

The Initial Boundary Value Problem for the Equations of Ideal Magneto-Hydrodynamics

Taku YANAGISAWA

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§ 1. Introduction and Results.

We consider the initial boundary value problem for the equations of ideal magneto-hydrodynamics :

$$\begin{aligned}
 (1.1) \quad & \begin{cases} (a) & \rho_p(\partial_t + (u \cdot \nabla))p + \rho \operatorname{div} u = 0 \\ (b) & \rho(\partial_t + (u \cdot \nabla))u + \nabla p + \mu H \times \operatorname{curl} H = 0 \\ (c) & \partial_t H - \operatorname{curl} (u \times H) = 0 \end{cases} & \text{in } [0, T] \times \Omega, \\
 (1.2) \quad & (p, u, H)|_{t=0} = (p_0, u_0, H_0) & \text{in } \Omega, \\
 (1.3) \quad & u \cdot n = 0, H \times n = g & \text{on } [0, T] \times \Gamma.
 \end{aligned}$$

Here Ω is a bounded domain in R^3 with C^∞ boundary Γ , T a given positive constant and $n = n(x) = {}^t(n_1, n_2, n_3)$ denotes the unit outward normal at $x \in \Gamma$. Pressure $p = p(t, x)$, velocity $u = u(t, x) = {}^t(u_1, u_2, u_3)$ and the magnetic field $H = H(t, x) = {}^t(H_1, H_2, H_3)$ are unknowns. The permeability μ is supposed to be constant. We also suppose that density $\rho > 0$ is a smooth known function of $p > 0$ i.e. $\rho = \rho(p)$; and $\rho_p = \partial \rho / \partial p$. $g = g(t, x) = {}^t(g_1, g_2, g_3)$ is a given function on $[0, T] \times \Gamma$. $\partial_t = \partial / \partial t$, $\partial_i = \partial / \partial x_i (i=1, 2, 3)$, $\nabla = (\partial_1, \partial_2, \partial_3)$, $(u \cdot \nabla) = \sum_{i=1}^3 u_i \cdot \partial_i$ and \cdot, \times denote scalar and vector product, respectively.

We assume that the initial data p_0 and H_0 satisfy

$$(1.4) \quad \inf_{x \in \Omega} \{\rho(p_0), \rho_p(p_0)\} \geq c_1 > 0,$$

$$(1.5) \quad \operatorname{div} H_0 = 0 \text{ in } \Omega,$$

$$(1.6) \quad \inf_{x \in \Gamma} |H_0 \cdot n| \geq c_2 > 0.$$

Here c_1 and c_2 are positive constants. The assumptions (1.4), (1.5) guarantee the equations (1.1) to be a quasilinear symmetric hyperbolic system.

Our purpose of this paper is to show a local in time existence theorem for the initial boundary value problem (1.1)-(1.3).

THEOREM. *Let m be an integer ≥ 3 . Assume that $g \in \dot{Y}_m(T)$ and the*

initial data $(p_0, u_0, H_0) \in H^m(\Omega)$ satisfies the assumptions (1.4), (1.5), (1.6) and the compatibility conditions of order m .

Then there exists a positive constant T_0 depending only on $\|(p_0, u_0, H_0)\|_m$, $\|g\|_{Y_m(T)}$, c_1 , c_2 , m , Ω such that the initial boundary value problem (1.1), (1.2) and (1.3) has a unique solution which belongs to $X_m(T_0)$.

As to the definitions of the compatibility conditions and the function spaces $X_m(T)$ and $Y_m(T)$, see (2.3), (2.4), (2.5) and (2.6) in § 2.

We remark that this result can be extended to the nonisentropic case without any essential modification of the proof.

The initial boundary value problem for the compressible Euler equations under the solid-wall boundary condition $u \cdot n = 0$ (i.e. (1.1)-(1.3) with $H = 0$) is a typical characteristic initial boundary value problem for quasilinear symmetric hyperbolic systems and has been studied by many authors. (For example, c.f. [1], [3], [10] and [11].)

If one takes account of the effect of the magnetic field, one must consider the equations (1.1). However there seems to be no literature on the existence problem for (1.1). Here, we make some remarks on the boundary conditions. As the boundary conditions for H , we take $H \times n = g$ so that the solution is unique (see, Lemma 5.3). The assumption (1.6), imposed on H_0 , is needed to guarantee the rank of the boundary matrix to be 6 on the boundary (see, (3.8)). It should be remarked that the assumptions (1.5) and (1.6) imply that Ω^c consists of at least two components.

The proof of Theorem proceeds via iteration scheme which involves the following steps. At first, following [10], we modify the equations (1.1) to make the boundary noncharacteristic (see, (3.9) _{ϵ}). We next establish the uniform estimates of the solution of the modified equations under the same initial boundary conditions (1.2) and (1.3). These uniform estimates are achieved by making use of a special structure of the modified equations and the fact that the rank of the boundary matrix is 6 on the boundary. Finally by taking a limit of these solutions, we get the solution of the original initial boundary value problem (1.1)-(1.3).

§ 2. Preliminaries.

In this section, we define some function spaces and the compatibility conditions and present the basic inequalities used in this paper.

$H^m(\Omega)$ denotes a usual scalar or vector valued Sobolev space of order m equipped with inner product $(\cdot, \cdot)_m$ and norm $\|\cdot\|_m$

$$(f, g)_m = \sum_{|\alpha| \leq m} \int_{\Omega} \partial^\alpha f \cdot \partial^\alpha g \, dx,$$

$$\|f\|_m^2 = \sum_{|\alpha| \leq m} \int_{\Omega} |\partial^\alpha f|^2 dx,$$

where $\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \partial_3^{\alpha_3}$ for $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$.

We also use Sobolev space of order s (s : real number) on the boundary $H^s(\Gamma)$ with norm $\|\cdot\|_{H^s(\Gamma)}$.

The following well-known inequalities in three dimension are basic in this paper (c. f. [4] and Appendix B of [10]):

$$(2.1) \quad \|f\|_{C^j(\bar{\Omega})} \leq c \|f\|_{j+2} \text{ for } f \in H^{j+2}(\Omega) \text{ with } j \geq 0,$$

and

$$(2.2) \quad \left\{ \begin{array}{l} \text{(a)} \quad \|fg\|_m \leq c(\|f\|_{m-1}\|g\|_m + \|f\|_m\|g\|_{m-1}) \\ \quad \text{for } f, g \in H^m(\Omega) \text{ with } m \geq 3, \\ \text{(b)} \quad \|fg\|_m \leq c\|f\|_2\|g\|_m \text{ for } f, g \in H^m(\Omega) \text{ with } m \leq 2, \\ \text{Furthermore,} \\ \quad \|fg\|_0 \leq c\|f\|_1\|g\|_1 \\ \text{(c)} \quad \|A(f)\|_m \leq c\|\partial A/\partial f\|_{C^{m-1}(G_1)}\|f\|_m^m \\ \quad \text{for a smooth function } A \text{ on a phase space } G \text{ and } f \in H^m(\Omega) \\ \quad \text{with } m \geq 2 \text{ such that } f(x) \in G_1, \bar{G}_1 \subset G. \end{array} \right.$$

Here $\|\cdot\|_{C^j(K)}$ denotes C^j norm on the set K and c is a constant depending only on m and Ω .

For a Banach space B , let $C^j(0, T; B)$ be the set consisting of j -times continuously differentiable functions of $t \in [0, T]$ with values in B . Then we define

$$(2.3) \quad X_m(T) = \bigcap_{j=0}^m C^j(0, T; H^{m-j}(\Omega)),$$

$$(2.4) \quad Y_m(T) = \bigcap_{j=0}^m C^j(0, T; H^{m-j+1/2}(\Gamma)) \cap C^{m+1}(0, T; H^{1/2}(\Gamma)),$$

with norms

$$\begin{aligned} \|u\|_{X_m(T)} &= \sup_{t \in [0, T]} \|u(t)\|_m = \sup_{t \in [0, T]} \left(\sum_{j=0}^m \|\partial_t^j u(t)\|_{m-j} \right), \\ \|g\|_{Y_m(T)} &= \sup_{t \in [0, T]} \left(\sum_{j=0}^m \|\partial_t^j g(t)\|_{H^{m-j+1/2}(\Gamma)} + \|\partial_t^{m+1} g(t)\|_{H^{1/2}(\Gamma)} \right), \end{aligned}$$

where $u(t)$ stands for $u(t, x)$ for fixed t . Moreover we define

$$(2.5) \quad \dot{Y}_m(T) = \{g \in Y_m(T); g \cdot n = 0 \text{ on } [0, T] \times \Gamma\}.$$

Finally we say that the initial data (p_0, u_0, H_0) satisfy the compatibility conditions of order m for the equations (1.1) and the boundary conditions (1.3), if

$$(2.6) \quad \partial_t^k u(0) \cdot n = 0, \quad \partial_t^k H(0) \times n = \partial_t^k g(0) \quad \text{on } \Gamma, \\ \text{for } k=0, 1, \dots, m-1.$$

Here the terms $\partial_t^k u(0)$, $\partial_t^k H(0)$ are calculated from (1.1) and (1.2), and are then expressed by initial data and their derivatives. For instance,

$$\partial_t u(0) = -(u_0 \cdot \nabla) u_0 - \rho(p_0)^{-1} (\nabla p_0 + \mu H_0 \times \text{curl } H_0), \\ \partial_t H_0 = \text{curl } (u_0 \times H_0).$$

§ 3. A modification of the equations (1.1) to the noncharacteristic case.

At first, we remark that (1.1)(c) is equivalent to

$$(1.1)(c)' \quad (\partial_t + (u \cdot \nabla))H - (H \cdot \nabla)u + H \text{ div } u = 0 \quad \text{in } [0, T] \times \Omega,$$

under the first condition of (1.3) and (1.5); This can be seen from well-known formulae from vector analysis

$$-\text{curl } (u \times H) = (u \cdot \nabla)H - (H \cdot \nabla)u + H \text{ div } u - u \text{ div } H, \\ \text{div curl} = 0.$$

Thus we consider the system of the equations (1.1)(a)(b)(c)' which is symmetric.

In order to make the boundary conditions homogeneous we construct an extension G of g as follows.

LEMMA 3.1. *For a function $g \in \dot{Y}_m(T)$, there exists an extension $G \in \bigcap_{j=0}^m C^j(0, T; H^{m-j+1}(\Omega)) \cap C^{m+1}(0, T; H^1(\Omega))$ such that*

- (a) $\text{div } G = 0$ in $[0, T] \times \Omega$,
- (b) $G \cdot n = 0$, $G \times n = g$ on $[0, T] \times \Gamma$,
- (c) $\sup_{t \in [0, T]} \left(\sum_{j=0}^m \|\partial_t^j G(t)\|_{m-j+1} + \|\partial_t^{m+1} G(t)\|_1 \right) \leq c \|g\|_{Y_m(T)}.$

PROOF. Since $(n \times g) \cdot n = 0$, it follows from the results of L.

Cattabriga [2] that there exists an extension $G \in \bigcap_{j=0}^m C^j(0, T; H^{m-j+1}(\Omega)) \cap C^{m+1}(0, T; H^1(\Omega))$ satisfying $G = n \times g$ on $[0, T] \times \Gamma$, (a) and (c). Furthermore since $g \cdot n = 0$ we can see that $(n \times g) \times n = g$, which completes the proof of (b).

Set $\tilde{H} = H - G$. Then (p, u, \tilde{H}) is a solution of the initial boundary value problem:

$$(3.1) \quad \begin{cases} (a) & \rho_p(\partial_t + (u \cdot \nabla))p + \rho \operatorname{div} u = 0 \\ (b) & \rho(\partial_t + (u \cdot \nabla))u + \nabla p + (\tilde{H} + G) \times \operatorname{curl} \tilde{H} + \tilde{H} \times \operatorname{curl} G \\ & = -G \times \operatorname{curl} G \\ (c)' & (\partial_t + (u \cdot \nabla))\tilde{H} - ((\tilde{H} + G) \cdot \nabla)u + (\tilde{H} + G) \operatorname{div} u + (u \cdot \nabla)G \\ & = -\partial_t G \end{cases}$$

in $[0, T] \times \Omega$,

$$(3.2) \quad (p, u, \tilde{H})|_{t=0} = (p_0, u_0, \tilde{H}_0), \quad \tilde{H}_0 = H_0 - G(0) \text{ in } \Omega,$$

$$(3.3) \quad u \cdot n = 0, \quad \tilde{H} \times n = 0 \text{ on } [0, T] \times \Gamma.$$

Here we assumed, without loss of generality, that $\mu = 1$; take a normalization if necessary.

By (1.5), (1.6), (2.6) and Lemma 3.1, we know that the initial data $U_0 = (p_0, u_0, \tilde{H}_0)$ satisfy (1.4) and

$$(3.4) \quad \operatorname{div} \tilde{H}_0 = 0 \text{ in } \Omega,$$

$$(3.5) \quad \inf_{x \in \Gamma} |\tilde{H}_0 \cdot n| \geq c_2,$$

$$(3.6) \quad \partial_t^k u(0) \cdot n = 0, \quad \partial_t^k \tilde{H}(0) \times n = 0 \text{ on } \Gamma, \text{ for } k = 0, 1, \dots, m-1.$$

For simplicity we will write again H instead of \tilde{H} . To clarify the structure of this initial boundary value problem it is convenient to express the system (3.1) in the matrix form:

$$(3.7) \quad A_0(U) \partial_t U + \sum_{j=1}^3 A_j(U) \partial_j U + B(t, x) U = F(t, x).$$

Here $U = {}^t(p, u, H)$ and $A_0(U)$, $A_j(U)$ ($j=1, 2, 3$), $F(t, x)$ are given explicitly by

$$\begin{aligned} A_0(U) &= \begin{bmatrix} \rho^{-1} \rho_p & 0 \\ & \rho I_3 \\ 0 & I_3 \end{bmatrix}, \quad A_j(U) = \begin{bmatrix} \rho^{-1} \rho_p u_j & {}^t e_j & 0 \\ e_j & \rho u_j I_3 & {}^t D_j \\ 0 & D_j & u_j I_3 \end{bmatrix}, \\ D_1 &= \begin{bmatrix} 0 & 0 & 0 \\ H_2 + G_2 & -H_1 - G_1 & 0 \\ H_3 + G_3 & 0 & -H_1 - G_1 \end{bmatrix}, \quad D_2 = \begin{bmatrix} -H_2 - G_2 & H_1 + G_1 & 0 \\ 0 & 0 & 0 \\ 0 & H_3 + G_3 & -H_2 - G_2 \end{bmatrix}, \\ D_3 &= \begin{bmatrix} -H_3 - G_3 & 0 & H_1 + G_1 \\ 0 & -H_3 - G_3 & H_2 + G_2 \\ 0 & 0 & 0 \end{bmatrix}, \quad F(t, x) = \begin{bmatrix} 0 \\ -G \times \operatorname{curl} G \\ -\partial_t G \end{bmatrix}, \end{aligned}$$

where I_k is the $k \times k$ unit matrix, $e_1 = {}^t(1, 0, 0)$ e. t. c., and $B(t, x)$ is the matrix which consists of first order derivatives of G .

By definition, the boundary matrix $A_n(U)$ is given by

$$(3.8) \quad A_n(U) = \sum_{j=0}^3 n_j \cdot A_j(U) \\ = \begin{bmatrix} \rho^{-1} \rho_p u \cdot n & {}^t n & 0 \\ n & \rho u \cdot n I_3 & {}^t D_n \\ 0 & D_n & u \cdot n I_3 \end{bmatrix} \text{ on } \Gamma,$$

where $D_n = (- (H + G) \cdot n \delta_{ij} + n_i (H_j + G_j) | i \rightarrow 1, 2, 3, j \downarrow 1, 2, 3)$. Because of the boundary condition $u \cdot n = 0$, $A_n(U)$ is singular on Γ , namely the boundary is characteristic for (3.1). Thus we consider the modified equations of (3.1) with a positive parameter ε under the same initial boundary conditions (3.2) and (3.3) (cf. [10]):

$$(3.9)_\varepsilon \quad \begin{cases} (a) & \partial \rho_\varepsilon / \partial p (\partial_t + (u_\varepsilon \cdot \nabla)) p_\varepsilon + \rho_\varepsilon \operatorname{div} u_\varepsilon = 0, \\ (b) & \rho_\varepsilon (\partial_t + (u_\varepsilon \cdot \nabla)) u_\varepsilon + \nabla p_\varepsilon + (H_\varepsilon + G) \times \operatorname{curl} H_\varepsilon \\ & \quad + H_\varepsilon \times \operatorname{curl} G = -G \times \operatorname{curl} G, \\ (c)' & (\partial_t + (u_\varepsilon \cdot \nabla)) H_\varepsilon - ((H_\varepsilon + G) \cdot \nabla) u + (H_\varepsilon + G) \operatorname{div} u_\varepsilon \\ & \quad + (u_\varepsilon \cdot \nabla) G + \varepsilon (n \cdot \nabla) H_\varepsilon = -\partial_t G + \varepsilon (n \cdot \nabla) \hat{H}, \\ & \rho_\varepsilon = \rho(p_\varepsilon). \end{cases}$$

Here n is an extension in $C^\infty(\bar{\Omega})$ of normal vector n to $\bar{\Omega}$; and $\hat{H} \in H^m([0, T] \times \Omega)$ is an extension of $\partial_t^k H(0) \in H^{m-k}(\Omega)$, $k=0, 1, \dots, m-1$, to $[0, T] \times \Omega$ such that

$$(3.10) \quad \begin{cases} \partial_t^k \hat{H}(0) = \partial_t^k H(0) \text{ on } \Omega \text{ for } k=0, 1, \dots, m-1, \\ \|\hat{H}\|_{H^m([0, T] \times \Omega)} \leq c \|U_0\|_m, \end{cases}$$

where $\partial_t^k H(0)$ is recursively defined through the equations (3.1) and the data (3.2). (For the construction of this extension, see Theorem 2.5.7 of [5].)

REMARK 3.1. We note that the compatibility conditions (3.6) is still satisfied for $(3.9)_\varepsilon$.

Now the boundary matrix $A_n^\varepsilon(U)$ for the modified equations $(3.9)_\varepsilon$ is expressed by

$$A_n^\varepsilon(U) = \begin{bmatrix} \rho^{-1} \rho_p u \cdot n & {}^t n & 0 \\ n & \rho u \cdot n I_3 & {}^t D_n \\ 0 & D_n & (u \cdot n + \varepsilon) I_3 \end{bmatrix}.$$

We shall show that the $A_n^\varepsilon(U)$ has the following properties.

LEMMA 3.2. Let $U'(t, x) = (p'(t, x), u'(t, x), H'(t, x))$ satisfy that $u' \cdot n|_\Gamma = 0$, $H' \times n|_\Gamma = 0$ and $\inf_{x \in \Gamma} |H' \cdot n| \geq \tilde{c} > 0$ on $[0, T]$.

Then the following holds :

- (1) For $\varepsilon > 0$, $A_n^\varepsilon(U')$ is nonsingular on Γ .
 (2) For $\varepsilon \geq 0$ and fixed $(t, x) \in [0, T] \times \Gamma$, the linear subspace,
 $N = \{U = {}^t(p, u, H) | u \cdot n(x) = 0, H \times n(x) = 0\}$ of R^7 is maximally non-negative with respect to $A_n^\varepsilon(U'(t, x))$.

PROOF. Let $U = {}^t(p, u, H)$ satisfy $A_n^\varepsilon(U')U = 0$ on $[0, T] \times \Gamma$. Since

$$A_n^\varepsilon(U')U = \begin{bmatrix} n \cdot u \\ pn - (H' \cdot n)H + ((H' + G) \cdot H)n \\ -(H' \cdot n)u + (n \cdot u)(H' + G) + \varepsilon H \end{bmatrix}$$

by Lemma 3.1 (b), $G \cdot n = 0$ and the assumption $u' \cdot n = 0$, we get

$$\begin{cases} \text{(i)} & pn - (H' \cdot n)H + ((H' + G) \cdot H)n = 0, \\ \text{(ii)} & -(H' \cdot n)u + \varepsilon H = 0. \end{cases}$$

First taking inner product of (ii) with n yields $H \cdot n = 0$. Next taking inner product of (i) with n gives, by the assumption $H' \times n = 0$, we have $p = -G \cdot H$. The substitution of this into (i) yields $(H' \cdot n)H = 0$. From the assumption $\inf_{x \in \Gamma} |H' \cdot n| \geq \tilde{c}$ on $[0, T]$, it follows that $H = 0$. Hence using (i) and (ii) just proved, we can see $p = 0$ and $u = 0$. This completes the proof of (1).

Next we prove the statement of (2). If $U = {}^t(p, u, H) \in N$,

$$\begin{aligned} {}^tUA_n^\varepsilon(U')U &= 2[p(u \cdot n) + \{(H' + G) \cdot H\}(u \cdot n) - (H' \cdot n)(H \cdot u)] \\ &\quad + \varepsilon |H|^2 = \varepsilon |H|^2 \geq 0. \end{aligned}$$

Namely N is nonnegative with respect to $A_n^\varepsilon(U'(t, x))$ if $\varepsilon \geq 0$. To show "maximally" part, we consider the linear subspaces properly including N which are divided into the following cases: (α) For the space generated by N and the element such that $u \cdot n(x) \neq 0$ and $H \times n(x) = 0$, we can choose $U = (1, -n(x), n(x))$ in it, then ${}^tUA_n^\varepsilon(U')U = -2 < 0$, because $G \cdot n = 0$ from Lemma 3.1 (b). (β) For the space generated by N and the element such that $H \times n(x) \neq 0$ and $u \cdot n(x) = 0$, we can choose $U = (1, (\varepsilon + 1)\tau/2\tilde{c}^2, \tau/(H' \cdot n(x)))$ in it, where τ is a unit vector orthogonal to $n(x)$, then ${}^tUA_n^\varepsilon(U')U = -(\varepsilon + 1)/\tilde{c}^2 + \varepsilon/|H' \cdot n(x)|^2 < 0$, because the assumption $\inf_{x \in \Gamma} |H' \cdot n| \geq \tilde{c}$. (γ) For the space generated by N and the elements such that $u \cdot n(x) \neq 0$ and $H \times n(x) \neq 0$, we can choose $U = (G \cdot \tau, -(\varepsilon + 1)n(x)/G \cdot \tau, \tau)$ in it if $G \neq 0$, then ${}^tUA_n^\varepsilon(U')U = -3\varepsilon - 4 < 0$, because $H' \times n = 0$. If $G = 0$, we can choose $U = (\varepsilon - 1, n(x), \tau)$ in it, then ${}^tUA_n^\varepsilon(U')U = -\varepsilon - 2 < 0$. Hence if $\varepsilon \geq 0$, the null space of the boundary condition, N , is maximally nonnegative. The proof of Lemma 3.2 is completed.

Thus the added terms $\varepsilon(\mathbf{n} \cdot \nabla)H_\varepsilon$ in the left side of (3.9) $_\varepsilon(c)'$ make the boundary matrix nonsingular, keeping the boundary conditions maximally nonnegative.

§ 4. Iteration Scheme.

To define the iteration scheme for the initial boundary value problem (1.1)-(1.3), we first approximate the initial data $U_0 \in H^m(\Omega)$ by the functions $U_0^j \in H^{m+2}(\Omega)$ that satisfy the compatibility conditions of order $m+1$. Before constructing these approximations of U_0 , we take the mollifier, $G^k \in C^\infty([0, T] \times \bar{\Omega})$ of G for $k=0, 1, \dots$, so that there exists a positive constant $\gamma' \ll 1$ such that

$$\sup_{t \in [0, T]} \left(\sum_{j=0}^m \|\partial_t^j (G^k - G)(t)\|_{m-j+1} + \|\partial_t^{m+1} (G^k - G)(t)\|_1 \right) < \gamma' 2^{-k}$$

and $G^k \cdot \mathbf{n} = 0$ on $[0, T] \times \Gamma$ for $k=0, 1, \dots$. For this G^k , we construct the approximations of U_0 , $\{U_0^j\}_{j=0}^\infty$, as follows.

LEMMA 4.1. *There exists a sequence $\{U_0^j\}_{j=0}^\infty$ in $H^{m+2}(\Omega)$ such that each U_0^j satisfies the compatibility conditions of order $m+1$ in the following meanings :*

U_0^0 satisfies the compatibility conditions of order $m+1$ for the equations (3.1) in which G is replaced by G^0 and the boundary conditions (3.3). U_0^j ($j=1, 2, \dots$) satisfies the compatibility conditions of order $m+1$ for the equations $A_0(U^{j-1})\partial_t U + \sum_{i=1}^3 A_i(U^{j-1})\partial_i U + BU = F$ in which G is replaced by G^j and the boundary conditions (3.3), where U^{j-1} satisfies that $\partial_t^k U^{j-1}(0) = \partial_t^k U_0^{j-1}$ for $k=1, 2, \dots, m+1$ and $\partial_t^k U_0^{j-1}$ denotes k -th time derivatives of U_0^{j-1} which are determined in the preceding steps and the estimates hold :

$\|U_0 - U_0^j\|_m \leq \gamma 2^{-j}$ for $j=0, 1, \dots$, where γ is a positive constant.

PROOF. In Lemma 3.3 of [9], such approximations has been constructed for the linear equations with the nonsingular boundary matrix. The method given there can be extended to the present case by the same arguments as in pp. 52-53 in [10], if we can check the relation

Range $M = \text{Range } M(A_n(U_0))^k$ on Γ for $k=1, 2, \dots, m$, where M is the matrix giving the boundary conditions (3.3) :

$$M = \begin{bmatrix} 0 & {}^t n & 0 \\ 0 & 0 & n \times \end{bmatrix},$$

where

$$n \times = \begin{bmatrix} 0 & n_3 & -n_2 \\ -n_3 & 0 & n_1 \\ n_2 & -n_1 & 0 \end{bmatrix}.$$

Since $A_n(U_0)$ is a symmetric matrix, $\text{Range } (A_n(U_0))^k = \text{Range } A_n(U_0)$ for $k=1, 2, \dots$. So it is sufficient to show that

$$\text{Range } M = \text{Range } MA_n(U_0) \text{ on } \Gamma.$$

We easily know that $\text{Range } M = R^3$. Since $G \cdot n = 0$ on Γ , straightforward computations yield

$$MA_n(U) = \begin{bmatrix} |n| & 0 & h_n \\ 0 & H_n & 0 \end{bmatrix} \text{ on } \Gamma,$$

where $h_n = (-(H \cdot n)n_i + |n|^2(H_i + G_i)|i \rightarrow 1, 2, 3)$ and $H_n = (n_i((H + G) \times n)_j |i \rightarrow 1, 2, 3, j \downarrow 1, 2, 3) - (H \cdot n)n \times$. Therefore we can see from routine calculations $\text{Range } MA_n(U_0) = R^3$ on Γ , since $\inf_{x \in \Gamma} |H_0 \cdot n| \geq c_2$. Now the proof of Lemma 4.1 is completed.

Let us make some remarks which are necessary to construct an invariant subset for iteration scheme in § 5. At first, let R be an arbitrary small but fixed constant. Next, let $U = {}^t(p, u, H)$ be any function on $[0, T] \times \Omega$ satisfying

$$(4.1) \quad \sup_{x \in \Omega} |p_0 - p| < R, \quad \sup_{x \in \Omega} |H_0 - H| < R, \quad \sup_{x \in \Omega} |u_0 - u| < R, \\ \text{for fixed } t \in [0, T].$$

Then, from the assumptions (1.4), (3.5) and (3.6) together with Lemma 3.1 (b), there exist a positive constant k_0 , depending only on c_1 , and an open covering, $\{\Omega_i\}_{i=0}^N$, of Ω with $\Omega_0 \cap \Gamma = \emptyset$, $\Omega_i \cap \Gamma \neq \emptyset$ for $i=1, 2, \dots, N$, such that

$$(4.2) \quad \begin{aligned} (a) \quad & k_0 I_7 \leq A_0(U(t)) \leq k_0^{-1} I_7 \text{ in } \Omega, \\ (b) \quad & \inf_{x \in \Omega \cap \Omega_i} |(H + G)(t) \cdot n| \geq c_2/2 \quad \text{for } i=1, 2, \dots, N, \\ (c) \quad & \sup_{x \in \Omega \cap \Omega_i} |u(t) \cdot n| \leq \tilde{c} \quad \text{for } i=1, 2, \dots, N. \end{aligned}$$

Here n is the extended one and c_2 is the constant in (1.6), and \tilde{c} a positive constant which will be determined later. We also may take γ in Lemma 4.1 so that $\gamma = c^{-1}R/2$ for R in (4.1). Then the estimates for $\{U_0^j\}_{j=0}^\infty$ in Lemma 4.1 become

$$(4.3) \quad \|U_0 - U_0^j\|_m \leq c^{-1}R2^{-j-1} \text{ for } j=0, 1, \dots.$$

Furthermore from the estimates of G^k and (4.2)(b), we can see that for

any function H which satisfies (4.1) the following estimates hold :

$$(4.4) \quad \inf_{x \in \Omega \cap \Omega_i} |(H + G^k)(t) \cdot \mathbf{n}| \geq c_2/4 \text{ for } i=1, 2, \dots, N, \quad k=0, 1, \dots.$$

Hereafter $\hat{H}^j \in H^{m+2}([0, T] \times \Omega)$ denotes the function in $[0, T] \times \Omega$ which is constructed by the same manner as (3.10), where we use U_0^j instead of U_0 . Then we rewrite the modified equations $(3.9)_\varepsilon$ in the matrix form, which corresponds to (3.7) :

$$A_0(U_\varepsilon) \partial_t U_\varepsilon + \sum_{i=1}^3 A_{i,k}^\varepsilon(U_\varepsilon) \partial_i U_\varepsilon + B_k(t, x) U_\varepsilon = F_{\varepsilon,k}(t, x).$$

Here G is replaced by G^k and $U_\varepsilon = {}^t(p_\varepsilon, u_\varepsilon, H_\varepsilon)$. For the $A_{i,k}^\varepsilon(U_\varepsilon)$ and $B_k(t, x)$, we define for convenience the following differential operator $L_{\varepsilon,k}(U)$:

$$L_{\varepsilon,k}(U) = A_0(U) \partial_t + \sum_{i=1}^3 A_{i,k}^\varepsilon(U) \partial_i + B_k(t, x).$$

We also use the notation $F_{\varepsilon,k}^j(t, x)$ to denote the term $F_{\varepsilon,k}(t, x)$ with $\varepsilon(\mathbf{n} \cdot \nabla) \hat{H}$ replaced by $\varepsilon(\mathbf{n} \cdot \nabla) \hat{H}^j$. We remark that this $F_{\varepsilon,k}^j(t, x)$ belongs to $H^{m+1}([0, T] \times \Omega)$.

Next we construct the first approximating solution U^0 of the iteration scheme as follows.

LEMMA 4.2. *There exists $U^0 = {}^t(p^0, u^0, H^0) \in X_{m+1}(T)$ such that*

$$(4.5) \quad \begin{cases} u^0 \cdot \mathbf{n} = 0, \quad H^0 \times \mathbf{n} = 0 \text{ on } \Gamma \times [0, T], \\ \partial_t^k U^0(0) = \partial_t^k U_0^0 \text{ for } k=0, 1, \dots, m, \text{ where } \partial_t^k U_0^0 \text{ denotes } k\text{-th time} \\ \quad \text{derivative at } t=0 \text{ of a solution for the Cauchy problem} \\ \quad (3.1), \text{ in which } G \text{ is replaced by } G_0, \text{ with the initial} \\ \quad \text{data } U_0^0. \end{cases}$$

REMARK 4.1. It follows from (4.3) and (4.5) that there exists a positive constant T' so that for R in (4.1)

$$\sup_{x \in \Omega} |p_0 - p^0| < R, \quad \sup_{x \in \Omega} |H_0 - H^0| < R, \quad \sup_{x \in \Omega} |u_0 - u^0| < R, \quad \text{on } [0, T'].$$

PROOF OF LEMMA 4.2. Let ε be a positive constant, and U^0 a solution of the following linear mixed problem :

$$\begin{cases} L_{\varepsilon,0}(U_0^0) U^0 = K_\varepsilon(t, x) & \text{in } [0, T] \times \Omega, \\ U^0|_{t=0} = U_0^0 & \text{in } \Omega, \\ u^0 \cdot \mathbf{n} = 0, \quad H^0 \times \mathbf{n} = 0 & \text{on } [0, T] \times \Gamma. \end{cases}$$

Here K_ε is a function in $[0, T] \times \Omega$ which satisfies the following equality at $t=0$,

$$\begin{aligned}\partial_t^k K_\varepsilon(0) &= \partial_t^k F_{\varepsilon,0}^0(0) - [\partial_t^k, A_0(U^0)] \partial_t U^0|_{t=0} \\ &\quad - \sum_{j=1}^3 [\partial_t^k, A_{j,0}^\varepsilon(U^0)] \partial_j U^0|_{t=0} \text{ for } k=0, 1, \dots, m,\end{aligned}$$

where $[\cdot, \cdot]$ denotes commutator. The K_ε is constructed by the same manner as (3.10); and belongs to $H^{m+1}([0, T] \times \Omega)$ since the right side of (4.5) belongs at least to $H^{m+1-k-1/2}(\Omega)$. Since $U^0(t) = U_0^0$, it follows, by inductions, from the definition of $F_{\varepsilon,0}^0(0)$ and the above equality of K_ε at $t=0$ that $\partial_t^k U^0(0) = \partial_t^k U_0^0$ for $k=1, \dots, m$. Therefore, according to the results of the linear hyperbolic mixed problems, we can show from Lemma 3.2 and Lemma 4.1 together with (4.2)(a), (4.3) and the inequality (2.1) that there exists a unique solution of the mixed problem (4.4) belonging to $X_{m+1}(T)$. (c. f. [9] and Appendix A of [10].) The proof of Lemma 4.2 is completed.

Now we can define iteration scheme.

$$(4.6)_j \quad \begin{cases} L_{\varepsilon_j, j}(U_{\varepsilon_{j-1}}^{j-1}) U_{\varepsilon_j}^j = F_{\varepsilon_j, j}^j & \text{in } [0, T] \times \Omega, \\ U_{\varepsilon_j}^j|_{t=0} = U_0^j & \text{in } \Omega, \\ u_{\varepsilon_j}^j \cdot n = 0, H_{\varepsilon_j}^j \times n = 0 & \text{on } [0, T] \times \Gamma, \end{cases}$$

for $j=1, 2, \dots$.

Here $U_{\varepsilon_j}^j = {}^t(p_{\varepsilon_j}^j, u_{\varepsilon_j}^j, H_{\varepsilon_j}^j)$, $U_{\varepsilon_0}^0 = U^0$ in Lemma 4.2 and $\varepsilon_j = 2^{-j} \theta(j)^{-1}$, where $\theta(j)$ is a positive monotone increasing function, tending to infinity as $j \rightarrow \infty$, such that $\|U_0^j\|_{m+2} \leq \theta(j) \|U_0\|_m$.

REMARK 4.2. We can show from the construction of \hat{H}^j and the definition of $\theta(j)$ that

$$(4.7) \quad \varepsilon_j \|(\mathbf{n} \cdot \nabla) \hat{H}^j\|_{H^{m+1}([0, T] \times \Omega)} \leq \varepsilon_j c \|U_0^j\|_{m+2} \leq c 2^{-j} \|U_0\|_m.$$

§ 5. Proof of Theorem.

To construct invariant set for iteration scheme, we define for positive constants \tilde{T} and M

$S(\tilde{T}, M) = \{U = {}^t(p, u, H) \mid U \in X_{m+1}(\tilde{T}), \|U\|_{X_m(\tilde{T})} \leq M, u \cdot n = 0, H \times n = 0 \text{ on } [0, \tilde{T}] \times \Gamma, U \text{ satisfies (4.1) for all } t \in [0, \tilde{T}]\}$.

We now state the main Lemma.

LEMMA 5.1. *There exist positive constants T_* and M_* depending only on $\|U_0\|_m$, k_0 , c_2 , $\|g\|_{Y_m(T)}$, m and Ω such that if there exist a solution $U_{\varepsilon_k}^k$ of (4.6)_k for $k=0, \dots, j-1$ which belongs to $S(T_*, M_*)$ then there exists a unique solution $U_{\varepsilon_j}^j$ of (4.6)_j which again belongs to $S(T_*, M_*)$.*

PROOF. Suppose that $U_{\varepsilon_{j-1}}^{j-1} \in S(T', M)$, where T' is the positive constant in REMARK 4.1 and M is a positive constant such that $\|U^0\|_{X_m(T')} \leq M$.

Then, it is easy to see from Lemma 3.2, Lemma 4.1, Lemma 4.2 and (4.2) (a) for $U_{\varepsilon_{j-1}}^{j-1}$ on $[0, T']$ that there exists a unique local in time solution $U_{\varepsilon_j}^j$ for the linear equations (4.6)_j belonging to $X_{m+1}(T')$.

Therefore we have only to show the following key a priori estimate for $U_{\varepsilon_j}^j$:

$$(5.1) \quad \|U_{\varepsilon_j}^j(t)\|_m \leq (k_a + k_b t) \exp(k_c t) \quad \text{for } 0 \leq t \leq \min(T', T_a, 1),$$

where k_a , k_b and k_c are positive constants independent of j . The constant k_a depends only on $\|U_0\|_m$, k_0 , c_2 , $\|g\|_{Y_m(T)}$, m and Ω . Moreover, k_b and k_c depend only on M , $\|U_0\|_m$, $\|g\|_{Y_m(T)}$, k_0 , c_2 , m and Ω . Furthermore, T_a is a positive constant depending only on M , $\|U_0\|_m$, $\|g\|_{Y_m(T)}$, k_0 , c_2 , ε_0 , m and Ω .

Then, taking $M_* = \max(6k_a, M)$ and $T_1 = \min(k_b^{-1}(M_*)k_a, k_c^{-1}(M_*), T_a(M_*), T', 1)$, where $k_b(M_*)$, $k_c(M_*)$, $T_a(M_*)$ denotes k_b , k_c and T_a with M replaced by M_* , we can see from the estimate (5.1) that

$$\|U_{\varepsilon_j}^j\|_{X_m(T_1)} \leq M_*.$$

Therefore this estimate together with (2.1) leads that there exists a positive constant T_* independent of j such that

$$\begin{aligned} \sup_{x \in \Omega} |p_0^j - p_{\varepsilon_j}^j| &< R/2, \quad \sup_{x \in \Omega} |H_0^j - H_{\varepsilon_j}^j| < R/2, \\ \sup_{x \in \Omega} |u_0^j - u_{\varepsilon_j}^j| &< R/2 \quad \text{on } [0, T_*], \end{aligned}$$

because for example $p_{\varepsilon_j}^j - p_0^j = \int_0^t \partial_\tau p_{\varepsilon_j}^j(\tau) d\tau$ and $\sup_{t \in [0, T_*]} \|\partial_t p_{\varepsilon_j}^j(t)\|_{m-1} \leq M_*$.

Accordingly, from (4.3) and (2.1) we see that

$$(5.2) \quad \begin{aligned} \sup_{x \in \Omega} |p_0 - p_{\varepsilon_j}^j| &< R, \quad \sup_{x \in \Omega} |H_0 - H_{\varepsilon_j}^j| < R, \\ \sup_{x \in \Omega} |u_0 - u_{\varepsilon_j}^j| &< R \quad \text{on } [0, T_*] \end{aligned}$$

and $U_{\varepsilon_j}^j \in S(T_*, M_*)$.

Now we begin to establish the estimate (5.1). For simplicity we use such abbreviations as U^j for $U_{\varepsilon_j}^j$ and use $k_i(M, k_0, \dots)$ ($i=1, 2, \dots$) as a positive constant independent of j which depends on M, k_0, \dots .

At first, we give the estimate of $\operatorname{div} H^j$ which is uniform with respect to j by making use of a special structure of (3.9)_ε.

Note that the equations of H^j corresponding to (3.9)_ε(c)' are

$$\begin{aligned} (\partial_t + (u^{j-1} \cdot \nabla)) H^j - ((H^{j-1} + G^j) \cdot \nabla) u^j + (H^{j-1} + G^j) \operatorname{div} u^j \\ + (u^{j-1} \cdot \nabla) G^j + \varepsilon_j (\mathbf{n} \cdot \nabla) H^j = \varepsilon_j (\mathbf{n} \cdot \nabla) \hat{H}^j - \partial_t G^j. \end{aligned}$$

Accordingly, it follows from a well-known formula from vector analysis that

$$\begin{aligned} & (\partial_t + (u^{j-1} \cdot \nabla)) H^j - (u^j \cdot \nabla)(H^{j-1} + G^j) + u^j \operatorname{div} (H^{j-1} + G^j) \\ & - \operatorname{curl}(u^j \times (H^{j-1} + G^j)) + (u^{j-1} \cdot \nabla) G^j + \varepsilon_j (\mathbf{n} \cdot \nabla) H^j \\ & = \varepsilon_j (\mathbf{n} \cdot \nabla) \hat{H}^j - \partial_t G^j. \end{aligned}$$

Therefore taking the divergence of these equations and noting that $\operatorname{div} \operatorname{curl} = 0$, we get

$$(5.3) \quad \begin{aligned} & \partial_t \operatorname{div} H^j + (u^{j-1} \cdot \nabla) \operatorname{div} H^j + \varepsilon_j (\mathbf{n} \cdot \nabla) \operatorname{div} H^j + l. o. t. \\ & = \operatorname{div}(\varepsilon_j (\mathbf{n} \cdot \nabla) \hat{H}^j - \partial_t G^j), \end{aligned}$$

where *l. o. t.* means the lower order terms of H^j and u^j . We remark that their coefficients contain at most first order derivatives of U^{j-1} and second order derivatives of G^j because the terms which contain second order derivatives of H^{j-1} are canceled each other. Operating ∂^α on (5.3) (this ∂^α contains both time and space derivatives), multiplying (5.3) by $2\partial^\alpha \operatorname{div} H^j$, summing over all α with $|\alpha| \leq m-1$, and applying integration by parts and inequalities (2.1), (2.2), we obtain by Lemma 3.1(c)

$$(5.4) \quad \begin{aligned} & \partial_t \|\operatorname{div} H^j(t)\|_{m-1}^2 \\ & \leq (k_1(M, \|g\|_{Y_m(T)}, m, \Omega)) \|U^j(t)\|_m \\ & + (k_2(\|g\|_{Y_m(T)}, m, \Omega) + \varepsilon_j \|\operatorname{div}((\mathbf{n} \cdot \nabla) \hat{H}^j(t))\|_{m-1}) \|\operatorname{div} H^j(t)\|_{m-1} \\ & - \varepsilon_j \sum_{|\alpha| \leq m-1} \int_{\Gamma} |\partial^\alpha \operatorname{div} H^j|^2 d\Gamma, \end{aligned}$$

since $u^{j-1} \cdot \mathbf{n} = 0$ on Γ . Therefore we get the following estimate of $\operatorname{div} H^j$ which is crucial in this proof:

$$(5.5) \quad \begin{aligned} & \|\operatorname{div} H^j(t)\|_{m-1} \leq k_3(\|U_0\|_m, m, \Omega) + k_2 t \\ & + \int_0^t (k_1 \|U^j(\tau)\|_m + \varepsilon_j \|\operatorname{div}((\mathbf{n} \cdot \nabla) \hat{H}^j(\tau))\|_{m-1}) d\tau \quad \text{for } 0 \leq t \leq T'. \end{aligned}$$

Because of (4.7), this estimate is uniform with respect to j .

Next, let us take a partition of unity, $\{\phi_i\}_{i=0}^N$, with respect to the open covering $\{\Omega_i\}_{i=0}^N$ in (4.2)(b) such that $\sum_{i=0}^N \phi_i^2(x) = 1$ in a neighborhood of Ω and each ϕ_i is C^∞ -function with compact support in Ω_i . Then we may suppose that $\Gamma \cap \Omega_i$ ($i \neq 0$) is given by the smooth function ψ_i such that

$$\begin{aligned} \Gamma \cap \Omega_i &= \{(x_1, x_2, x_3) | x_3 = \psi_i(x_1, x_2)\}, \\ \Omega \cap \Omega_i &= \{(x_1, x_2, x_3) | x_3 = \psi_i(x_1, x_2)\}. \end{aligned}$$

For this ψ_i , we define the change of independent variables in Ω_i :

$$(5.6) \quad \begin{cases} y_1 = x_1, \\ y_2 = x_2, \\ y_3 = x_3 - \psi_i(x_1, x_2). \end{cases}$$

Then we see easily that there exists a positive smooth function $\kappa_i(x)$ in Ω_i ($i=1, 2, \dots, N$) such that

$$\kappa_i(x)\mathbf{n}(x) = (-\partial_1\psi_i, -\partial_2\psi_i, 1) \text{ on } \Omega \cap \Omega_i,$$

and $\Gamma \cap \Omega_i$ is mapped to the subset of $\{(y_1, y_2, y_3) | y_3=0\}$. Moreover we change the dependent variables $\phi_i U^j$ by an orthogonal matrix valued transform $T_i(x)$ which makes the boundary operator M constant on Γ (c.f. Proposition 3.1 of [6]), and rewrite for simplicity this new variables $T_i \phi_i U^j$ by \tilde{U}^j in Ω_i . Then note that this \tilde{U}^j satisfies the following equations in Ω_i :

$$(5.7) \quad \begin{aligned} & \tilde{A}_0(U^{j-1})\partial_t \tilde{U}^j + \tilde{A}_{1,j}^{\varepsilon_j}(U^{j-1})\partial_{y_1} \tilde{U}^j + \tilde{A}_{2,j}^{\varepsilon_j}(U^{j-1})\partial_{y_2} \tilde{U}^j \\ & + \kappa_i \tilde{A}_{n,j}^{\varepsilon_j}(U^{j-1})\partial_{y_3} \tilde{U}^j + \tilde{B}_j \tilde{U}^j - \sum_{k=1}^2 \tilde{A}_{k,j}^{\varepsilon_j}(U^{j-1})\partial_{y_k} \{\phi_i T_i\} U^j \\ & - \kappa_i \tilde{A}_{n,j}^{\varepsilon_j}(U^{j-1})\partial_{y_3} \{\phi_i T_i\} U^j = \tilde{F}_{\varepsilon_j,j}^j, \end{aligned}$$

where $\partial_{y_i} = \partial/\partial y_i$, $\tilde{A}_{k,j}^{\varepsilon_j}(U^{j-1}) = T_i \tilde{A}_{k,j}^{\varepsilon_j}(U^{j-1})^t T_i$ for $k=0, 1, 2$, \mathbf{n} , $\tilde{B}_j = T_i B_j^t T_i$, $\tilde{F}_{\varepsilon_j,j}^j = T_i \phi_i F_{\varepsilon_j,j}^j$. Applying tangential derivative operator ∂_{\tan}^α with $|\alpha| \leq m$ to the equations (5.7), we get

$$(5.8) \quad \begin{aligned} & \tilde{A}_0(U^{j-1})\partial_t \tilde{U}_{\tan}^{j,\alpha} + \tilde{A}_{1,j}^{\varepsilon_j}(U^{j-1})\partial_{y_1} \tilde{U}_{\tan}^{j,\alpha} + \tilde{A}_{2,j}^{\varepsilon_j}(U^{j-1})\partial_{y_2} \tilde{U}_{\tan}^{j,\alpha} \\ & + \kappa_i \tilde{A}_{n,j}^{\varepsilon_j}(U^{j-1})\partial_{y_3} \tilde{U}_{\tan}^{j,\alpha} + \tilde{B}_j \tilde{U}_{\tan}^{j,\alpha} + l. o. t. = \partial_{\tan}^\alpha \tilde{F}_{\varepsilon_j,j}^j, \end{aligned}$$

where $\tilde{U}_{\tan}^{j,\alpha} = \partial_{\tan}^\alpha \tilde{U}^j$, $\partial_{\tan}^\alpha = \partial_t^{\alpha_0} \partial_{y_1}^{\alpha_1} \partial_{y_2}^{\alpha_2}$ for $\alpha = (\alpha_0, \alpha_1, \alpha_2)$, $|\alpha| = \alpha_0 + \alpha_1 + \alpha_2$. And here $l. o. t.$ means the lower order terms of U^j with their coefficients which contain at most m -th order derivatives of U^{j-1} and $(m+1)$ -th order derivatives of G^j . Multiplying (5.8) by $\tilde{U}_{\tan}^{j,\alpha}$ and integrating over x lead to

$$\begin{aligned} & \partial_t (\tilde{U}_{\tan}^{j,\alpha}, \tilde{A}_0(U^{j-1}) \tilde{U}_{\tan}^{j,\alpha})_0 \\ & = - \int_{y_3=0} \tilde{U}_{\tan}^{j,\alpha} \kappa_i \tilde{A}_{n,j}^{\varepsilon_j}(U^{j-1}) \tilde{U}_{\tan}^{j,\alpha} d\Gamma + (\tilde{U}_{\tan}^{j,\alpha}, \{\partial_t \tilde{A}_0(U^{j-1}) \\ & + \partial_{y_1} \tilde{A}_{1,j}^{\varepsilon_j}(U^{j-1}) + \partial_{y_2} \tilde{A}_{2,j}^{\varepsilon_j}(U^{j-1}) + \partial_{y_3} (\kappa_i \tilde{A}_{n,j}^{\varepsilon_j}(U^{j-1})) \\ & + 2\tilde{B}_j\} \tilde{U}_{\tan}^{j,\alpha})_0 + (\tilde{U}_{\tan}^{j,\alpha}, l. o. t.)_0 + 2(\tilde{U}_{\tan}^{j,\alpha}, \partial_{\tan}^\alpha \tilde{F}_{\varepsilon_j,j}^j)_0. \end{aligned}$$

Here we note that $\tilde{U}_{\tan}^{j,\alpha} \kappa_i \tilde{A}_{n,j}^{\varepsilon_j}(U^{j-1}) \tilde{U}_{\tan}^{j,\alpha} \geq 0$ on $y_3=0$, because the null space of the boundary conditions is nonnegative with respect to $\tilde{A}_{n,j}^{\varepsilon_j}(U^{j-1})$ (c.f. Lemma 3.2). We also note that $M \partial_{\tan}^\alpha \tilde{U}^j = \partial_{\tan}^\alpha M \tilde{U}^j = 0$ on $y_3=0$ since the transformed M is constant. Summing over all α with $|\alpha| \leq m$, applying the inequalities (2.1) and (2.2), we can get by Lemma 3.1 (c)

$$(5.9) \quad \begin{aligned} & \partial_t \sum_{|\alpha| \leq m} (\tilde{A}_0(U^{j-1}) \tilde{U}_{\tan}^{j,\alpha}, \tilde{U}_{\tan}^{j,\alpha})_0 \\ & \leq (k_4(\|g\|_{Y_m(T)}, m, \Omega) + \varepsilon_j \|T_i \phi_i(\mathbf{n} \cdot \nabla) \hat{H}^j(t)\|_{m, \tan} \\ & + k_5(M, \|g\|_{Y_m(T)}, m, \Omega) \|U^j(t)\|_m) \|\tilde{U}^j(t)\|_{m, \tan}, \end{aligned}$$

where $\|\tilde{U}^j(t)\|_{m, \tan}^2 = \sum_{|\alpha| \leq m} \|\partial_{\tan}^\alpha \tilde{U}^j(t)\|_0^2$. Therefore (4.2)(a) yields

$$(5.10) \quad \begin{aligned} & \|\tilde{U}^j(t)\|_{m, \tan} \\ & \leq k_6(\|U_0\|_m, m, \Omega) + k_0^{-1}k_4t \\ & + k_0^{-1} \int_0^t (k_5\|U^j(\tau)\|_m + \varepsilon_j\|T_i\phi_i(\mathbf{n} \cdot \nabla)\hat{H}^j(\tau)\|_m) d\tau \quad \text{for } 0 \leq t \leq T'. \end{aligned}$$

Here and hereafter we use \tilde{U}^j again to represent $\phi_i U^j$ instead of $T_i \phi_i U^j$.

Next we shall give the estimates of normal derivatives of \tilde{U}^j which are independent of j . For this purpose we should note that the rank of the boundary matrix $\tilde{A}_{n,j}^{\varepsilon_j}(U^{j-1})$ is 6 on Γ because of (4.4) for $H_{\varepsilon_{j-1}}^{j-1}$. Therefore we can see the estimate for the normal derivatives of \tilde{U}^j by returning to the equations (5.7) and combining the uniform estimates (5.5) and (5.10). To see this procedure precisely, we first note that divergent operator can be locally written as

$$\begin{aligned} \operatorname{div} \tilde{H}^j &= \partial_{y_1} \tilde{H}_1^j + \partial_{y_2} \tilde{H}_2^j + \partial_{y_3} \tilde{H}_3^j - \partial_1 \psi_i \partial_{y_3} \tilde{H}_1^j \\ &\quad - \partial_2 \psi_i \partial_{y_3} \tilde{H}_2^j \quad \text{in } \Omega \cap \Omega_i. \end{aligned}$$

Hence we get the identity :

$$(5.11) \quad \begin{aligned} & \kappa_i(\mathbf{n}_1 \partial_{y_3} \tilde{H}_1^j + \mathbf{n}_2 \partial_{y_3} \tilde{H}_2^j + \mathbf{n}_3 \partial_{y_3} \tilde{H}_3^j) \\ &= \operatorname{div} \tilde{H}^j - \partial_{y_1} \tilde{H}_1^j - \partial_{y_2} \tilde{H}_2^j \quad \text{in } \Omega \cap \Omega_i. \end{aligned}$$

Moreover, the equations for \tilde{U}^j are written as

$$(5.12) \quad \begin{aligned} \kappa_i \tilde{A}_{n,j}^{\varepsilon_j}(U^{j-1}) \partial_{y_3} \tilde{U}^j &= -\{A_0(U^{j-1}) \partial_t \tilde{U}^j + \sum_{k=1}^2 \tilde{A}_{k,j}^{\varepsilon_j}(U^{j-1}) \partial_{y_k} \tilde{U}^j\} \\ &+ \sum_{k=1}^2 \tilde{A}_{k,j}^{\varepsilon_j}(U^{j-1}) \partial_{y_k} \{\phi_i\} U^j + \kappa_i \tilde{A}_{n,j}^{\varepsilon_j}(U^{j-1}) \partial_{y_3} \{\phi_i\} U^j + \tilde{F}_{\varepsilon_j,j}^j \quad \text{in } \Omega \cap \Omega_i. \end{aligned}$$

Then the equations of the second, the third, the fourth lines of (5.12) and the identity (5.11) can be written as

$$(5.13) \quad \begin{aligned} & \Xi_a \partial_{y_3}^t (\tilde{p}^j, \tilde{H}_1^j, \tilde{H}_2^j, \tilde{H}_3^j) \\ &= (4\text{-dimensional vector valued functions consisting of the right} \\ & \text{side of (5.11) and corresponding lines of (5.1) and } \rho(p^{j-1})(u^{j-1} \cdot \\ & \mathbf{n}) \partial_{y_3} u_i^j \text{ for } i=1, 2, 3). \end{aligned}$$

Here

$$\Xi_a = \kappa_i \begin{bmatrix} \mathbf{n} & D_n^{j-1} \\ 0 & {}_t \mathbf{n} \end{bmatrix},$$

where, $D_n^{j-1} = (-(H^{j-1} + G^j) \cdot \mathbf{n} \delta_{l,k} + \mathbf{n}_k (H_l^{j-1} + G_l^j) | l \rightarrow 1, 2, 3, k \downarrow 1, 2, 3)$. Then from (4.4) and the assumption $\sup_{x \in \Omega} |H^{j-1} - H_0| < R$, it follows that

$\inf_{x \in \Omega \cap \Omega_i} |(H^{j-1} + G^j) \cdot \mathbf{n}| \geq c_2/4$ on $[0, T']$, which implies

$$\det \Xi_\alpha = \kappa_i^4 ((H^{j-1} + G^j) \cdot \mathbf{n})^2 \geq \kappa_i^4 c_2^2/16$$

in $[0, T'] \times \Omega \cap \Omega_i$.

Next note that the equations of the first, the fifth, the sixth line of (5.12) can be written as

$$(5.14) \quad \begin{aligned} & \Xi_\beta \partial_{y_3}^t (\tilde{u}_1^j, \tilde{u}_2^j, \tilde{u}_3^j) \\ & = (3\text{-dimensional vector valued functions consisting of the right} \\ & \text{side of corresponding lines of (5.12),} \\ & \varepsilon_j \partial_{y_3} \tilde{H}_i^j + (u^{j-1} \cdot \mathbf{n}) \partial_{y_3} \tilde{H}_i^j \text{ for } i=1, 2 \text{ and} \\ & \rho^{-1}(p^{j-1}) \rho_p(p^{j-1}) (u^{j-1} \cdot \mathbf{n}) \partial_{y_3} p^j). \end{aligned}$$

Here

$$\Xi_\beta = \kappa_i \begin{bmatrix} \mathbf{n}_1 & \mathbf{n}_2 & \mathbf{n}_3 \\ -(H^{j-1} + G) \cdot \mathbf{n} + (H_1^{j-1} + G_1) \mathbf{n}_1 & (H_1^{j-1} + G_1) \mathbf{n}_2 & (H_1^{j-1} + G_1) \mathbf{n}_3 \\ (H_2^{j-1} + G_2) \mathbf{n}_1 & -(H^{j-1} + G) \cdot \mathbf{n} + (H_2^{j-1} + G_2) \mathbf{n}_2 & (H_2^{j-1} + G_2) \mathbf{n}_3 \end{bmatrix},$$

where $G = G^j$. Then $\inf_{x \in \Omega \cap \Omega_i} |(H^{j-1} + G^j) \cdot \mathbf{n}| \geq c_2/4$ on $[0, T']$ again yields

$$\det \Xi_\beta = \kappa_i^2 ((H^{j-1} + G^j) \cdot \mathbf{n})^2 \geq \kappa_i^2 c_2^2/16 \text{ in } [0, T'] \times \Omega \cap \Omega_i.$$

Then, taking \tilde{c} in (4.2)(c) small enough for given $\|U_0\|_m$, $\|g\|_{Y_m(T)}$, c_2^{-1} , and ε_0 , taking a positive constant T_a small enough for this \tilde{c} and M , and then applying the mean value theorem and the inequalities (2.1) and (2.2), we can show from (5.13) and (5.14) that

$$(5.15) \quad \begin{aligned} & \|\tilde{U}^j(t)\|_m \\ & \leq k_7(c_2^{-1}, \|U^{j-1}(t)\|_{m-1}, \|g\|_{Y_m(T)}, m, \Omega) \\ & \times (\|\operatorname{div} H^j(t)\|_{m-1} + \|\tilde{U}^j(t)\|_{m, \tan} + \|U^j(t)\|_{m-1}) \\ & \leq k_7 \times (\|\operatorname{div} H^j(t)\|_{m-1} + \|\tilde{U}^j(t)\|_{m, \tan} \\ & + k_8(\|U_0\|_m, m, \Omega) + \int_0^t \|U^j(\tau)\|_m d\tau) \text{ for } 0 \leq t \leq \min(T', T_a). \end{aligned}$$

Here the constant k_7 depends smoothly on $\|U^{j-1}(t)\|_{m-1}$. Therefore we get

$$\begin{aligned} & k_7(c_2^{-1}, \|U^{j-1}(t)\|_{m-1}, \|g\|_{Y_m(T)}, m, \Omega) \\ & \leq k_9(c_2^{-1}, \|U_0\|_m, \|g\|_{Y_m(T)}, m, \Omega) + k_{10}(c_2^{-1}, M, \|g\|_{Y_m(T)}, m, \Omega)t \end{aligned}$$

On the other hand, in the same arguments as in the case of Cauchy problem, we can show from (4.7)

$$(5.16) \quad \|\phi_0 U^j(t)\|_m \leq k_{11} + k_{12}t + k_{13} \int_0^t \|U^j(\tau)\|_m d\tau$$

for $0 \leq t \leq \min(T', 1)$,

where $k_{11} = k_{11}(\|U_0\|_m, m, \Omega)$, $k_{12} = k_{12}(M, k_0^{-1}, \|U_0\|_m, \|g\|_{Y_m(T)}, m, \Omega)$ and $k_{13} = k_{13}(M, k_0^{-1}, \|g\|_{Y_m(T)}, m, \Omega)$.

Accordingly, since

$$\|U^j(t)\|_m^2 \leq \|\phi_0 U^j(t)\|_m^2 + \sum_{i=1}^N \|T_i \phi_i U^j(t)\|_m^2,$$

it follows from (4.7), (5.5), (5.10), (5.15), and Gronwall's inequality that

$$(5.17) \quad \|U^j(t)\|_m \leq (k_{14} + k_{15}t) \exp(k_{16}t) \quad \text{for } 0 \leq t \leq \min(T', T_a, 1),$$

where $k_{14} = k_{14}(c_2^{-1}, \|U_0\|_m, \|g\|_{Y_m(T)}, m, \Omega)$, $k_{15} = k_{15}(M, k_0^{-1}, c_2^{-1}, \|U_0\|_m, \|g\|_{Y_m(T)}, m, \Omega)$ and $k_{16} = k_{16}(M, k_0^{-1}, c_2^{-1}, \|g\|_{Y_m(T)}, m, \Omega)$. Therefore the proof of Lemma 5.1 is completed.

Next we show that the sequence $\{U_{\varepsilon_j}^j\}_{j=0}^\infty$ converges strongly in $X_0(T)$ for a sufficiently small constant T .

LEMMA 5.2. *There exists a positive constant T_0 such that*

$$(5.18) \quad \sup_{t \in [0, T_0]} \sum_{j=1}^\infty \|U_{\varepsilon_j}^j(t) - U_{\varepsilon_{j-1}}^{j-1}(t)\|_0 < \infty.$$

PROOF. For simplicity, we also use U^j for $U_{\varepsilon_j}^j$ in this proof. Direct computations yields the equations for $U^j - U^{j-1}$:

$$\begin{aligned} & A_0(U^{j-1}) \partial_t (U^j - U^{j-1}) + \sum_{i=1}^3 A_{i,j}^{\varepsilon_j}(U^{j-1}) \partial_i (U^j - U^{j-1}) \\ & + B_j(U^j - U^{j-1}) - (F_{\varepsilon_j,j}^j - F_{\varepsilon_{j-1},j-1}^{j-1}) = -A_0(U^{j-1}) \sum_{i=1}^3 (\tilde{A}_{i,j}^{\varepsilon_j}(U^{j-1}) - \\ & \tilde{A}_{i,j-1}^{\varepsilon_{j-1}}(U^{j-2})) \partial_i U^{j-1} - A_0(U^{j-1}) (A_0^{-1}(U^{j-1}) B_j - A_0^{-1}(U^{j-2}) B_{j-1}) U^{j-1} \\ & + A_0(U^{j-1}) F_{\varepsilon_{j-1},j-1}^{j-1} (A_0^{-1}(U^{j-1}) - A_0^{-1}(U^{j-2})), \end{aligned}$$

where $\tilde{A}_{i,k}^{\varepsilon_k}(U^{k-1}) = A_0^{-1}(U^{k-1}) A_{i,k}^{\varepsilon_k}(U^{k-1})$. Noting that the estimates (4.7) and

$$\|U^{j-1}\|_{X_m(T_*)} \leq M, \quad \|U^{j-2}\|_{X_m(T_*)} \leq M, \quad k_0 I_7 \leq A_0(U^{j-1}) \leq k_0^{-1} I_7$$

in $[0, T_*] \times \Omega$, we conclude by standard energy estimates for symmetric hyperbolic systems and the mean value theorem that

$$\begin{aligned} & \sup_{\tau \in [0, T_0]} \|U^j(\tau) - U^{j-1}(\tau)\|_0 \leq (\|U_0^j - U_0^{j-1}\|_0 \\ & + c'(2^{-j} + \sup_{\tau \in [0, T_0]} \|U^{j-1}(\tau) - U^{j-2}(\tau)\|_0) T_0) \exp c'' T_0 \end{aligned}$$

for $T_0 \leq T_*$, where c' and c'' are positive constants depending only on M , $\|U_0\|_m$, $\|g\|_{Y_m(T)}$, k_0 , m , and Ω . Therefore taking T_0 small enough, we have, by (4.3)

$$\sup_{\tau \in [0, T_0]} \|U^j(\tau) - U^{j-1}(\tau)\|_0 \leq \alpha \sup_{\tau \in [0, T_0]} \|U^{j-1}(\tau) - U^{j-2}(\tau)\|_0 + \beta_j,$$

where $0 < \alpha < 1$ and $\{\beta_j\}_{j=2}^\infty$ with $\sum_{j=2}^\infty |\beta_j| < \infty$. This estimate immediately yields (5.18).

By Lemma 5.1 and Lemma 5.2, together with the sobolev space interpolation inequalities (c.f. pp. 39 of [8]), we can conclude that there exists a function U such that for any positive constant δ the sequence $\{U_{\varepsilon_j}^j\}_{j=0}^\infty$ converges to U strongly in $H^{m-\delta}(\Omega)$ uniformly on $[0, T_0]$ and the sequence $\{\partial_t U_{\varepsilon_j}^j\}_{j=0}^\infty$ converges to $\partial_t U$ strongly in $H^{m-1-\delta}(\Omega)$ uniformly on $[0, T_0]$. Accordingly, we get

$$(5.19) \quad U \in C^0(0, T_0; H^{m-\delta}(\Omega)) \cap C^1(0, T_0; H^{m-1-\delta}(\Omega)).$$

Because of $m \geq 3$, this U is a classical solution of the initial boundary value problem (1.1)-(1.3).

The uniqueness of the solution of (1.1)-(1.3) follows directly from the next a priori estimate.

LEMMA 5.3. Assume that U and $U' = {}^t(p', u', H')$ are solutions of (1.1) belonging to $X_m(T)$ ($m \geq 3$) which satisfy only the solid boundary condition $u \cdot n = 0$ on $[0, T] \times \Gamma$.

Then the following a priori estimate holds:

$$(5.20) \quad \|U(t) - U'(t)\|_0^2 \leq c_3 \exp(c_4 t) \{ \|U(0) - U'(0)\|_0^2 \\ + \int_0^t \int_\Gamma |(H(\tau) \cdot n) \{ (H(\tau) - H'(\tau)) \cdot (u(\tau) - u'(\tau)) \}| d\Gamma d\tau \}$$

for $0 \leq t \leq T$, where c_3, c_4 are positive constants which depend on $|\rho|_{L^\infty}$ and $|\nabla U|_{L^\infty}, |\nabla U'|_{L^\infty}$, respectively. Here $|\cdot|_{L^\infty}$ denotes essential supremum norm on $[0, T] \times \Omega$.

PROOF. Direct computations yield the equations for $U - U'$:

$$A_0(U) \partial_t (U - U') + \sum_{j=1}^3 A_j(U) \partial_j (U - U') = (A_0(U)) \partial_t U' \\ + \sum_{j=1}^3 (A_j(U') - A_j(U)) \partial_j U',$$

where $A_0(U), A_j(U)$ are the matrices of (3.7) with $G=0$.

Note that under the condition $u \cdot n = 0$ on Γ

$$\int_{\Gamma}^t (U - U') A_n(U) (U - U') d\Gamma = 2 \int_{\Gamma} H \cdot n \{ (H - H') \cdot (u - u') \} d\Gamma$$

(c. f. proof of Lemma 3.2). Then (5.20) follows from the standard method for symmetric hyperbolic systems.

Finally we shall remove δ in (5.19). Since $\{U_{\varepsilon_j}^j\}_{j=0}^{\infty}$ converges to U strongly in $H^{m-\delta}(\Omega)$ uniformly on $[0, T_0]$ and $\|U_{\varepsilon_j}^j\|_{X_m(T_0)} \leq M$ for $j=0, 1, \dots$ and since $(H^{m-\delta}(\Omega))'$ is dense in $(H^m(\Omega))'$, (the dual space of $H^m(\Omega)$; c. f. pp. 84 of [7]), we can show by an elementary argument that

$$(5.21) \quad U \in C_w(0, T_0; H^m(\Omega)),$$

where C_w means continuity on the interval $[0, T_0]$ with values in the weak topology of $H^m(\Omega)$ (c. f. pp. 40 of [8]).

Next, we obtain from (1.1)(c), (1.5) and $\operatorname{div} \operatorname{curl} = 0$ that

$$\operatorname{div} H(t) = 0 \quad \text{for } 0 \leq t \leq T_0.$$

Furthermore, applying the tangential molifier for the principal parts of the equations corresponding to (5.7), carrying out the similar estimates to (5.9) for $\phi_i T_i U$ ($i=1, \dots, N$), we obtain the following estimate: for some positive constant k ,

$$\begin{aligned} & \| \phi_i T_i U(t) \|_{m, \tan}^2 - \| \phi_i T_i U(s) \|_{m, \tan}^2 \\ & \leq k(M, \|g\|_{Y_m(T)}, k_0^{-1}, m, \Omega) |t-s| \text{ for } 0 \leq t, s \leq T_0, \quad i=1, \dots, N, \end{aligned}$$

since $\|U\|_{X_m(T_0)} \leq M$. Accordingly, we get from (5.21)

$$\partial_{\tan}^{\alpha}(\phi_i T_i U) \in C(0, T_0; L^2(\Omega)) \quad \text{with } |\alpha| \leq m \text{ for } i=1, \dots, N.$$

Hence, by using the similar relations to (5.13) and (5.14) for U we can show that

$$\phi_i U \in C(0, T_0; H^m(\Omega)) \quad \text{for } i=1, \dots, N,$$

since T_i^{-1} are smooth. Since ϕ_i are smooth and since $\phi_0 U \in C(0, T_0; H^m(\Omega))$, we get

$$U \in C(0, T_0; H^m(\Omega))$$

Therefore, by returning to the equations (1.1) we get

$$U \in X_m(T_0).$$

Now the proof of THEOREM 1.1 is completed.

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Department of Mathematics,
Nara Women's University