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NON-EXISTENCE OF HIGHER ORDER NON-SINGULAR IMMERSIONS OF COMPLEX HYPERSURFACES INTO EUCLIDEAN SPACES

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0. Introduction

Pohl [9, 10] and Feldman [2, 3] studied the differential geometry of higher order. We know many informations on non-existences of higher order non-singular immersions of projective spaces, lens spaces or Dold manifolds into euclidean spaces, projective spaces or lens spaces. They are seen in Suzuki [12, 13], Kobayashi [5, 6], Yoshioka [14] and Ôike [7]. We denote by $V_n(q)$ $(q\geqq 1)$ a non-singular complex hypersurface of begree q in the $(n+1)$-dimensional complex projective space P_{n+1}. In this note, we study non-existences of higher order non-singular immersions of $V_n(q)$ into the N-dimensional euclidean space $\boldsymbol{R}^N$ by means of Stiefel-Whitney classes of higher order tangent bundles of $V_n(q)$. Our main result is Corollary 2.4. The computations of higher order tangent bundles of $V_n(q)$ heavily depend upon symmetric power operations in K-theory that Suzuki [12] firstly used to compute higher order tangent bundles of real projective spaces. On the other hand, in [9] Pohl also formulated and studied the complex analytic geometry of higher order. In [8] Ôike studied of non-existences of higher order non-singular holomorphic immersions of P_n and V_n (q) into P_N.

1. Preliminaries

Let M^n de a compact connected n-dimensional smooth manifold and $(x^1, \dots, x^n)$ be a local coordinate of M^n. Let $\tau_p(M^n)_x$ $(x\in M^n,\ p\geqq 1)$ be the real vector space spanned by

$$\left\{\frac{\partial^k}{\partial x^{\alpha_1}\dots\partial x^{\alpha_k}}\ ;\ 1\leqq k\leqq p,\ 1\leqq\alpha_1\leqq\dots\leqq\alpha_k\leqq n\right\}$$

and put

$$\tau_p(M^n)=\bigcup_{X\in M^n}\tau_p(M^n)_x,$$

where $\tau_1(M^n)=\tau(M^n)$ is the tangent bundle of M^n. Then we have that

$\tau_p(M^n)$ is a smooth vector bundle of rank $\nu(n, p)$ over M^n, where

$$\nu(n, p)=\binom{n+p}{p}-1.$$

$\tau_p(M^n)$ is called the p-th order tangent bundle of M^n (see [9, p. 189]). For a smooth vector bundle $\eta \longrightarrow M^n$ of rank m, we denote the k-fold symmetric thensor product of η by $O^k\eta$ $(O^0\eta=1,\ O^1\eta=\eta)$, where 1 is a trivial line bundle over M^n. We have a short exact sequence

$$0 \longrightarrow \tau_{p-1}(M^n) \longrightarrow \tau_p(M^n) \longrightarrow O^p\tau(M^n) \longrightarrow 0,$$

for $p \geqq 2$ (see [9, Theorem 2.1]). Then we have the following lemma.

LEMMA 1.1 (Suzuki [12, Lemma 2.2])

$$\tau_p(M^n)+1=O^p(\tau(M^n)+1).$$

Let f be a smooth map of M^n into the N-dimensional euclidean space $\boldsymbol{R}^N$. Let $\tau_p(f): \tau_p(M^n) \longrightarrow \tau_p(\boldsymbol{R}^N)$ be the p-th order differential of f (see [2, p 190]) and $D_k: \tau_k(\boldsymbol{R}^N) \longrightarrow \tau_{k-1}(\boldsymbol{R}^N)$ $(2 \leqq k)$ be the vector bundle homomorphism which are defined by

$$D_k\left(V_{k-1}+\sum a_{\alpha_1 \dots \alpha_k} \frac{\partial^k}{\partial x^{\alpha_1} \dots \partial x^{\alpha_k}}\right)=V_{k-1},$$

where $V_{k-1} \in \tau_{k-1}(\boldsymbol{R}^N)$ (see [9, p 176]). Then $\delta_p(f)=D_2 \dots D_p\tau_p(f): \tau_p(M^n) \longrightarrow \tau_1(\boldsymbol{R}^N)=\tau(\boldsymbol{R}^N)=\hat{\boldsymbol{R}}^N$ is a vector bundle homomorphism covering f, where $\hat{\boldsymbol{R}}^N$ is a product bundle $\boldsymbol{R}^N \times \boldsymbol{R}^N$ over $\boldsymbol{R}^N$. If $\delta_p(f)$ is of maximal rank (i. e., of rank min $\{\nu(n, p),\ N\}$) on each fibre, where $\delta_1(f)=\tau_1(f)=\tau(f)$, we say that f is p-th order non-singular. If there exists a p-th order non-singular immersion of M^n into $\boldsymbol{R}^N$, we write $M^n \subseteq_p \boldsymbol{R}^N$ and if there exists no such immersion, we write $M^n \not\subseteq_p \boldsymbol{R}^N$. We have the following proposition by means of [2, p 217].

PROPOSITION 1.2. *If* $n>1$, *then* $M^n \subseteq_p R^{\nu(n, p)+n}$ *and if* $p>1$, *then* $M^n \subseteq_p R^{\nu(n, p)-n}$.

Suppose that an immersion $M^n \longrightarrow \boldsymbol{R}^N$ is p-th order non-singular. Then the cokernel Coker $\delta_p(f)$ or the kernel Ker $\delta_p(f)$ of $\delta_p(f): \tau_p(M^n) \longrightarrow \hat{\boldsymbol{R}}^N$ is a smooth vector bundle of rank $N-\nu(n, p)$ or $\nu(n, p)-N$ over M^n as $\nu(n, p) \leqq N$ or $\nu(n, p) \geqq N$, respectively. Let $w(\tau_p(M^n))$, $\bar{w}(\tau_p(M^n))$, $w_k(\tau_p(M^n))$ or $\bar{w}_k(\tau_k(M^n))$ be the total, the total dual, the k-th or the k-th dual Stiefel-Whitney class of $\tau_p(M^n)$, respectively. Then the total Stiefel-Whitney class of Coker $\delta_p(f)$, Ker $\delta_p(f)$ is given by

$$w(\text{Coker } \delta_p(f)) = w(\hat{\boldsymbol{R}}^N - \tau_p(M^n)) = w(-\tau_p(M^n))$$
$$= (w(\tau_p(M^n))^{-1} = \bar{w}(\tau_p(M^n)),$$
$$w(\text{Ker } \delta_p(f)) = w(\tau_p(M^n) - \hat{\boldsymbol{R}}^N) = w(\tau_p(M^n)),$$

respectively, where $\hat{\boldsymbol{R}}^N$ is a product bundle over M^n with fibre $\boldsymbol{R}^N$. Hence we have the following proposition.

PROPOSITION 1.3. *Suppose that M^n is a compact connected n-dimensional smooth manifold and that $f: M^n \longrightarrow \boldsymbol{R}^N$ is a p-th order non-singular immersion for $p \geqq 2$.*

(i) *If $N \geqq \nu(n, p)$, then for $k > N - \nu(n, p)$,*
$w_k(\text{Coker } \delta_p(f)) = \bar{w}_k(\tau_p(M^n)) = 0,$

where $w_k(\text{Coker } \delta_p(f))$ is the k-th Stiefel-Whitney class of Coker $\delta_p(f)$.

(ii) *If $N \leqq \nu(n, p)$, then for $k > \nu(n, p) - N$,*
$w_k(\text{Ker } \delta_p(f)) = w_k(\tau_p(M^n)) = 0,$

where $w_k(\text{Ker } \delta_p(f))$ is the k-th Stiefel-Whitney class of Ker $\delta_p(f)$.

Let G be a compact connected Lie group and F be the real number field $\boldsymbol{R}$ or the complex number field $\boldsymbol{C}$. Let V be a finite dimensional G-vector space over F and $[V]$ be a G-isomorphism class of V. The dimension of V is said to be the degree of $[V]$. We denote the k-fold symmetric tensor product of V by $O^k V$ ($O^0 V = 1$, $O^1 V = V$, where 1 is the 1-dimensional G-vector space F with a trivial G-action). Then $O^k V$ is regarded canonically as a G-vector space over F. Especially if V is a 1-dimensional G-vector space, then $O^k V$ is G-isomorphic to the k-fold tensor product $\otimes^k V$. Let $M_F(G)$ be a semiring which consists of all G-isomorphism classes of finite dimensional G-vector spaces over F. The additon and multiplication in $M_F(G)$ are induced by the direct sum and tensor product of finite dimensional G-vector spaces over F. We define $O^k[V]$ for $[V] \in M_F(G)$ by $O^k[V] = [O^k V]$, then $O^k (k = 0, 1, 2, \ldots)$ induce operations of $M_F(G)$ having following properties:

i) $O^0(x) = 1$, $O^1(x) = x$ for $x \in M_F(G)$,

ii) $O^k(x+y) = \sum_{i+j=k} O^i(x) O^j(y)$ for $x, y \in M_F(G)$,

iii) $O^k(x) = x^k$ for $x \in M_F(G)$: of degree 1.

Let $R_F(G)$ be a ring completion of $M_F(G)$ and $\alpha: M_F(G) \longrightarrow R_F(G)$ be a natural semiring homomorphism. We set

$$O_t(x)=\sum_{k=0}^{\infty}(O^k(x))t^k$$

for an indeterminate t and each $x\in M_F(G)$. Let $A_F(G)$ denote the multiplicative group of formal power series in t with coefficients in $R_F(G)$ and constant term 1. Then the properties i), ii) assert that O_t defines a homomorphism of $M_F(G)$ into $A_F(G)$ which turns the former addition into the latter multiplication. Hence we get such a homomorphism $O_t: R_F(G) \longrightarrow A_F(G)$. Taking the coefficients of O_t, this defines operators $O^k: R_F(G) \longrightarrow R_F(G)(k=0, 1, 2, \ldots)$ which is called the symmetric power operations. Properties i), ii) continue to hold for these O^k but the property iii) holds only in $\alpha(M_F(G))$. Symmetric power operations O^k are applied to calculations of higher order tangent bundles of real projective spaces firstly by Suzuki [12], complex and quaternion projective spaces thereafter by Ôike [7]. Note that for $x\in M_F(G)$ of degree 1,

$$O_t(x)=(1-xt)^{-1}.$$

Let r, c, ψ_C^{-1} be the following operations

$$\begin{aligned} r: R_C(G)&\longrightarrow R_R(G) && \text{real restriction,}\\ c: R_R(G)&\longrightarrow R_C(G) && \text{complexification,}\\ \psi_C^{-1}: R_C(G)&\longrightarrow R_C(G) && \text{complex conjugation.}\end{aligned}$$

Then we have the following lemma (see [1]).

LEMMA 1.4. i) *r is a group homomorphism and c, ψ_C^{-1} are ring homomorphisms.*

ii) $rc=2$, $cr=1+\psi_C^{-1}$.
iii) *c is injective.*
iv) $cO^k=O^kc$.

The following proposition is obtained by means of [1, 3.77 Corollary].

PROPOSITION 1.5. *$R_C(U(1))$ equals the ring $Z[z, z^{-1}]$ of all finite Laurent series with integer coefficients, where z is the $U(1)$-isomorphism class of the 1-dimensional complex vector space $\boldsymbol{C}$ (the field of complex numbers) with the natural $U(1)$-action and $z^{-1}=\psi_C^{-1}(z)$.*

Set $\eta=r(z)-2\in R_R(U(1))$, then we have the following lemma (see [7, Lemma 1.4] or [5, Lemma(4.3) and Appendix]).

LEMMA 1.6. i) $\psi_R^k(\eta)=r(z^k)-2$, $\psi_R^0(\eta)=0$, $\psi_R^{-k}(\eta)=\psi_R^k(\eta)$,

ii) $\eta^j=\sum_{i=1}^{j}(-1)^{j-i}\binom{2j}{j-i}\psi_R^i(\eta)$,

iii) $\psi_R^k(\eta)=\sum_{j=1}^{k}A_j^k\eta^j$,

iv) $\psi_R^i(\eta)\psi_R^j(\eta)=\psi_R^{i+j}(\eta)+\psi_R^{j-i}(\eta)-2(\psi_R^i(\eta)+\psi_R^j(\eta))$,

where ψ_R^k is the real Adams operation and

$$A_j^k=\frac{2}{(2j)!}\prod_{i=0}^{j-1}(k^2-i^2)=\frac{k}{j}\binom{k+j-1}{2j-1}.$$

The following lemma is Lemma 2.1 of [7].

LEMMA 1.7.

$$O^j((n+2)r(z))=F_j(\eta)+\binom{2n+3+j}{j},$$

where

$$F_j(\eta)=\sum_{i=0}^{\left[\frac{j-1}{2}\right]}\binom{n+1+i}{i}\binom{n+1+j-i}{j-i}\psi_R^{j-2i}(\eta).$$

2. Our results and their proofs.

Let $\gamma_{n+1}\longrightarrow P_{n+1}$ be the universal line bundle over P_{n+1} and put $\xi_n=\gamma_{n+1}|_{V_n(q)}$. We denote holomorphic tangent bundles of P_{n+1}, $V_n(q)$ by $\theta(P_{n+1})$, $\theta(V_n(q))$, respectively. Hirzebruch has shown that the holomorphic normal bundle $\nu(V_n(q))$ of $V_n(q)$ in P_{n+1} is isomorphic to ξ_n^{-q} (see [4, p. 69]). Hence we have a holomorphic short exact sequence

$$0\longrightarrow\theta(V_n(q))\longrightarrow\theta(P_{n+1})|_{V_n(q)}\longrightarrow\xi_n^{-q}\longrightarrow 0.$$

It is well known that $\theta(P_{n+1})+1=(n+2)\gamma_{n+1}^{-1}$ in K-group $K(P_{n+1})$ of P_{n+1}. Thus we have that $\theta(V_n(q))=(n+2)\xi_n^{-1}-\xi_n^{-q}-1$ in K-group $K(V_n(q))$ of $V_n(q)$. Therefore the tangent bundle $\tau(V_n(q))$ of $V_n(q)$ is given by $\tau(V_n(q))=(n+2)r(\xi_n^{-1})-r(\xi_n^{-q})-2$ in KO-group $KO(V_n(q))$ of $V_n(q)$. Since $r(\xi_n^{-k})=r(\xi_n^k)$ for each natural number k, we have that in $KO(V_n(q))$

$$\tau(V_n(q))+1=(n+2)r(\xi_n)-r(\xi_n^q)-1.$$

Set $y=r(\xi_n)-2$. We show that the k-th order tangent bundle of $V_n(q)$ is given by the following formula.

THEOREM 2.1.

$$\tau_k(V_n(q))+1=F_k(y)-(F_{k-1}(y)+G_{k-1}^q(y))$$

$$+(F_{k-2}(y)+G_{k-2}^{q}(y))-F_{k-3}(y)$$
$$+(-1)^{k}\binom{n+1+[\frac{k-1}{2}]}{[\frac{k-1}{2}]}^{2}\psi_{\boldsymbol{R}}^{q}(y)+\binom{2n+k}{k},$$

where

$$F_{j}(y)=\sum_{i=0}^{[\frac{j-1}{2}]}\binom{n+1+i}{i}\binom{n+1+j-i}{j-i}\psi_{\boldsymbol{R}}^{j-2i}(y),$$

$$G_{j}^{q}(y)=\sum_{i=0}^{[\frac{j-1}{2}]}\binom{n+1+i}{i}\binom{n+1+j-i}{j-i}(\psi_{\boldsymbol{R}}^{j-2i+q}(y)$$
$$+\psi_{\boldsymbol{R}}^{j-2i-q}(y)).$$

PROOF. $O_t(c((n+2)r(z)-r(z^q)-1))=O_t(c(n+2)r(z)))(1-tz^q)(1-tz^{-q})(1-t)=(1+\sum_{j=1}^{\infty}t^jcO^j)(1-tc(r(z^q)+1)+t^2c(r(z^q)+1)-t^3)$, where $O^j=O^j((n+2)r(z))$. Hence by Lemma 1.7, we have $O^k((n+2)r(z)-r(z^q)-1)=O^k-(r(z^q)+1)(O^{k-1}-O^{k-2})-O^{k-3}=F_k(\eta)-(\psi_{\boldsymbol{R}}^{q}(\eta)+3)F_{k-1}(\eta)+(\psi_{\boldsymbol{R}}^{q}(\eta)+3)F_{k-2}(\eta)-F_{k-3}(\eta)-\binom{2n+1+k}{k-1}\psi_{\boldsymbol{R}}^{q}(\eta)+\binom{2n+k}{k}$.

By iv) of Lemma 1.6, the following formula holds

$$(\psi_{\boldsymbol{R}}^{q}(\eta)+3)F_j(\eta)=F_j(\eta)+G_j^q(\eta)$$
$$-2\sum_{i=0}^{[\frac{j-1}{2}]}\binom{n+1+i}{i}\binom{n+1+j-i}{j-i}\psi_{\boldsymbol{R}}^{q}(\eta).$$

In general, for natural number j, we have that

$$2\sum_{i=0}^{[\frac{j-1}{2}]}\binom{n+1+i}{i}\binom{n+1+j-i}{j-i}=\binom{2n+3+j}{j},$$

if j is odd and that if j is even,

$$2\sum_{i=0}^{[\frac{j-1}{2}]}\binom{n+1+i}{i}\binom{n+1+j-i}{j-i}$$

$$=\binom{2n+3+j}{j}-\binom{n+1+\frac{j}{2}}{\frac{j}{2}}^2.$$

These formulas complete the proof. q. e. d.

Now we caluculate the Stiefel-Whitney class of $\tau_k(V_n(q))$. Let $x\in H^2(V_n(q)\,;\,Z)$ be the first Chern class of the complex line bundle ξ_n^{-1}. Then the additive order of x^m is infinite for $1\leqq m\leqq n$ and $x^{n+1}=0$. The following proposition is shown in [11, p. 172].

Proposition 2.2. *Let $j: V_n(q)\longrightarrow P_{n+1}$ the canonical inclusion. Then*

i) *$j^*: H^k(P_{n+1}\,;\,Z)\longrightarrow H^k(V_n(q)\,;\,Z)$ is an isomorphism if $k<n$ (similarly in homology) and if $k\neq n$, $H^k(V_n(q)\,;\,Z)$ is isomorphic to $H^k(P_n\,;\,Z)$;*

ii) *$H^n(V_n(q)\,;\,Z)$ is a free abelian group and*

$$rank(H^n(V_n(q)\,;\,Z))=\begin{cases}\dfrac{(q-1)^{n+2}-q+1}{q} & (n\,;\,odd),\\[2ex] \dfrac{(q-1)^{n+2}+2q-1}{q} & (n\,;\,even)\,;\end{cases}$$

iii) *x generates a truncated polynomial subalgebra of $H^*(V_n(q)\,;\,Z)$ and $x^k=q\cdot$(generator of $H^{2k}(V_n(q)\,;\,Z)$), if $2k\geqq n$.*

Set $\bar{x}=x \pmod 2\in H^2(V_n(q)\,;\,Z_2)$. Then we have that for $\frac{n}{2}\leqq k\leqq n$,

$$\bar{x}^k\begin{cases}\neq 0, & \text{if } q \text{ is odd,}\\ =0, & \text{if } q \text{ is even.}\end{cases}$$

The Stiefel-Whitney class of $\psi^j_{\boldsymbol{R}}(y)=r(\xi^j_n)-2$ is given by $w(\psi^j_{\boldsymbol{R}}(y))=c(\xi^j_n)$ (mod 2) $=1+j\bar{x}$, where $c(\xi^j_n)$ is the Chern class of ξ^j_n. Hence we have

$$w(\psi^j_{\boldsymbol{R}}(y))=\begin{cases}1+\bar{x} & (j\,;\,\text{odd}),\\ 1 & (j\,;\,\text{even}).\end{cases}$$

Therefore the following corollary is easily obtained with elementary calculations from the above Theorem 2.1.

Corollary 2.3. i) *If q is odd, then*

$$w(\tau_k(V_n(q)))=\begin{cases}(1+\bar{x})^{a(n,k)} & (k\,;\,odd),\\ (1+\bar{x})^{-b(n,k)} & (k\,;\,even),\end{cases}$$

where

$$a(n,k)=\frac{1}{2}\binom{2n+1+k}{k},\quad b(n,k)=\frac{1}{2}\binom{2n+k}{k-1}.$$

ii) *If q is even, then*

$$w(\tau_k(V_n(q)))=\begin{cases}(1+\bar{x})^{c(n,k)} & (k;\ odd),\\ (1+\bar{x})^{-d(n,k)} & (k;\ even),\end{cases}$$

where

$$c(n,k)=\frac{1}{2}\left\{\binom{2n+3+k}{k}+3\binom{2n+1+k}{k-2}\right\},$$
$$d(n,k)=\frac{1}{2}\left\{3\binom{2n+2+k}{k-1}+\binom{2n+k}{k-3}\right\}.$$

By this corollary, we have that

$$\bar{w}_{2j-1}(\tau_k(V_n(q)))=0,\quad w_{2j-1}(\tau_k(V_n(q)))=0,$$
$$\bar{w}_{2j}(\tau_k(V_n(q)))=\begin{cases}\binom{\alpha(n,k)+j-1}{j}\bar{x}^j & (k;\ \text{odd}),\\ \binom{\beta(n,k)}{j}\bar{x}^j & (k;\ \text{even}),\end{cases}$$
$$w_{2j}(\tau_k(V_n(q)))=\begin{cases}\binom{\alpha(n,k)}{j}\bar{x}^j & (k;\ \text{odd}),\\ \binom{\beta(n,k)+j-1}{j}\bar{x}^j & (k;\ \text{even}),\end{cases}$$

where $\alpha=a(q;\ \text{odd})$ or $c(q;\ \text{even})$, $\beta=b(q;\ \text{odd})$ or $d\ (q;\ \text{even})$. In case $w(\tau_k(V_n(q)))\neq 1$, we set

$$n^+_{a(n,k)}=\max\left\{1\leqq j\leqq n;\ \binom{a(n,k)+j-1}{j}\equiv 1 \pmod 2\right\},$$
$$n^-_{a(n,k)}=\max\left\{1\leqq j\leqq n;\ \binom{a(n,k)}{j}\equiv 1 \pmod 2\right\},$$
$$n^+_{b(n,k)}=\max\left\{1\leqq j\leqq n;\ \binom{b(n,k)}{j}\equiv 1 \pmod 2\right\},$$
$$n^-_{b(n,k)}=\max\left\{1\leqq j\leqq n\ \binom{b(n,k)+j-1}{j}\equiv 1 \pmod 2\right\},$$
$$n^+_{c(n,k)}=\max\left\{1\leqq j<\frac{n}{2};\ \binom{c(n,k)+j-1}{j}\equiv 1 \pmod 2\right\},$$
$$n^-_{c(n,k)}=\max\left\{1\leqq j<\frac{n}{2};\ \binom{c(n,k)}{j}\equiv 1 \pmod 2\right\},$$
$$n^+_{d(n,k)}=\max\left\{1\leqq j<\frac{n}{2};\ \binom{d(n,k)}{j}\equiv 1 \pmod 2\right\},$$

$$n^-_{d(n,k)}=\max\left\{1\leqq j<\frac{n}{2};\ \binom{d(n,k)+j-1}{j}\equiv 1 \pmod 2\right\}.$$

The following example is easily shown.

Example 1.

i) $n^+_{a(n,k)}=n^-_{a(n,k)}=n$ for $n=2^r$, $k=2^{r+1}+2^r-1$ $(r\geqq 1)$,

ii) $n^+_{b(n,k)}=n^-_{b(n,k)}=n$ for $n=2^r$, $k=2^{r+1}+2^r$ $(r\geqq 1)$,

iii) $n^+_{c(n,k)}=n^-_{c(n,k)}=\frac{n}{2}-1$ for $n=2^r+2$, $k=2^{r+1}+2^{r-2}-5$ $(r\geqq 5)$,

iv) $n^+_{d(n,k)}=n^-_{d(n,k)}=\frac{n}{2}-1$ for $n=2^r+2$, $k=2^{r+1}+2^{r-2}-4$ $(r\geqq 5)$.

By the above Corollary 2.3 and Proposition 1.3, we have the following corollary.

COROLLARY 2.4. *Suppose that* $w(\tau_k(V_n(q)))\neq 1$ *and that natural number* N *satisfys one of following inequalities*;

a) *for* q; *odd and* k; *odd*
$\nu(2n,k)-2n^-_{a(n,k)}<N<\nu(2n,k)+2n^+_{a(n,k)}$,

b) *for* q; *odd and* k; *even*
$\nu(2n,k)-2n^-_{b(n,k)}<N<\nu(2n,k)+2n^+_{b(n,k)}$,

c) *for* q; *even and* k; *odd*
$\nu(2n,k)-2n^-_{c(n,k)}<N<\nu(2n,k)+2n^+_{c(n,k)}$,

d) *for* q; *even and* k; *even*
$\nu(2n,k)-2n^-_{d(n,k)}<N<\nu(2n,k)+2n^+_{d(n,k)}$.

Then

$$V_n(q)\not\subseteq_k \boldsymbol{R}^N.$$

By Proposition 1.2, Example 1 and Corollary 2.4, we have the following example.

Example 2.

$$V_n(q)\subseteq_k \boldsymbol{R}^{\nu(2n,k)+2n},\quad V_n(q)\subseteq_k \boldsymbol{R}^{\nu(2n,k)-2n}\quad (k>1).$$

But if q is odd, $n=2^r$, $k=2^{r+1}+2^r-1$ or $2^{r+1}+2^r$ $(r\geqq 1)$, then

$$V_n(q)\not\subseteq_k \boldsymbol{R}^N$$

for $\nu(2n,k)-2n<N<\nu(2n,k)+2n$.

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