following characteristic functions

$$(6.3)\qquad \sigma_i=t-\Psi_{k_i}(y),$$

where the map $i\to k_i$ is defined according to (6.2)i) and Ψ_{k_i} is the characteristic function determined according to lemma 3.1. Having in view our construction of solution to (5.2) we shall look for the solution to (6.1) in the form

$$(6.4)\qquad v=\sum_{j=1}^{q} v_j(\sigma_j, y)e_j^+(y)+\sum_{j=1}^{r-q}\sum_{s=1}^{q} v_{js}(\sigma_s, y)e_j(y).$$

Substituting (6.4) into the first equation of (6.1) and using (6.2) we are going to the following equations

$$(6.5)\qquad \begin{cases} \partial_{y_n} v_i = b_i(\sigma_i, y, \partial_{y'}) v_i + \sum_{j=1}^{r-q} b_{ji}(\sigma_i, y, \partial_{y'}) v_{ji}, \\ \partial_{\sigma_i} v_{ki} = b_{ki}(\sigma_i, y, \partial_y) v_i + \sum_{j=1}^{r-q} b^k_{ji}(\sigma_i, y, \partial_y) v_{ji}, \\ v_i(\sigma_i, y) = h_i(t, y') + \sum_{s=1}^{q} v_{is}(\sigma_s, y) \text{ for } y_n = 0, \\ v_i(\sigma_i, y) = v_{ki}(\sigma_i, y) = 0 \text{ for } \sigma_i \le 0. \end{cases}$$

Here $i=1,\dots, q$, $k=1,\dots, r-q$, $b_i(\sigma_i, y, \partial_{y'})$, $b_{ji}(\sigma_i, y, \partial_{y'})$, $b^k_{ji}(\sigma_i, y, \partial_y)$ are differential operators of first order whose coefficients are real analytic for $|y-y^0|+|\sigma_i|\le R$. Now we can use the main idea of the proof of Cauchy-Kovalewska theorem (see theorem 4.1 in [20]).

Let $\sum_\alpha c_\alpha(y(y-y^0)^\alpha$ and $\sum_\alpha C_\alpha(y-y^0)^\alpha$ be the series expansions of the real analytic functions $f(y)$ and $F(y)$ near y^0. We shall say that $f(y)$ is majorated by $F(y)$ if $|c_\alpha|\le C_\alpha$ for any $\alpha \in \mathbf{Z}^n_+$.

The construction of the previous section shows that the series (5.6) can be represented in the form (6.4) with respect to the basis $e^+_1(y), \dots, e^+_q(y)$, $e_1(y), \dots, e_{r-q}(y)$ so that

$$(6.6)\qquad v_i(\sigma_i, y) = \sum_{p=1}^{\infty} w^p_i(y)\sigma^p_i,\quad v_{ki}(\sigma_i, y) = \sum_{p=1}^{\infty} w^p_{ki}\sigma^p_i.$$

The construction of section 5 guarantees that the above series expansions satisfy formally (6.5). More precisely, if we substitute the series (6.6) into (6.5) we obtain that the coefficients in front of σ^p_i in the both sides of the equations in (6.5) are equal to each other for any $p\ge 1$.

In order to prove the convergence of the series (6.6) we consider the following problem, which "majorates" (6.5)

$$(6.7)\qquad \begin{cases} \partial_{y_n} V_i = B_i(\sigma_i, y, \partial_{y'}) V_i + \sum_{j=1}^{r-q} B_{ji}(\sigma_i, y, \partial_{y'}) V_{ji}, \\ \partial_{\sigma_i} V_{ki} = B_{ki}(\sigma_i, y, \partial_y) V_i + \sum_{j=1}^{r-q} B^k_{ji}(\sigma_i, y, \partial_y) V_{ji}, \\ V_i(\sigma_i, y) > H_i(t, y') + \sum_{s=1}^{q} V_{is}(\sigma_s, y) \text{ for } y_n = 0, \\ V_i(\sigma_i, y) > 0,\ V_{ki}(\sigma_i, y) > 0. \end{cases}$$

Here $i=1,\dots, q$, $k=1,\dots, r-q$, $H_i(t, y')$ majorates $h_i(t, y')$, the coeffients of the operators B_i, B_{ji}, B^k_{ji} majorate the corresponding coefficients of b_i, b_{ji}, b^k_{ji}. Moreover $F>f$ means that the series expansion of F majorates the series expansion of f. The crucual role in our considerations is played by the following

LEMMA 6.1. *Suppose* $V_i(\sigma_i, y)$, $V_{ki}(\sigma_i, y)$, $i=1,\dots, q$, $k=1,\dots, r-q$, *are real analytic functions satisfying* (6.7) *and* (6.8) $V_i(\sigma_i, y)=\sum_{p=0}^{\infty} W_i^p\sigma_i^p$, $V_{ki}(\sigma_i, y)=\sum_{p=0}^{\infty} W_{ki}^p\sigma_i^p$. *Then* $W_i^p(y)$ *majorates* $w_i^p(y)$ *and* W_{ki}^p *majorates* w_{ki}^p *for any* $p\geq 1$.

PROOF: Substitution of the series (6.6) into (6.5) leads to the equations

$$(6.9)\quad \begin{cases} \text{a)}\quad \partial_{y_n} w_i^p=\sum_{s=1}^{p} d_{is}^p(y, \partial_{y'})w_i^s+\sum_{s=1}^{p}\sum_{j=1}^{r-q} d_{ij}^{ps}(y, \partial_{y'})w_{ji}^s, \\ \text{b)}\quad w_{ki}^1=0, \\ \text{c)}\quad (1+p)w_{ki}^{p+1}=\sum_{s=1}^{p} d_{ki}^{ps}(y, \partial_y)w_i^s+\sum_{s=1}^{p}\sum_{j=1}^{r-q} d_{ij}^{skp}(y, \partial_y)w_{ji}^s, \\ \text{d)}\quad w_i^p(y)=h_i^p(y')+\sum_{j=1}^{q} w_{ij}^p \text{ for } y_n=0. \end{cases}$$

Here $i=1,\dots, q$, $k=1,\dots, r-q$, and $p\geq 1$. In a similar manner the substitution of (6.8) into (6.7) leads to the equations

$$(6.10)\quad \begin{cases} \text{a)}\quad \partial_{y_n} W_i^p=\sum_{s=0}^{p} D_{is}^p(y, \partial_{y'})W_i^s+\sum_{s=0}^{p}\sum_{j=0}^{r-q} D_{ij}^{ps}(y, \partial_{y'})W_{ji}^s, \\ \text{b)}\quad W_{ki}^1>0, \\ \text{c)}\quad (p+1)W_{ki}^{p+1}=\sum_{s=0}^{p} D_{ki}^{ps}(y, \partial_y)W_i^s+\sum_{s=0}^{p}\sum_{j=1}^{r-q} D_{ij}^{skp}(y, \partial_y)W_{ji}^s, \\ \text{d)}\quad W_i^p(y)>H_i^p(y')+\sum_{j=1}^{q} W_{ij}^p \text{ for } y_n=0. \end{cases}$$

The equalities (6.9)b) and (6.10)b) show that $W_{ki}^1>w_{ki}^1=0$ for $k=1,\dots, r-q$, $i=1,\dots, q$. Since w_i^1 and W_i^1 for $i\leq q$ can be compared from (6.9) a)d) and (6.10)a)d) by using the Cauchy-Kovalewska calculation, we obtain that w_i^1 is majorated by W_i^1. Then from (6.9)c) and (6.10)c) we conclude that W_{ki}^2 majorates w_{ki}^2. Again utilyzing (6.9)a)d) and (6.10)a)d) and the Cauchy-Kovalewska calculation we derive that W_i^2 majorates w_i^2 etc. Repeating the above procedure many times, we obtain that w_{ki}^p is majorated by W_{ki}^p, while w_i^p is majorated by W_i^p. This proves the lemma.

In order to complete the proof of the convergence of the series expansion (5.7) it is sufficient to construct a real analytic solution to (6.7) and then apply lemma 6.1. Choosing $M>0$ sufficiently large and $R>0$ sufficiently small, we can assume that $H_i(t, y)$ and the coefficients of the operators B_i, B_{ji}, B_{ji}^k have the form $M/(1-z_i)$, where $z_i=(y_1+\dots+y_{n-1}+\rho y_n+\rho^2\sigma_i)/R$. Here $\rho>0$ will be chosen later. We snall look for the solution to (6.7) in the following form

$$V_{ki} = V(z_i) \text{ for } k \le r-q,\ i \le q,$$
$$V_i = \rho^{-\frac{1}{2}q} V(z_i) + M/(1-z_i) + \tilde{V}(z_i)$$

Then the problem (6.7) can be reduced to the following system of ordinary differential equations with respect to $U = (V, \tilde{V})$

$$(6.11) \qquad (I - z_i I - \Gamma_1(\rho)) \frac{dU}{dZ_i} = \Gamma_2(\rho) U(z_i) + g_3(\rho),$$

where $\Gamma_1(\rho)$, $\Gamma_2(\rho)$ are (2×2) matrices and $\lim_{\rho\to\infty} \Gamma_1(\rho) = 0$. Choosing $\rho > 0$ sufficiently large, we conclude that there exists a real analytic solution to (6.11) satisfying the initial condition $V(0) \ge 0$, $\tilde{V}(0) \ge 0$. Applying lemma 6.1 we complete the proof of the convergence of the series (5.7).

Finally, we shall verify that $v(t, x)$ is a solution to the problem (5.2) (see definition 1 in section 5). Since the set $K(t) = \bar{\Omega} \cap \bigcap_{k=1+m/2}^{m} \{\Psi_k(x) \le t\}$ is a compact set we conclude that $v(t, x) \in D_\varepsilon$ (see the equality (5.3) for the definition of the space D_ε). Let $w(t, x)$ be a smooth function with a compact support in $\boldsymbol{R}^{n+1}$. Then the Green's equality leads to the relation

$$\int_0^\varepsilon \int_\Omega < v, (\partial_t - \sum_{j=1}^n A_j \partial_{x_j}) w > dx\, dt$$

$$= -\sum_{j=1+m/2}^{m} \Big[\int_0^\varepsilon \int_{K_j(t)} < (\partial_t - \sum_1^n A_k \partial_{x_k}) v_j, w > dx\, dt$$

$$+ \int_\Omega < v_j(\varepsilon, x), w(\varepsilon, x) > dx - \int_\Omega < v_j(0, x), w(0, x) > dx$$

$$- \int_0^\varepsilon \int_{\partial\Omega} < A(\nu) v_j, w > dS_x\, dt$$

$$+ \int_0^\varepsilon \int_{t=\Psi_j(x)} < (I + A(\nabla\Psi_j)) v_j, w > dS_x\, dt \Big],$$

where $K_j(t) = \{x ; \Psi_j(x) \le t\}$. Our construction of the solution $v(t, x)$ in the region $K_j(t)$ implies that $(\partial_t - \sum_k A_k \partial_{x_k}) v_j = 0$ in $K_j(t)$, $v_j(t, x) = 0$ for $t = \Psi_j(x)$ and $v(0, x) = 0$. Having in view these properties, we are going to the equality

$$\int_0^\varepsilon \int_\Omega < v, (\partial_t - \sum_{k=1}^n A_k \partial_{x_k}) w > dx\, dt$$

$$= \int_{\Omega} < v(\varepsilon, x), w(\varepsilon, x) > dx - \int_0^{\varepsilon} \int_{\partial\Omega} < A(\nu) v, w > dS_x \, dt.$$

This completes the proof of the proposition 5. 1.

7. The first moment when disappearing solution vanishes

Given any D. S. $u(t, x) = V(t)f$ it is important to obtain some information about the first moment $T(f) = \inf \{t ; V(t)f = 0\}$ when D. S. vanishes. Our goal is to estimate $T(f)$ provided the obstacle $K = \boldsymbol{R}^n \backslash \Omega$ and the support of the initial data supp f contain in the ball $B_R = \{x ; |x| < R\}$. Set $\Omega_R = \Omega \cap B_R$, $S_R = \{x ; |x| = R\}$. First step in this section is to define the maximal distance in Ω_R between the points $x \in \partial\Omega$ and the sphere S_R. Any two points $x \in \partial\Omega$, $y \in S_R$ can be connected by a broken path $P(x, y)$ in Ω_R determined by the points $x_0 = x$, $x_1 \in \Omega_R$, $x_2 \in \Omega_R, \dots, x_{N-1} \in \Omega_R$, $x_N = y$, such that the open segments (x_i, x_{i+1}) lie in Ω_R for $i = 0, 1, \dots, N-1$. Such a path exists, since the boundary $\partial\Omega$ and the domain Ω_R are connected. Any broken path $P(x, y)$ determined as above has a length $|P(x, y)| = \sum_{i=0}^{N-1} |x_{i+1} - x_i|$. The distance in Ω_R between $x \in \partial\Omega$, $y \in S_R$ is $d(x, y) = \min_P |P(x, y)|$, where the minimum is taken over all paths in Ω_R connecting x and y. Now we can define the distance in Ω_R between any point $x \in \partial\Omega$ and S_R by the equality $d(x, S_R) = \min_{y \in S_R} d(x, y)$. Finally, the maximal distance in Ω_R between the points $x \in \partial\Omega$ and the sphere S_R is

$$M(R) = \max_{x \in \partial\Omega} d(x, S_R). \tag{7.1}$$

To simplify our considerations we shall assume that

$$(\mathrm{H}_4) \begin{cases} \text{the matrix } A(\xi) = \sum_j A_j \xi_j \text{ is an invertible one} \\ \text{for } \xi \in \boldsymbol{R}^n \backslash \{0\}. \end{cases}$$

After this preparation work we can turn to the

Proof of theorem 5: The first tool in the proof is the finite propagation speed arguments for hyperbolic problem

$$(\partial_t - \sum_{j=1}^n A_j \partial_{x_j}) u = 0 \text{ in } \{(x, t) ; |t - t_0| < \frac{\varepsilon}{C_{\max}}, |x - x^0| < \varepsilon\}, \tag{7.2}$$

where $\varepsilon > 0$, $(x^0, t_0) \in \boldsymbol{R}^n \times \boldsymbol{R}$ and

$$c_{\max}=\max\ \{|\tau_j(\xi)|\,;1\leq j\leq m,\ \xi\in S^{n-1}\}.$$

More precisely, we need the following

LEMMA 7.1. *(see [14], chapter VI) Suppose that $u(t, x)$ is a solution to (7.2) and $u(t_0, x)=0$ for $|x-x^0|<\varepsilon$. Then $u(t, x)=0$ for $|x-x^0|<\varepsilon-c_{\max}|t-t_0|$.*

The above lemma shows that there exists a compact set $Q\subset \boldsymbol{R}^n$, such that

(7.3) $\quad \mathrm{supp}_x u(t, x)\subset Q$ for $t\geq 0$.

The second lemma we use is the following form of the Holmgren's uniqueness theorem, obtained in [8]

LEMMA 7.2. *(see theorem 1.1 in [8]) Suppose that $\varepsilon>0$, $\varepsilon_1>0$, $u(t, x)$ is a solution to the problem*

$$(\partial_t-\sum_j A_j\partial_{x_j})u=0 \text{ for } |x-x^0|<\varepsilon+\varepsilon_1,\ t\geq t_0$$

and $u(t, x)=0$ for $|x-x^0|<\varepsilon$, $t\geq t_0$. Then $u(t, x)=0$ for $|x-x^0|<\varepsilon+\varepsilon_1$, $t\geq t_0+\varepsilon_1\backslash c_{\min}$.

Combining lemma 7.1 and 7.2 one can check the property

(7.4) $\quad u(t, x)=0$ for $|x|\geq R$.

Indeed, setting

(7.5) $\quad R_1=\max\ \{|x|\,;x\in\mathrm{supp}_x u(t, x),\ t\geq 0\}$

and assuming $R_1>R$, one can choose ε, $0<\varepsilon<R_1-R$, and $x^0\in\{x\,;|x|=R_1+\varepsilon\}$. Our choice of R_1 and x^0 leads to the property $u(t, x)=0$ for $|x-x^0|\leq\varepsilon$, $t\geq 0$. Applying lemma 7.2 we get

(7.6) $\quad u(t, x)=0$ for $|x-x^0|\leq 2\varepsilon$, $t>\varepsilon\backslash c_{\min}$.

Since x^0 is an arbitrary point on the sphere $\{x\,;|x|=R_1+\varepsilon\}$ the property (7.6) implies that

(7.7) $\quad u(t, x)=0$ for $|x|>R_1-\varepsilon$, $t>\varepsilon\backslash c_{\min}$.

On the other hand, lemma 7.1 and the inclusion supp $f\subset B_R$ guarantee that

$$(7.8)\qquad \begin{cases} \text{given any } t>0,\ x\in\mathrm{supp}_x u(t, x) \text{ with } |x|>R, \text{ we have} \\ u(t, x)=0 \text{ for } |x|>R+tc_{\max}. \end{cases}$$

Any point $(x, t)\in\{(x, t)\,;\,R_1-\varepsilon<|x|<R+tc_{\max}\}$ lies in $\{(x, t)\,;\,|x|>R_1-\varepsilon,\ t>\varepsilon\backslash c_{\min}$ provided $R_1>R$ and $\varepsilon>0$ is a sufficiently small number. This fact together with the properties (7.7), (7.8) yield $u(t, x)=0$ for $|x|>R_1-\varepsilon,\ t\geq 0$. But this condition contradicts to (7.5) The contradiction shows that $R_1=R$ and the property (7.4) is verified.

The proof of the theorem will be complete if we verify the property

(7.9) $\quad u(t, x)=0$ for $x\in\Omega_R,\ t\geq M(R)\backslash c_{\min}$.

Let $\varepsilon>0$ and $x^\wedge\in\Omega_R$ be fixed. The equality (7.1) enables us to find $y^\wedge\in S_R$ and a broken path $P(x^\wedge, y^\wedge)$ in Ω_R such that

(7.10) $\quad |P(x^\wedge, y^\wedge)|>M(R)-\varepsilon$.

Let the segments $(x_i, x_{i+1})\subset\Omega_R$, $i=0, 1, \ldots, N$, form the path $P(x^\wedge, y^\wedge)$ and $x_0=y^\wedge$, $x_{N+1}=x^\wedge$. Without lose of generality we can assume that $\{|x-x_i|<r_i+r_{i+1}\}\subset\Omega$ and $r_i<\varepsilon$, where $i=0, \ldots, N$ and $r_i=|x_i-x_{i+1}|$. The property (7.4) and the inequality $r_0<\varepsilon$ guarantee that $u(t, x)=0$ for $|x-x_0|<r_0$, $t>0$. Applying lemma 7.2 we get $u(t, x)=0$ for $|x-x_0|<r_0+r_1$, $t>r_1\backslash c_{\min}$. Hence, we have the property $u(t, x)=0$ for $|x-x_1|<r_1$, $t>r_1\backslash c_{\min}$. Then applying lemma 7.2 we obtain inductively $u(t, x)=0$ for $|x-x_i|<r_i+r_{i+1}$, $t>(r_1+\ldots+r_{i+1})\backslash c_{\min}$. for $i=1, \ldots, N$. Choosing $i=N$ and using the inclusion

$$\{x\,;\,|x-x_N|<r_N+r_{N+1}\}\subset\{x\,;\,|x-x^\wedge|<r_{N+1}\}\subset\Omega,$$

we are going to the property

$$u(t, x)=0 \text{ for } |x-x^\wedge|<r_{N+1},\ t>(r_1+\ldots+r_N)\backslash c_{\min}.$$

Combining the above property and (7.10) we obtain the property (7.9).

This completes the proof of the theorem.

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