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KO-theory of Hermitian symmetric spaces

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Dedicated to Professor Haruo Suzuki on his 60th birthday

§ 1. Introduction

Our purpose of this paper is the determination of KO-theory of the compact irreducible Hermitian symmetric spaces. The spaces are classified by E. Cartan as follows:

$AIII$	$M_{m,n}=U(m+n)/(U(m)\times U(n))$	
BDI	$Q_n \ =SO(n+2)/(SO(n)\times SO(2))$	$(n\geq 3)$
CI	$Sp(n)/U(n)$	$(n\geq 3)$
$DIII$	$SO(2n)/U(n)$	$(n\geq 4)$
$EIII$	$=E_6/(Spin(10)\cdot T^1)$	$(Spin(10)\cap T^1\cong \boldsymbol{Z}_4)$
$EVII$	$=E_7/(E_6\cdot T_1)$	$(E_6\cap T^1\cong \boldsymbol{Z}_3)$.

Bott showed their cohomology rings have no torsion and no odd dimensional part. The integral cohomology rings are determined by [2], [9] and [10], while the actions of the squaring operations on them are determined in [5]. In [6], we compute the KO-theory of $M_{m,n}$. Here we show:

THEOREM 1. *Let X be a compact irreducible Hermitian symmetric space, then its Atiyah-Hirzebruch spectral sequence for $KO^*(X)$:*

$$E_r^{*,*}(X)\Rightarrow KO^*(X)$$

has nontrivial differential d_r only for $r=2$.

Let $H^*(X)$ be the modulo 2 cohomology ring of X. When the odd dimensional parts of $H^*(X)$ are trivial, $Sq^2Sq^2(=Sq^3Sq^1)$ vanishes on $H^*(X)$, and $(H^*(X), Sq^2)$ is a differential module. For the proof of Theorem 1 we compute the (co)homology group $H(H^*(X); Sq^2)$, which is isomorphic to $E_3^{*,-1}(X)$, and show the differentials d_r $(r\geq 3)$ are trivial for each X.

By Theorem 1, KO^* (X) is obtained from $E_3^{*,*}(X)$. Consequently the groups $H^*(X)$ and $H(H^*(X), Sq^2)$ determine $KO^*(X)$ in the following corollary.

COROLLARY 2. *The $KO^i(X)$ is given by the following table:*

i	KO^i
0	$t_0\boldsymbol{Z}\oplus s_1\boldsymbol{Z}_2$
-1	$s_0\boldsymbol{Z}_2$
-2	$t_1\boldsymbol{Z}\oplus s_0\boldsymbol{Z}_2$
-3	$s_3\boldsymbol{Z}_2$
-4	$t_0\boldsymbol{Z}\oplus s_3\boldsymbol{Z}_2$
-5	$s_2\boldsymbol{Z}_2$
-6	$t_1\boldsymbol{Z}\oplus s_2\boldsymbol{Z}_2$
-7	$s_1\boldsymbol{Z}_2$

where

$$t_\delta=\dim_{Z_2}\oplus_{i\equiv 2\delta(\mathrm{mod}\ 4)}H^i(X)$$
$$s_\varepsilon=\dim_{Z_2}\oplus_{i\equiv 2\varepsilon(\mathrm{mod}\ 8)}H^i(H^*(X)\,;\,Sq^2).$$

We discribe $H^*(X)$, $H(H^*(X):Sq^2)$, t_δ and s_ε for each X later.

§ 1. Preliminaries

In this section we recall the result of our previous papers and prepare a lemma for the proof of the theorem.

LEMMA 1.1. *Let X be a CW complex of finite type, such that its cohomology has no torsion and is concentrated in even dimensions, and let $E_r^{*,*}$ be the Atiyah-Hirzebruch spectral sequence for KO-theory.*

(1) *We have an isomorphism:*

$$\iota: E_3^{*,q}\xrightarrow{\cong} H(H^*(X)\,;\,Sq^2),\ q\equiv -1\pmod 8.$$

(2) *Suppose there is a nontrivial differential $d_r: E_r^{*,*}\longrightarrow E_r^{*+r,*+1-r}$ $(r\geq 3)$. For the smallest r $(E_r^{*,*}=E_3^{*,*})$, the next conditions are satisfied.*

(i) $1-r\equiv -1\pmod 8$.

(ii) *If p is the smallest integer such that $d_r(E_r^{p,*})\neq\{0\}$, there is an element $x\in E_r^{p,0}$ such that $\eta\cdot x\neq 0$ and $\eta\cdot d_rx\neq 0$, where η is the generator of the coefficient group $KO^{-1}\cong\boldsymbol{Z}_2$.*

(iii) *$\iota(\eta\cdot x)$ is indecomposable as a element of $H(H^*(X)\,;\,Sq^2)$.*

(3) *Moreover if X is a Hopf space, $H(H^*(X)\,;\,Sq^2)$ is a Hopf algebra and, in (2), $\iota(d_rx)$ is a primitive element.*

PROOF: (1). This is given in [3].

(2). (i) and (ii) are demonstrated in [6]. (iii). Suppose $\eta\cdot x$ is a decomposable element. It is written as $\sum\eta\cdot x'x''$ with x', $x''\in E_r^{*,0}$ and deg x', deg $x''<$deg x. By the assumption on p, $d_rx'=d_rx''=0$. We obtain $d_rx=\Sigma\,\eta\cdot d_r(x')x''+\Sigma\,\eta\cdot x'd_r(x'')=0$. This is a contradicitn. Thus $\eta\cdot x\in E_r^{*,-1}$ is an indecomposable element of the algebra $E_r^{*,*}$. Let A be $H(H^*(X);Sq^2)$. Because $H^*(\mathrm{X})$ has trivial odd dimensional parts, Sq^2 acts as a derivation on it and its homology group A is an algebra. The product of A is compatible with that of $E_r^{*,*}$, that is, the next diagram is commutative:

$$(1-1)\qquad \begin{array}{ccccc} E_r^{*,0}(X)\otimes E_r^{*,-1}(X) & \xrightarrow{\times} & E_r^{*,-1}(X\times X) & \xrightarrow{\Delta^*} & E_r^{*,-1}(\mathrm{X}) \\ \downarrow \pi\otimes\iota & & \downarrow\cong & & \iota\downarrow\cong \\ A\otimes A & \xrightarrow{\cong} & H(H^*(X\times X);Sq^2) & \xrightarrow{\Delta^*} & A, \end{array}$$

where $\times$ is the external product, Δ is the diagonal map, and π is the natural projection:

$$E_r^{*,0}(X)=E_3^{*,0}(X)\cong\mathrm{Ker}[Sq^2\pi_2: H^*(X;\boldsymbol{Z})\to H^{*+2}(X)]\to H(H^*(X);Sq^2).$$

(π_2 is the modulo 2 reduction $H^*(X;\boldsymbol{Z})\to H^*(X)$.) The map $\pi\otimes\iota$ is epimorphic. This proves that: if $\eta\cdot x$ is indecomposable then $\iota(\eta\cdot x)$ is also indecomposable.

(3). Since $H^*(X)$ is a Hopf algebra and Sq^2 is a derivation and commutes with the coproduct, A has a Hopf algebra structure. Let ψ : $E_r^{*,*}(X)\to E_r^{*,*}(X\times X)$ be the map induced by the multiplication of X. Consider the commutative diagram:

$$(1-2)\qquad \begin{array}{ccccc} E_r^{*-r}(X) & \xrightarrow{\psi} & E_r^{*,1-r}(X\times X) & & \\ \uparrow d_r & & \uparrow d_r & \searrow\eta\cdot & \\ E_r^{*,0}(X) & \xrightarrow{\psi} & E_r^{*,0}(X\times X) & & E_r^{*,-r}(X\times X) \\ & \searrow\eta\cdot & & \searrow\eta\cdot & \uparrow d_r \\ & & E_r^{*,-1} & \xrightarrow{\psi} & E_r^{*,-1}(X\times X) \end{array}$$

As the external product map $\times$ is an epimorphism by the diagram (1—1), $\psi(\eta\cdot x)\in E_r^{*,-1}(X\times X)$ can be expressed as:

$$\psi(\eta\cdot x)=\eta\cdot x\otimes 1+1\otimes\eta\cdot x+\sum x'\otimes x''\quad (x'\in E_r^{*,-1}(X))$$

($\times$ is omitted.) By assumption on p, $d_rx'=d_rx''=0$, thus we have $d_r\psi(\eta\cdot x)=d_r(\eta\cdot x)\otimes 1+1\otimes d_r(\eta\cdot x)$ and $\eta\cdot d_r\psi(x)=\eta\cdot(d_rx\otimes 1+1\otimes d_rx)$. Since the multiplication by $\eta: E_r^{*,1-r}(X\times X)\to E_r^{*,-r}(X\times X)$ is a monomorphism, we obtain $d_r\psi(x)=d_rx\otimes 1+1\otimes d_rx=\psi(d_rx)$. It follows that $\iota(d_rx)$ is primitive as an element of A.

§ 2. Type CI, DIII and BDI

In this section we show the collapsing of the Atiyah-Hirzebruch spectral sequence for the spaces of the classical types. (For the case of $M_{m,n}$, it is done in [6].)

First we consider the space of type CI, $Sp(n)/U(n)$. Recall that the modulo 2 cohomology of $Sp(n)/U(n)$ is:

(2—1) $\quad H^*(Sp(n)/U(n))\cong\wedge(c_1, c_2, \cdots, c_n),$

where c_i is the i-th Chern class.

Define the differential submodules M_j of $H^*(Sp(n)/U(n))$ by

$$M_j=\wedge(c_{2j}, c'_{2j+1}),$$

where $c'_{2j+1}=Sq^2c_{2j}=c_{2j+1}+c_{2j}c_1$, $j\geq 1$. Let $m=[n/2]$, then we have

$$H^*(Sp(n)/U(n))\cong\begin{cases}\wedge(c_1, c_{2m})\otimes M_1\otimes M_2\otimes\cdots\otimes M_{m-1}, & \text{if } n=2m,\\ \wedge(c_1)\otimes M_1\otimes M_2\otimes\cdots\otimes M_m, & \text{if } n=2m+1.\end{cases}$$

As $H(M_j; Sq^2)\cong\wedge([c_{2i}c'_{2i+1}])$, we have:

(2—2)

$$H(H^*(Sp(n)/U(n)); Sq^2)\cong\begin{cases}\wedge([c_1], [c_2c'_3], \cdots, [c_{2m-2}c'_{2m-1}]), & \text{if } n=2m,\\ \wedge([c_1], [c_2c'_3], \cdots, [c_{2m}c'_{2m+1}]), & \text{if } n=2m+1.\end{cases}$$

It is easy to see the case for $n=\infty$

$$H(H^*(Sp/U); Sp^2)\cong\wedge([c_1], [c_2c'_3], \cdots, [c_{2j}c'_{2j+1}], \cdots).$$

Since Sp/U has a homotopy commutative Hopf space structure (in fact, it is an infinite loop space) and all generators have the form of $[c_{2j}c'_{2j+1}]$, whose degrees are in $8j+2$. By Lemma 1.1, if the Atiyah-Hirzebruch spectral sequence has nontrivial differentials d_r, for the smallest r, there must be elements of $H(H^*(Sp/U); Sq^2)$, x, y, corresponding to the source and the target of d_r respectively, such that $deg\, y-deg\, x=r\equiv 2 \pmod 8$ and x is a generator and y is a primitive element. But, as the all primitive elements are in the dimensions of the generators [**Prop. 4.23**, 8], that is impossible. Thus $E_r^{*,*}(Sp/U)$ collapses for $r\geq 3$. Since the canonical map $E_r^{*,*}(Sp/U)\to E_r^{*,*}(Sp(n)/U(n))$ is an epimorphism. $E_r^{*,*}(Sp(n)/U(n))$, $r\geq 3$ collapses. Thus we have:

THEOREM 2.1. *The Atiyah-Hirzebruch spectral sequence for KO theory of $Sp(n)/U(n)$ collapses for $r\geq 3$ and*

$$E_\infty^{*,-1}(Sp(n)/U(n))\cong\wedge(x_2, x_{10}, \cdots, x_{8k+2}),$$

where $k=\left[\frac{n-1}{2}\right]$ *and* $\deg x_i=i$.

Next we consider the space of type DIII, $SO(2n)/U(n)$. It is known that ([5]):

$$(2-3)\qquad H^*(SO(2n)/U(n))\cong\triangle(e_2, e_4, \cdots, e_{2n-2}),\ \deg e_i=i$$

where $e_{2i}{}^2=e_{4i}$, and $e_{2j}=0(j\geq n)$, and the action of Sq^2 is given by $Sq^2e_{2i}=i\cdot e_{2i+2}$.

Define the differential submodules M_i of $H^*(SO(2n)/U(n))$ by

$$M_i=Z_2\langle 1, e_{4i-2}, e_{4i}, e'_{8i-2}\rangle,$$

where $e'_{8i-2}=e_{4i-2}e_{4i}+e_{8i-2}$. Then $H^*(SO(2n)/U(n))$ splits as

$$H^*(SO(2n)/U(n))=\begin{cases}M_1\otimes M_2\otimes\cdots\otimes M_{m-1}\otimes\wedge(e_{4m-2}) & \text{, if } n=2m,\\ M_1\otimes M_2\otimes\cdots\otimes M_m & \text{, if } n=2m+1.\end{cases}$$

Using $Sq^2e_{4i-2}=e_{4i}$, $Sq^2e'_{8i-2}=0$, we get $H(M_i;Sq^2)=Z_2\langle 1, [e_{8i-2}]\rangle$. Thus we obtain

(2—4)

$$H(H^*(SO(2n)/U(n));Sq^2)=\begin{cases}\triangle([e'_6], [e'_{14}], \cdots, [e'_{8m-10}], [e_{4m-2}]), & \text{if } n=2m,\\ \triangle([e'_6], [e'_{14}], \cdots, [e'_{8m-2}]), & \text{if } n=2m+1.\end{cases}$$

On the other hand, since $e'_{8i-2}{}^2=Sq^2(e_{8i-6}e_{8i}+e_{16i-6})$, we have $[e'_{8i-2}]^2=0$, and the algebra of (2—4) is an external algebra.

Now we show the Atiyah-Hirzebruch spectral sequence $E_r^{*,*}$ collapses for $r>3$. As the similar way for the type CI, consider the case $n=\infty$, then the space SO/U is a Hopf space and $H((H^*(SO(2n)/U(n));Sq^2)\cong\wedge_{j\geq 1}([e'_{8j-2}])$. Since the generators and primitives are concentrated in the degrees $\{8j-2\}$, by Lemma 1.1, there is no nontrivial differential d_r on $E_r^{*,*}(SO/U)$ for $(r>3)$. For finite n, consider the map

$$E_r^{*,*}(SO/U)\to E_r^{*,*}(SO(2n)/U(n)),$$

which is induced by inclusion $SO(2n)/U(n)\to SO/U$. The elements $[e'_{8j-2}]$ in (2—4) is in the image of this map. Therefore the possible nontrivial differential in the minimal degree occurs only on $[e_{4m-2}]$. We show the next lemma later.

LEMMA 2.2. $[e_{4m-2}]\in E_3^{*,-1}(SO(4m)/U(2m))$ *is a permanent cycle.*

This completes the proof of this theorem.

THEOREM 2.3. *The Atiyah-Hirzebruch spectral sequence for KO theory of* $SO(2n)/U(n)$ *collapses for* $r\geq 3$ *and*

$$E_\infty^{*,-1}(SO(2n)/U(n))\cong\begin{cases}\wedge(x_6, x_{14}, \cdots, x_{8m-10}, y_{4m-2}), & \text{if } n=2m,\\ \wedge(x_6, x_{14}, \cdots, x_{8m-2}), & \text{if } n=2m+1\end{cases}$$

where deg $x_i=i$ *and deg* $y_i=i$.

Lastly we consider the space of type BDI:

$$Q_n=SO(n+2)/(SO(n)\times SO(2))$$

Let $t\in H^2(BSO(2);Z)$ be the canonical generator and put $t=\iota^*(1\times t)\in H^*(Q_n;Z)$, where ι comes from the fibration:

$$Q_n \xrightarrow{\iota} BO(n)\times BO(2)\longrightarrow BO(n+2).$$

When $n=2m$, let $\chi\in H^{2m}(BSO(2m);Z)$ be the Euler class. There is an element $s_{2m}\in H^{2m}(Q_{2m};Z)$ such that $2s_{2m}=\iota^*(\chi\times 1+1\times t^n)$. The same symbol $s_{2m}\in H^{2m}(Q_{2m})$ denotes its modulo 2 reduction. When $n=2m-1$, $s_{2m}\in H^{2m}(Q_{2m-1})$ denotes its image by the map induced by the inclusion i: $Q_{2m-1}\to Q_{2m}$.

It is known that ([5]):

$$(2\text{—}5)\qquad H^*(Q_n)=\begin{cases}Z_2[t, s_{4m}]/(t^{2m+1}, s_{4m}{}^2-s_{4m}t^{2m}), & \text{if } n=4m,\\ Z_2[t, s_{4m}]/(t^{2m}, s_{4m}{}^2), & \text{if } n=4m-1,\\ Z_2[t, s_{4m-2}]/(t^{2m}, s_{4m-2}{}^2), & \text{if } n=4m-2,\\ Z_2[t, s_{4m-2}]/(t^{2m-1}, s_{4m-2}{}^2), & \text{if } n=4m-3.\end{cases}$$

and

$$Sq^2 s_{2k}=(k+1)s_{2k}t.$$

From this, we can easily compute the Sq^2 cohomology of them.

$$(2\text{—}6)\qquad H(H^*(Q_n);Sq^2)=\begin{cases}\wedge[t^{2m}s_{4m}]), & \text{if } n=4m,\\ \wedge[t^{2m-1}]), & \text{if } n=4m-1,\\ \wedge[t^{2m-1}], [s_{4m-2}]), & \text{if } n=4m-2,\\ \wedge([s_{4m-2}]), & \text{if } n=4m-3.\end{cases}$$

In the cases other than $n=4m-2$, it is trivial that the Atyah-Hirzegruch spectral sequence $E_r^{*,*}(r\geq 3)$ collapses. If $n=4m-2$, the inclusion map $i: Q_{4m-2}\to Q_{4m-1}$ maps $[t^{2m-1}]$ to $[t^{2m-1}]$, thus we see $[t^{2m-1}]$

is a permanent cycle. Therefore it is enough to show that:

LEMMA 2.4. *In* $H(H^*(Q_{4m-2});Sq^2)$, $[s_{4m-2}]$ *is a permanent cycle.*

We demonstrate it later. Thus we have

THEOREM 2.5. *The Atiyah-Hirzebruch spectral sequence for KO theory of* Q_n *collapses for* $r\geq 3$ *and*

$$E_\infty^{*,-1}(Q_n)\cong\begin{cases}\wedge(z_{8m}), & \text{if } n=4m,\\ \wedge(x_{4m-2}), & \text{if } n=4m-1,\\ \wedge(x_{4m-2},\, y_{4m-2}), & \text{if } n=4m-2,\\ \wedge(y_{4m-2}), & \text{if } n=4m-3.\end{cases}$$

where $\deg x_i=\deg y_i=\deg z_i=i$

Now we prove Lemma 2.2 and Lemma 2.4 simultaneously.

PROOF OF LEMMA 2.2. AND LEMMA 2.4: Consider the diagram:

$$\begin{array}{ccc} SO(4m)/U(2m-1)\times U(1) & \xrightarrow{q} & SO(4m)/U(2m) \\ {\scriptstyle p}\downarrow & & \\ Q_{4m-2}=SO(4m)/SO(4m-2)\times SO(2) & & \end{array}$$

where p and q are the canonical maps. It is easy to see that $H^*(q)$ is an injection and $H^*(SO(4m)/U(2m-1)\times U(1))=H^*(SO(4m)/U(2m))\otimes \boldsymbol{Z}_2[t]/(t^{2m})$, $\deg t=2$.

Apply $H(H^*(\quad);Sq^2)$ to that diagram:

$$\begin{array}{ccc} \wedge([e'_6],\cdots,[e'_{8m-10}],\,[e_{4m-2}])\otimes\wedge([t^{2m-1}]) & \xleftarrow{q^*} & \wedge([e'_6],\cdots,[e'_{8m-10}],\,[e_{4m-2}]) \\ {\scriptstyle p^*}\uparrow & & \\ \wedge([t^{2m-1}],\,[s_{4m-2}]). & & \end{array}$$

Here p^* and q^* are monomorphisms, and by [5],

$$\begin{aligned} p^*s_{4m-1}&=\sum_{i=0}^{2m-2} e_{4m-2-2i}t^i \\ &=e_{4m-2}+\sum_{i=0}^{m-2} Sq^2(e_{4m-6-4i}t^{2i+1}). \end{aligned}$$

Thus we obtain $p^*[s_{4m-2}]=[e_{4m-2}]$, and we can take t so as satinfy $p^*t=t$.

Suppose there is a nontrivial differential on $E_r^{*,*}(Q_{4m-2})(r\geq 3)$. Then, by the differential, $[s_{4m-2}]$ corresponds to $[t^{2m-1}][s_{4m-2}]$. Thus $[e_{4m-2}]$ corresponds to $[t^{2m-1}][e_{4m-2}]$ in $H(H^*(SO(4m)/U(2m-1)\times U(1));Sq^2)$. On the other hand, $[e_{4m-2}]\in \operatorname{Im} q^*$, but $[t^{2m-1}][e_{4m-2}]\notin \operatorname{Im} q^*$, this is a contradiction. Therefore $[s_{4m-2}]$ is a permanent cycle.

By the same way, we can see the $[e_{4m-2}]$ is a permanent cycle.

§ 3. Exceptional types

The integral cohomology ring of *EIII* is obtained in [9], and the modulo 2 reduction is:

(3—1) $$H^*(EIII)=Z_2[t, w]/(t^9+w^2t, w^3+w^2t^4+wt^8),$$

where deg $w=8$ and deg $t=2$, and by [5], $Sq^2w=wt+t^5$.

Let $w'=w+t^4$, then

$$H^*(EIII)=\boldsymbol{Z}_2[t, w']/(w'^2t, w'^3+t^{12}), \text{ with } Sq^2w'=w't.$$

Thus we have

(3—2) $$H(H^*(EIII); Sq^2))=\boldsymbol{Z}_2[[w'^2]]/([w'^2]^3).$$

Its generators exist only in the degree 0 modulo 8. Lemma 1.1 assert that the Atiyah-Hirzebruch spectral sequence collapses. We obtain:

THEOREM 3.1. *The Atiyah-Hirzebruch spectral sequence for KO theory of EIII collapses for* $r\geq 3$ *and*

$$E_\infty^{*,-1}(EIII)\cong Z_2[x_{16}]/(x_{16}^3),$$

where deg $x_{16}=16$.

Lastly we consider the case *EVII*. From the result of [10], we have

(3—3) $$H^*(EVII)=Z_2[u, v, w]/(u^{14}, v^2, w^2),$$

where deg $u=2$, deg $v=10$, deg $w=18$, and the actions of cohomology operations are determined in [5],

$$Sq^2v=0, \quad Sq^4v=vu^2+u^7, \quad Sq^2v=w+vu^4+u^9,$$
$$Sq^2w=u^{10}, \quad Sq^4w=vu^6+u^{11}, \quad Sq^8w=vu^8+u^{13}, \quad Sq^{16}w=vu^{12}.$$

Let $w'=w+u^9$, then

$$H^*(EVII)=Z_2[u, v, w']/(u^{14}, v^2, w'^2), \text{ with } Sq^2v=0 \text{ and } Sq^2w'=0.$$

Thus we have

(3—4) $$H(H^*(EVII); Sq^2))=\wedge([u^{13}], [v], [w']).$$

In this algebra the generators are in degrees 26, 10 and 18. So we cannot apply Lemma 1.1 directly to this case. To use the Hopf algebra structure, consider a generating map

$$g : EVII \to \Omega E_7$$

which makes $EVII$ a generating variety of ΩE_7. In [1], Bott showes:

PROPOSITION 3.2. *Im $[g^* : H_*(EVII) \to H_*(\Omega E_7)]$ generates the Pontrjagin ring.*

On the other hand, ΩE_7 is a homotopy commutative Hopf space and $H_*(\Omega E_7)$ is completely given in [7]:

PROPOSITION 3.3.
(1) $H_*(\Omega E_7) \cong \wedge(x_2, x_4, x_8) \otimes \boldsymbol{Z}_2[x_{10}, x_{14}, x_{16}, x_{18}, x_{22}, x_{26}, x_{34}]$.
(2) *For the coproduct ϕ,*

$$\begin{aligned}
\phi x_4 &= x_4 \otimes 1 + x_2 \otimes x_2 + 1 \otimes x_4,\\
\phi x_8 &= x_8 \otimes 1 + x_2 x_4 \otimes x_2 + x_2 \otimes x_2 x_4 + 1 \otimes x_8,\\
\phi x_{16} &= x_{16} \otimes 1 + x_2 x_4 x_8 \otimes x_2 + x_4 x_8 \otimes x_4 + x_2 x_8 \otimes x_2 x_4 + x_8 \otimes x_8\\
&\quad + x_2 x_4 \otimes x_2 x_8 + x_4 \otimes x_4 x_8 + x_2 \otimes x_2 x_4 x_8 + 1 \otimes x_{16}.
\end{aligned}$$

Other generators are primitive.
(3) *The dual operations are completely determined by:*

$$\begin{array}{llll}
Sq_*^2 x_4 = x_2, & Sq_*^2 x_8 = x_2 x_4, & Sq_*^2 x_{16} = x_{14} + x_2 x_4 x_8, & Sq_*^2 x_{22} = x_{10}^2,\\
Sq_*^2 x_{34} = x_{16}^2, & & & \\
Sq_*^4 x_8 = x_4, & Sq_*^4 x_{14} = x_{10}, & Sq_*^4 x_{16} = x_4 x_8, & Sq_*^4 x_{26} = x_{22},\\
Sq_*^8 x_{16} = x_8, & Sq_*^8 x_{18} = x_{10}, & Sq_*^8 x_{22} = x_{14}, & Sq_*^8 x_{26} = x_{18},\\
\textit{otherwise}, & Sq_*^{2^i} x_j = 0, & &
\end{array}$$

We can easily compute its dual Hopf algebra from this. Let w_i be the dual element of x_i for the monomial basis of x_i's, (exceptionally w_{32} be the dual to x_{16}^2).

PROPOSITION 3.4.
(1) $H^* \Omega E_7 \cong Z_2[w_2]/(w_2^{16}) \otimes \Gamma(w_{10}, w_{14}, w_{18}, w_{22}, w_{26}, w_{32}, w_{34})$,
where $\Gamma(w)$ denotes the divided power algebra which has addtive basis $\{\gamma_n(w)\}$.

(2) *The generators indicated above except w_{32} are primitive.*
(3) *The cohomology operations are given by:*

$$\begin{array}{llll}
Sq^2 w_2 = w_2^2, & Sq^2 w_{14} = w_2^8, & Sq^2 \gamma_2(w_{10}) = w_{22}, & Sq^2 w_{32} = w_{34},\\
Sq^4 w_4 = w_8, & Sq^4 w_{10} = w_{14}, & Sq^4 w_{22} = w_{26}, & \\
Sq^8 w_8 = w_{16}, & Sq^8 w_{10} = w_{18}, & Sq^8 w_{14} = w_{22}, & Sq^8 w_{18} = w_{26}.
\end{array}$$

If x is the dual element of the generator of $H_*(\Omega E_7)$, then $g^*(x)$ is the non zero element, because Im g_* generates $H_*(\Omega E_7)$. We can determine

g_* as follows.

For dimensional reasons, $g^*(w_2)=u$. Since $g^*(w_{10})^2=g^*(w_{10}{}^2)=0$, we have:

$$(3-5) \qquad g^*(w_{10})=v.$$

From this g^* is determined, using squaring operations:

$$(3-6) \qquad \begin{aligned} &g^*(w_{14})=g^*(Sq^4 w_{10})=Sq^4 g^*(w_{10})=Sq^4 v=vu^2+u^7,\\ &g^*(w_{18})=g^*(Sq^8 w_{10})=Sq^8 g^*(w_{10})=Sq^8 v=w'+vu^4,\\ &g^*(w_{22})=g^*(Sq^8 w_{14})=Sq^8 g^*(w_{14})=Sq^8(vu^2+u^7)=w'u^2,\\ &g^*(w_{26})=g^*(Sq^4 w_{22})=Sq^4 g^*(w_{22})=Sq^4(w'u^2)=w'u^4+vu^8+u^{13}.\end{aligned}$$

By the way, from Proposition 3.4, we get the next isomorphism of the Hopf algebra.

(3—7)
$$H(H^*(\Omega E_7\} ;Sq^2))\cong\wedge([w_{10}],[w_{14}+w_2{}^7])\otimes\Gamma([w_{18}],[w_{26}],[\gamma_2(w_{22})],[\gamma_2(w_{34})]).$$

From (3—5) and (3—6), we have the correspondence of elements of Sq^2-cohomology groups.

$$(3-8) \qquad \begin{aligned} &H(g^*\,;\,Sq^2)([w_{10}])=[v],\\ &H(g^*\,;\,Sq^2)([w_{18}])=[w'+vu^4]=[w'],\\ &H(g^*\,;\,Sq^2)([w_{26}])=[w'u^4+vu^8+u^{13}]=[u^{13}].\end{aligned}$$

Suppose that there is a nontrivial differential of $E_r^{*,*}(EVII)$. By Lemma 1.1 and (3—4), it is given by:

$$d_r\alpha=\beta, \text{ with deg } \alpha=10 \text{ or } 18 \text{ or } 26, \text{ and deg } \beta=28 \text{ or } 36 \text{ or } 44.$$

Because $H(g^*\,;Sq^2)$ is epimorphic by (3—8), the differential must occur in the same dimensions of $E_r^{*,*}(\Omega E_7)$. Again by Lemma 1.1, the target must be primitive. Thus by (3—7), we conclude that:

$[\gamma_2(w_{22})]$ *is hit by* $[w_{10}]$ *or* $[w_{18}]$ *or* $[w_{26}]$.

To show that this is impossible, it is enough to prove that $[x_{22}{}^2]$ is a permanent cycle of the dual Atiyah-Hizebruch spectral sequence $E^r_{*,*}(\Omega E_7)$ for $KO_*(\Omega E_7)$. Here, we quote the result on $H_*(\Omega F_4)$ from [7] again.

PROPOSITION 3.5. $H_*(\Omega F_4)\cong\wedge(x_2)\otimes Z_2[x_4, x_{10}, x_{14}, x_{22}]$.
$Sq^2_* x_4=x_2$, $Sq^2_* x_{10}=x_4{}^2$, $Sq^2_* x_{22}=x_{10}^2$.

Thus we have

$$(3-9) \qquad E^3_{*,*}(\Omega F_4)\cong H(H_*(\Omega F_4)\,;\,Sq^2_*)\cong Z_2[[x_{14}],[x_{22}{}^2]].$$

As we discussed in Lemma 1.1, this spectral sequence $E^r_{*,*}(\Omega F_4)$ collapse for $r \geq 3$, by dimensional reason. So $[x_{22}^2]$ is a permanent cycle. By the canonical inclusion $F_4 \xrightarrow{i} E_7$, x_{22} maps to x_{22}. Hence $[x_{22}^2]$ is a permanent cycle in $E^r_{*,*}(\Omega E_7)$. This completes the proof of the next theorem.

THEOREM 3.6. *The Atiyah-Hirzebruch spectral sequence for KO theory of EVII collapses for* $r \geq 3$ *and*

$$E_\infty^{*,-1}(EVII) \cong \wedge(x_{10}, x_{18}, x_{26}),$$

where $\deg x_i = i$.

§ 4. Proof of Corollary 2 and lists

Suppose X is a finite complex such that $H^*(X; \boldsymbol{Z})$ has no torsion and no odd dimensional part. By the similar arguments of Lemma 2.1 and 2.2 of [4] we have

$$(4\text{—}1) \qquad \begin{aligned} KO^{2i+1}(X) &\cong s\boldsymbol{Z}_2, \\ KO^{2i}(X) &\cong r\boldsymbol{Z} \otimes s\boldsymbol{Z}_2 \end{aligned}$$

for some r and s, and

$$(4\text{—}2) \qquad \begin{aligned} \text{rank } KO^0(X) &= \text{rank } KO^{-4}(X) = t_0, \\ \text{rank } KO^{-2}(X) &= \text{rank } KO^{-6}(X) = t_1. \end{aligned}$$

By (4—1) the extension of $\oplus_{p+q=2i+1} E_\infty^{p,q}$ to $KO^{2i+1}(X)$ is trivial. Thus if X is a compact irreducible Hermitian symmetric space, we have

$$\begin{aligned} \dim_{Z_2} KO^{2i+1}(X) &= \dim_{Z_2} \bigoplus_{p+q=2i+1} E_\infty^{p,q} \\ &= \dim_{Z_2} \bigoplus_{p \equiv 2i+2 (\mathrm{mod}\ 8)} E_\infty^{p,-1} \\ &= \dim_{Z_2} \bigoplus_{p \equiv 2i+2 (\mathrm{mod}\ 8)} H^p(H^*(X); Sq^2) \\ &= s_{i+1}. \end{aligned}$$

The proof of Corollary 2 is done.

Now we list the order of $KO^*(X)$, which is determined by t_δ and s_ε as in Corollary 2.

We prepare a lemma for the cases $X = Sp(n)/U(n)$ and $X = SO(2n)/U(n)$. Let R_n^* be the exterior algebra over $\boldsymbol{Z}_2$ defined by

$$R_n^* = \wedge(e_1, e_2, \cdots, e_n), \text{ with } \deg e_i \equiv 1 \pmod 4,$$

and

$$\rho(n, i) = \dim \bigoplus_{d \equiv i (\mathrm{mod}\ 4)} R_n^d.$$

Of course, $\rho(n, i)$ equals to $\sum_{d\equiv i (\mathrm{mod}\, 4)} \binom{n}{d}$. But to get more concrete description, we solve the next equations:

$$\rho(1,0)=\rho(1,1)=1,\ \rho(1,2)=\rho(1,3)=0,$$
$$\rho(n+1, i)=\rho(n, i-1)+\rho(n, i).$$

We have

PROPOSITION 4.1.

$$\rho(n,0)=2^{n-2}+\frac{\sqrt{-1}}{2}(\alpha^{n-2}-\beta^{n-2}),$$
$$\rho(n,1)=2^{n-2}+\frac{1}{2}(\alpha^{n-2}+\beta^{n-2}),$$
$$\rho(n,2)=2^{n-2}-\frac{\sqrt{-1}}{2}(\alpha^{n-2}-\beta^{n-2}),$$
$$\rho(n,3)=2^{n-2}-\frac{1}{2}(\alpha^{n-2}+\beta^{n-2}),$$

where $\alpha=1+\sqrt{-1}$, $\beta=1-\sqrt{-1}$, *thus we have*

n	$\rho(n,0)$	$\rho(n,1)$	$\rho(n,2)$	$\rho(n,3)$
$4k$	$2^{4k-2}+(-1)^k 2^{2k-1}$	2^{4k-2}	$2^{4k-2}-(-1)^k 2^{2k-1}$	2^{4k-2}
$4k+1$	$2^{4k-1}+(-1)^k 2^{2k-1}$	$2^{4k-1}+(-1)^k 2^{2k-1}$	$2^{4k-1}-(-1)^k 2^{2k-1}$	$2^{4k-1}-(-1)^k 2^{2k-1}$
$4k+2$	2^{4k}	$2^{4k}+(-1)^k 2^{2k}$	2^{4k}	$2^{4k}-(-1)^k 2^{2k}$
$4k+3$	$2^{4k+1}-(-1)^k 2^{2k}$	$2^{4k+1}+(-1)^k 2^{2k}$	$2^{4k+1}+(-1)^k 2^{2k}$	$2^{4k+1}-(-1)^k 2^{2k}$

From this, in the case $X=Sp(n)/U(n)$, we get s_ε's by Theorem 2.1. t_δ's are obtained by (2—1).

THEOREM 4.2. *For* $X=Sp(n)/U(n)$,

$$t_0=t_1=2^{n-1},$$
$$s_\varepsilon=\rho\left(\left[\frac{n+1}{2}\right], \varepsilon\right).$$

n	s_0	s_1	s_2	s_3
$8k$	$2^{4k-2}+(-1)^k 2^{2k-1}$	2^{4k-2}	$2^{4k-2}-(-1)^k 2^{2k-1}$	2^{4k-2}
$8k+1$	$2^{4k-1}+(-1)^k 2^{2k-1}$	$2^{4k-1}+(-1)^k 2^{2k-1}$	$2^{4k-1}-(-1)^k 2^{2k-1}$	$2^{4k-1}-(-1)^k 2^{2k-1}$
$8k+2$	$2^{4k-1}+(-1)^k 2^{2k-1}$	$2^{4k-1}+(-1)^k 2^{2k-1}$	$2^{4k-1}-(-1)^k 2^{2k-1}$	$2^{4k-1}-(-1)^k 2^{2k-1}$
$8k+3$	2^{4k}	$2^{4k}+(-1)^k 2^{2k}$	2^{4k}	$2^{4k}-(-1)^k 2^{2k}$
$8k+4$	2^{4k}	$2^{4k}+(-1)^k 2^{2k}$	2^{4k}	$2^{4k}-(-1)^k 2^{2k}$
$8k+5$	$2^{4k+1}-(-1)^k 2^{2k}$	$2^{4k+1}+(-1)^k 2^{2k}$	$2^{4k+1}+(-1)^k 2^{2k}$	$2^{4k+1}-(-1)^k 2^{2k}$
$8k+6$	$2^{4k+1}-(-1)^k 2^{2k}$	$2^{4k+1}+(-1)^k 2^{2k}$	$2^{4k+1}+(-1)^k 2^{2k}$	$2^{4k+1}-(-1)^k 2^{2k}$
$8k+7$	$2^{4k+2}-(-1)^k 2^{2k+1}$	2^{4k+2}	$2^{4k+2}+(-1)^k 2^{2k+1}$	2^{4k+2}

When $X=SO(2n)/U(n)$, the next result is obtained by the similar computation as above from Theorem 2.3 and (2—3).

THEOREM 4.3. *For* $X=SO(2n)/U(n)$,

$$t_0=t_1=2^{n-2},$$

$$s_\varepsilon=\begin{cases}\rho\left(\left[\dfrac{n}{2}\right],1-\varepsilon\right), & \textit{if } n\equiv 2 \pmod 4,\\ \rho\left(\left[\dfrac{n}{2}\right],-\varepsilon\right), & \textit{otherwise.}\end{cases}$$

n	s_0	s_1	s_2	s_3
$8k$	$2^{4k-2}+(-1)^k 2^{2k-1}$	2^{4k-2}	$2^{4k-2}-(-1)^k 2^{2k-1}$	2^{4k-2}
$8k+1$	$2^{4k-2}+(-1)^k 2^{2k-1}$	2^{4k-2}	$2^{4k-2}-(-1)^k 2^{2k-1}$	2^{4k-2}
$8k+2$	$2^{4k-1}+(-1)^k 2^{2k-1}$	$2^{4k-1}+(-1)^k 2^{2k-1}$	$2^{4k-1}-(-1)^k 2^{2k-1}$	$2^{4k-1}-(-1)^k 2^{2k-1}$
$8k+3$	$2^{4k-1}+(-1)^k 2^{k-1}$	$2^{4k-1}-(-1)^k 2^{2k-1}$	$2^{4k-1}-(-1)^k 2^{2k-1}$	$2^{4k-1}+(-1)^k 2^{2k-1}$
$8k+4$	2^{4k}	$2^{4k}-(-1)^k 2^{2k}$	2^{4k}	$2^{4k}+(-1)^k 2^{2k}$
$8k+5$	2^{4k}	$2^{4k}-(-1)^k 2^{2k}$	2^{4k}	$2^{4k}+(-1)^k 2^{2k}$
$8k+6$	$2^{4k+1}+(-1)^k 2^{2k}$	$2^{4k+1}-(-1)^k 2^{2k}$	$2^{4k+1}-(-1)^k 2^{2k}$	$2^{4k+1}+(-1)^k 2^{2k}$
$8k+7$	$2^{4k+1}-(-1)^k 2^{2k}$	$2^{4k+1}-(-1)^k 2^{2k}$	$2^{4k+1}+(-1)^k 2^{2k}$	$2^{4k+1}+(-1)^k 2^{2k}$

When $X=Q_n$, by (2—5) and Theorem 2.5, we have

THEOREM 4.4. *For* $X=Q_n$,

n	t_0	t_1	s_0	s_1	s_2	s_3
$8k$	$4k+2$	$4k$	2	0	0	0
$8k+1$	$4k+1$	$4k+1$	1	1	0	0
$8k+2$	$4k+2$	$4k+2$	1	2	1	0
$8k+3$	$4k+2$	$4k+2$	1	1	0	0
$8k+4$	$4k+4$	$4k+2$	2	0	0	0
$8k+5$	$4k+3$	$4k+3$	1	0	0	1
$8k+6$	$4k+4$	$4k+4$	1	0	1	2
$8k+7$	$4k+4$	$4k+4$	1	0	0	1

For $X=EIII$, by (3—1) and Theorem 3.1 and for $X=EVII$, by (3—3) and Theorem 3.6, we have the following table.

THEOREM 4.5. *For the exceptional types,*

X	t_0	t_1	s_0	s_1	s_2	s_3
$EIII$	15	12	3	0	0	0
$EVII$	28	28	1	3	3	1

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