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RADONIFICATION THEOREM FOR F-CYLINDRICAL PROBABILITIES

Neven ELEZOVIĆ

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§ 1. Introduction

Let E, F and G be real Banach spaces. Let λ be a cylindrical measure on E of type p ([4]), and $w: E\rightarrow G$ a continuous linear operator. Then, for $1<p<\infty$, w is p-Radonifying (i. e. $w(\lambda)$ is a Radon probability on G of order p) if and only if it is p-summing (cf. [10], Theorem 1. 1). It is well known that p-left-nuclear operators $w: E\rightarrow G$ are p-Radonifying even in the case $0<p\leq 1$ (cf. [10], Proposition 2. 6).

Let α be a norm on $E\otimes F$. Denote by $E\otimes_{\alpha}F$ the normed space $(E\otimes F, \alpha)$ and by $E\hat{\otimes}_{\alpha}F$ its completion. The Radonification problem for the class of F-cylindrical probabilities on the tensor product $E\otimes F$ was considered by B. Maurey in [5]. He introduced (p, F)-Radonifying operators, which map every F-cylindrical probability on $E\otimes F$ of type (p, F) into a Radon probability of order p on some completion $G\hat{\otimes}_{\alpha}F$. It is shown that (p, F)-summing operators of the form $w\otimes 1_{F}: E\otimes F\rightarrow G\hat{\otimes}_{\alpha}F$ are (p, F)-Radonifying, for reflexive F, $1<p<\infty$ and under certain natural assumptions on the norm α (the conditions (1) and (2) in § 2). As an example, the operator $w\otimes 1_{F}: E\otimes F\rightarrow G\hat{\otimes}_{\varepsilon}F$ is (p, F)-summing, whenever $w: E\rightarrow G$ is p-summing. Here ε denotes the least reasonable norm. We refer to [7] for definitions and properties of all tensor norms used here.

In this paper it is proved that $\bar{r}_{p}$-nuclear operators $W: E\otimes F\rightarrow G$ are (p, F)-Radonifying, for reflexive F, $1\leq p<\infty$ and the same assumptions on the norm α as in [5]. $\bar{r}_{p}$-nuclear operators generalise classical p-left-nuclear operators, in a natural way, to operators which are defined on the tensor product of two Banach spaces (without any prescribed topology on this space) and such that an operator of the form $w\otimes 1_{F}$ may belong to this class. See § 3 for definition and [2] for more details. Furthermore, $\bar{r}_{p}$-nuclear operators are (p, F)-Radonifying from $E\otimes F$ into a third Banach space G which is not necessarily the completion of some tensor product. In the case when this space is actually the completion of some tensor product, we give some examples of (p, F)-Radonifying operators into the completion under a stronger norm than the ε norm. Namely, every $\bar{r}_p$-nuclear operator is (p, F)-summing. Thus, starting from p-left-nuclear operator $w: E \to G$ we obtain an $\bar{r}_p$-nuclear operator $w \otimes 1_F : E \otimes F \to G \hat{\otimes}_{d_p} F$ (cf. [2], Theorem 4), hence also a (p, F)-summing operator. The case $p=1$ is not covered in Maurey's theorem and represents one of the main aims of this paper. The importance of this case lies in the fact that the space $L_1 \hat{\otimes}_{d_1} F$ can be identified with the space $L_1(F)$, a result which is no longer true for the space $L_p \hat{\otimes}_{d_p} F$, $p>1$.

§ 2. F-cylindrical probabilities on E⊗F

$L(E, G)$ denotes the space of all continuous linear operators from E into G. For $w \in L(E, G)$, $\tilde{w} : E \otimes F \to G \otimes F$ denotes the operator $w \otimes 1_F$, defined by $(w \otimes 1_F)(x \otimes y) := wx \otimes y$.

Let us recall that F-cylindrical probability λ on the tensor product $E \otimes F$ is a projective system $\{\lambda_N, \tilde{\pi}_N\}$ of Radon probabilities on $(E/N) \otimes F$, where N runs over the family $FC(E)$ of all closed subspaces of E of the finite codimension, and $\pi_N : E \to E/N$ is the natural projection. $M_F^C(E \otimes F)$ denotes the space of all F-cylindrical probabilities.

A topology τ on $M_F^C(E \otimes F)$ is defined as the coarsest topology for which the mappings $\lambda \mapsto \tilde{\pi}_N(\lambda) = \lambda_N$ from $M_F^C(E \otimes F)$ into $M(E/N \otimes F)$ are continuous. The second space of all Radon probabilities on $E/N \otimes F$ is equipped with the topology of the weak convergence of measures. This topology τ is called the topology of F-cylindrical convergence.

Let $\psi : E \otimes_\alpha F \to L(E', F)$ be the canonical embedding. We suppose that α satisfies the condition

$$\psi : E \otimes_\alpha F \to L(E', F) \text{ is continuous and } \|\psi\| \leq 1 \tag{1}$$

Then there exists the extension $\hat{\psi}$ of this mapping to the completion $E \hat{\otimes}_\alpha F$.

Let μ be a Radon probability on $E \hat{\otimes}_\alpha F$. Then $\lambda_N := \tilde{\pi}_N(\mu)$ defines a cylindrical probability λ on $E \otimes F$ (cf [5]). We denote this connection by $\lambda = \check{\mu}$.

We say that a cylindrical probability λ on $E \otimes F$ is Radon on $E \hat{\otimes}_\alpha F$ if there exists a Radon probability μ on $E \hat{\otimes}_\alpha F$ for which it holds $\lambda = \check{\mu}$. Such measure need not be unique. However, if α satisfies

$$\hat{\psi} : E \hat{\otimes}_\alpha F \to L(E', F) \text{ is one to one} \tag{2}$$

then μ, if it exists, is unique ([5]).

It is well known that every reasonable crossnorm α satisfies (1). If E or F has the metric approximation property, then such norm satisfies (2).

The F-cylindrical topology on $M_F^C(E\otimes F)$ induces on the space $M(E\hat{\otimes}_\alpha F)$ the notion of convergence as follows: $\check{\mu}_\gamma\to\check{\mu}$ if and only if $\mu_\gamma\to\mu$ F-cylindrically. Such topology on $M(E\hat{\otimes}_\alpha F)$ is weaker than the topology of the weak convergence of measures.

Let $\lambda\in M_F^C(E\otimes F)$. For $w\in L(E, G)$ and $M\in FC(G)$ the following diagram commutes:

$$\begin{tikzcd} E\otimes F \arrow[r, "\tilde{w}"] \arrow[d, "\tilde{\pi}_{w^{-1}(M)}"'] & G\otimes F \arrow[d, "\tilde{\pi}_M"] \\ (E/w^{-1}(M))\otimes F \arrow[r, "w_M\otimes 1_F"'] & (G/M)\otimes F \end{tikzcd}$$

Since w_M is continuous and $w^{-1}(M)\in FC(E)$, we can define the F-cylindrical probability $\tilde{w}(\lambda)$ on $G\otimes F$ by

$$\tilde{w}(\lambda)_M := (w_M\otimes 1_F)(\lambda_{w^{-1}(M)})$$

For a linear mapping $W : E\otimes F\to G$, $W(\lambda)$ cannot be defined to be a cylindrical measure on G. Namely, for $M\in FC(G)$, $(E\otimes F)/W^{-1}(M)$ need not be of the form $(E/N)\otimes F$, so $\lambda_{W^{-1}(M)}$ may not have a sense. However, the notion " $W(\lambda)$ is a Radon probability on G " has a sense through the following:

DEFINITION. $W(\lambda)$ is a Radon probability on G if W has a factorization of the form

$$\begin{tikzcd} E\otimes F \arrow[rr, "W"] \arrow[dr, "\tilde{w}"'] & & G \\ & E_1\hat{\otimes}_\alpha F \arrow[ur, "v"'] & \end{tikzcd}$$

for some Banach space E_1 and a norm α which satisfies (1), (2) ; where w, v are continuous and $\tilde{w}(\lambda)$ is a Radon probability on $E_1\hat{\otimes}_\alpha F$.

A Radon probability μ on a Banach space G is of order p([4]) if

$$\|\mu\|_p := \{\int_G \|z\|^p\, d\mu(z)\}^{1/p}<\infty$$

Let λ be a F-cylindrical probability on $E\otimes F$. For each $x'\in E'$ the image $\lambda_{x'}: =\tilde{x}'(\lambda)$ represents a Radon probability on $\boldsymbol{R}\otimes F\simeq F$. We say that λ is of type (p, F), $p>0$, if $\lambda_{x'}$ is of order p for all $x'\in E'$ and

$$\|\lambda\|_{p,F}^{*}: =\sup_{\|x'\|\leq 1}\|\lambda_{x'}\|_{p}<\infty$$

$W: E\otimes F\rightarrow G$ is said to be (p, F)-Radonifying if it maps every F-cylindrical probability on $E\otimes F$ of type (p, F) into a Radon probability on G of order p.

§ 3. (p, F)-summing and $\bar{r}_p$-nuclear operators

$W: E\otimes F\rightarrow G$ is (p, F)-summing ([5]) if there exists $C\geq 0$ such that for every finite $\{u_j\}\subset E\otimes F$ one has

$$\{\sum\|W(u_j)\|^{p}\}^{1/p}\leq C\sup_{\|x'\|\leq 1}\{\sum\|[u_j, x']\|^{p}\}^{1/p} \tag{3}$$

where $[u, x']$ denotes the canonical action of an element $u\in E\otimes F$ on the vectors in E'. The norm of W is defined by $\tilde{\pi}_{p,F}(W): =\inf\{C: C$ satisfies $(3)\}$.

The following theorem is due by B. Maurey in [5]:

THEOREM 1. *Let F be reflexive, $1<p<\infty$, $w\in L(E, G)$ and $\lambda\in M_F^C(E\otimes F)$ of type (p, F). If α satisfies* (1) *and* (2), *and if $\tilde{w}: E\otimes F\rightarrow G\hat{\otimes}_{\alpha}F$ is (p, F)-summing, then $\tilde{w}(\lambda)$ is a Radon probability on $G\hat{\otimes}_{\alpha}F$ and*

$$\|\tilde{w}(\lambda)\|_{p}\leq\tilde{\pi}_{p,F}(\tilde{w})\ \|\lambda\|_{p,F}^{*}$$

We shall prove a strengthening of this result for a smaller class of $\bar{r}_p$-nuclear operators. Let us recall that $W: E\otimes F\rightarrow G$ is $\bar{r}_p$-nuclear if it has a representation

$$W=\sum_{j=1}^{\infty}x_j'\otimes v_j \tag{4}$$

such that for $\{x_j'\}\subset E'$ and $\{v_j\}\subset L(F, G)$ it holds

$$\bar{r}_p(W): =\inf\{(\sum\|x_j'\|^{p})^{1/p}\sup_{\|z'\|\leq 1}(\sum\|{}^{t}v_j z'\|^{p'})^{1/p'}\}<\infty\quad(1<p<\infty)$$

or

$$\bar{r}_1(W): =\inf\{\sum\|x_j'\|\cdot\sup_j\|{}^{t}v_j\|\}<\infty,$$

where the infimum is taken over all representations of the form (4), ${}^t v : G' \to F'$ is the adjoint operator, and p' is defined by $\frac{1}{p}+\frac{1}{p'}=1$. For $u \in E \otimes F$, $W(u)$ is defined by

$$W(u)=\sum_{j=1}^{\infty} v_j([u, x_j']) \tag{5}$$

Every $\bar{r}_p$-nuclear operator $W : E \otimes F \to G$ is (p, F)-summing, with $\tilde{\pi}_{p,F}(W) \leq \bar{r}_p(W)$ (see [3], Prop. 5). Further, if $w : E \to G$ is $\bar{p}$-left-nuclear, then $\tilde{w} : E \otimes F \to G \hat{\otimes}_{d_p} F$ is $\bar{r}_p$-nuclear and $\bar{r}_p(\tilde{w}) \leq g_p(w)$ holds (cf. [2], Theorem 4). By Proposition 2 of [5] (Expose II) it follows

Proposition 1. *Let $w : E \to G$ be p-left-nuclear, $1 \leq p < \infty$, and let μ be a Radon probability on $E \hat{\otimes}_\varepsilon F$ of type (p, F). Then*

$$\|\tilde{w}(\mu)\|_p \leq g_p(w) \|\mu\|_{p,F}^* \tag{6}$$

We are ready now to prove the main result:

Theorem 2. *Let F be reflexive and $1 \leq p < \infty$. If $W : E \otimes F \to G$ is $\bar{r}_p$-nuclear and λ F-cylindrical probability on $E \otimes F$ of type (p, F), then $W(\lambda)$ is a Radon probability on G of order p and*

$$\| W(\lambda) \|_p \leq \bar{r}_p(W) \|\lambda\|_{p,F}^* \tag{7}$$

Proof. The operator W has a factorization of the form

$$\begin{tikzcd} E \otimes F \arrow[rr, "W"] \arrow[d] & & G \\ l_\infty \otimes F \arrow[rr, "\tilde{d}"'] & & l_p(F) \arrow[u, "v"'] \end{tikzcd} \tag{8}$$

where w, v are continuous and $d : l_\infty \to l_p$ is the diagonal operator of the multiplication by an element $\{d_j\} \in l_p$. Denote by $e_j' = (0, \dots, 0, 1, 0, \dots)$ (1 is on the j-th place). The linear mapping $\tilde{d}$ is defined by

$$\tilde{d}(x \otimes y) := \{d_j \xi_j y\}, \ x=(\xi_j) \in l_\infty, \ y \in F$$

i.e. $\tilde{d}(u)=\{d_j[u, e_j']\}$, $u \in l_\infty \otimes F$ (cf. [2], Theorem 8). Further, $\tilde{d}$ is $\bar{r}_p$-nuclear, hence also (p, F)-summing.

Denote by Δ_p the norm on $l_p \otimes F$ induced from the space $l_p(F)$. Then $l_p \hat{\otimes}_{\Delta_p} F \simeq l_p(F)$. On $l_p \otimes F$ the following relation holds: $d_p \leq \Delta_p \leq g_p$ (see [1], Théorème 5 or [8], Théorème 2). Since l_p has the metric approximation property, the norms d_p and g_p satisfy (1) and (2) on the space $l_p \otimes F$. Hence, Δ_p has the properties (1) and (2) on $l_p \otimes F$.

Suppose now $p>1$. $\tilde{w}(\lambda)$ is a F-cylindrical probability on $l_\infty \otimes F$ of type (p, F), since it holds

$$\|\tilde{w}(\lambda)\|^*_{p,F} \le \|w\| \ \|\lambda\|^*_{p,F} \tag{9}$$

By Theorem 1, $(\tilde{d}\tilde{w})(\lambda)$ is a Radon probability on $l_p \hat{\otimes}_{\Delta_p} F \simeq l_p(F)$. Thus, $W(\lambda) = (v\ \tilde{d}\ \tilde{w})(\lambda)$ is Radon on G and

$$\begin{aligned} \|W(\lambda)\|_p &\le \|v\| \ \|(\tilde{d}\ \tilde{w})(\lambda)\|_p \\ &\le \|v\| \ \tilde{\pi}_{p,F}(\tilde{d}) \|\tilde{w}(\lambda)\|^*_{p,F} \\ &\le \|v\| \ g_p(d) \ \|w\| \ \|\lambda\|^*_{p,F} \\ &\le (\bar{r}_p(W) + \eta) \|\lambda\|^*_{p,F} \end{aligned}$$

since $\bar{r}_p(W) = \inf\{\|v\| \ g_p(d) \ \|w\|\}$, the infimum is taken over all factorizations of the form (8), and $\eta > 0$ being given in advance (cf. [2], Theorem 8). Thus, (7) follows.

Suppose now $p=1$. Take the representation (4) of W for which it holds $\|w\|=1$, $\|v\| = \sup_j \|{}^t v_j\| = 1$, $\|d\| = \sum |d_j| = \sum \|x'_j\| \le \bar{r}_1(W) + \eta$. We can write the sequence $\{d_j\} \in l_1$ in the form $d_j = b_j \cdot c_j$, where $\{b_j\} \in l_1$ and $\{c_j\} \in c_0$ are chosen such that it holds

$$\sum |b_j| \le \sum |d_j| + \eta \le \bar{r}_1(W) + 2\eta, \ \ |c_j| \le 1 \tag{10}$$

The diagonal operators $b : l_\infty \to l_1$ and $c : l_1 \to l_1$ defined by these sequences are nuclear and compact, respectively, with the norms $\|c\| \le 1$ and $g_1(b) = \sum |b_j|$ (cf. [6], Satz 7).

Since $(l_\infty)'$ has the metric approximation property, $\tilde{w}(\lambda)$ is the F-cylindrical limit of a net $\{\lambda_\gamma, \gamma \in \Gamma\}$ of Radon probabilities on $l_\infty \otimes F$ (each λ_γ is concentrated on some spaces of the form $(l_\infty/N) \otimes F$, where $N \in FC(l_\infty)$), of order 1, see [5], Expose II, Théorème 1. Further,

$$\|\lambda_\gamma\|^*_{1,F} \le \|\tilde{w}(\lambda)\|^*_{1,F} \le \|\lambda\|^*_{1,F}$$

The mapping $b : l_\infty \to l_1$ is nuclear and so $\tilde{b} : l_\infty \otimes F \to l_1 \hat{\otimes}_{d_1} F$ is $\bar{r}_1$-nuclear. It is well known that $l_1 \hat{\otimes}_{d_1} F$ is isometrically isomorphic to $l_1(F)$ (the norm d_1 coincides with the projective norm π). Let $\{e'_j\}$ be defined as before. Define a linear mapping $i : l_1 \hat{\otimes}_{d_1} F \to l_1(F)$ by $i(u) := \{[u, e'_j]\}$ and $\tilde{c} : l_1(F) \to l_1(F)$ by $\tilde{c}(\{y_j\}) = \{c_j y_j\}$. Then it holds $\|i\| \le 1$, $\|c\| \le 1$ and $\tilde{d} = \tilde{c}\ i\ \tilde{b}$.

Denote $\mu_\gamma := (i\ \tilde{b})(\lambda_\gamma)$. μ_γ are Radon probabilities on $l_1(F)$ for which it holds, by Proposition 1 and (10)

$$\begin{aligned}\|\mu_\gamma\|_1 &\leq \|i\| \ \|\tilde{b}(\lambda_\gamma)\|_1 \\ &\leq g_1(b)\,\|\lambda_\gamma\|^*_{1,F} \\ &= (\textstyle\sum |b_j|)\,\|\lambda_\gamma\|^*_{1,F} \\ &\leq (\bar{r}_1(W)+2\eta)\,\|\lambda\|^*_{1,F}\end{aligned}$$

Let j be the canonical embedding of $l_1(F)$ into $l_1(F)''$. Then $\|j\| \leq 1$. The measures μ_γ are Radon on $\sigma(l_1(F), l_1(F)')$, so, $j(\mu_\gamma)$ are Radon probabilities on $l_1(F)''_\sigma := \sigma(l_1(F)'', l_1(F)')$ for which it holds

$$\|j(\mu_\gamma)\|_1 \leq \|j\| \ \|\mu_\gamma\|_1 \leq \bar{r}_1(W)\,\|\lambda\|^*_{1,F}$$

Hence, $\overline{\{j(\mu_\gamma)\}}$ is compact in the topology of the weak convergence of measures ([4], Prop. 4), and hence F-cylindrically compact. There exists a Radon probability μ on $l_1(F)''_\sigma$ which lies in the closure of $\{j(\mu_\gamma)\}$. We may suppose that $\mu = \lim j(\mu_\gamma)$. Since $\{\lambda_\gamma\}$ converges F-cylindrically to $\tilde{w}(\lambda)$ and F-cylindrical convergence is preserved by continuous mappings, it follows $\mu = \lim\limits_{\gamma} (j\, i\, \tilde{b})(\lambda_\gamma) = (j\, i\, \tilde{b}\, \tilde{w})(\lambda)$.

Since the space F is reflexive and $c : l_1 \to l_1$ is compact, it follows from Eberlein-Šmulian theorem (with the help of the usual diagonal procedure) that $\tilde{c} : l_1(F) \to l_1(F)$ is weakly compact. Hence, ${}^{tt}\tilde{c}(l_1(F)'') \subset l_1(F)$ and it holds ${}^{tt}\tilde{c}\, j = \tilde{c}$. Thus, the following diagram is commutative:

$$\begin{tikzcd} l_\infty \otimes F \arrow[rr, "\tilde{d}"] \arrow[d, "\tilde{b}"] & & l_1(F) \\ l_1 \hat{\otimes}_{d_1} F \arrow[r, "i"'] & l_1(F) \arrow[ur, "\tilde{c}"] \arrow[r, "j"'] & l_1(F)'' \arrow[u, "{}^{tt}\tilde{c}"'] \end{tikzcd}$$

We conclude that ${}^{tt}\tilde{c}(\mu) = {}^{tt}\tilde{c}(j\, i\, \tilde{b}\, \tilde{w})(\lambda) = (\tilde{c}\, i\, \tilde{b}\, \tilde{w})(\lambda) = (\tilde{d}\, \tilde{w})(\lambda)$ is concentrated on sets which are relatively compact in $\sigma(l_1(F), l_1(F)')$, and represents, by Phillips theorem ([9], Theorem 3, p. 162) a Radon probability on $l_1(F)$. Thus, $W(\lambda)$ is a Radon probability on G and

$$\|W(\lambda)\|_1 \leq \|v\| \ \|(\tilde{d}\, \tilde{w})(\lambda)\|_1 \leq \bar{r}_1(W)\,\|\lambda\|^*_{1,F}$$

which proves the theorem.

COROLLARY 1. *Let F be reflexive, $1 \leq p < \infty$ and $\alpha \leq d_p$ a reasonable* *norm on* $G\otimes F$. *If* $w: E\to G$ *is p-left-nuclear and* λ *a F-cylindrical probability on* $E\otimes F$ *of type* (p, F), *then* $\tilde{w}(\lambda)$ *is a Radon probability on* $G\hat{\otimes}_\alpha F$ *of order* p *and*

$$\|\tilde{w}(\lambda)\|_p \leq g_p(w)\,\|\lambda\|^*_{p,F}$$

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Elektrotehnički fakultet
P. B. 170, Unska 3
Zagreb, Yugoslavia