Hokkaido Mathematical Journal Vol. 22 (1993) p. 99-103

Note on even tournaments whose automorphism groups contain regular subgroups

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(Received May 26, 1992)

§ 1. Introduction

A (0, 1)-matrix A of degree v is called a tournament of order v if A satisfies the following equation

(1) $\qquad A+A^{t}+I=J,$

where t denotes the transposition, and I and J are the identity and all one matrices of degree v respectively. In other words, a tournament is the adjacency matrix of a complete asymmetric digraph.

A tournament A is called even if the inner product of any two distinct row vectors of A is even.

A permutation matrix P such that $P^{t}AP=A$ is called an automorphism of A. The multiplicative group $\mathfrak{G}(A)$ of all automorphisms of A is called the automorphism group of A.

In the present note we consider a tournament A such that $\mathfrak{G}(A)$ contains a regular subgroup $\mathfrak{G}$. In previous two notes we considered the case where $\mathfrak{G}$ is cyclic (**1**) and (**2**). In such a case A is called a cyclic tournament. We obtained the following result in (**2**).

THEOREM. *An even cyclic tournament of order v exists if and only if v satisfies one of the following conditions*: (i) *v is congruent to* 3 *modulo* 8 *and the order of* 2 *modulo every prime divisor of v is singly even, where an even integer n is called singly even if n is not divisible by* 4; (ii) *v is cogruent to* 1 *modulo* 8 *and the order of* 2 *modulo every prime divisor of v is odd.*

Now since $\mathfrak{G}$ is regular, we label rows and columns of A by elements of $\mathfrak{G}$ so that

(2) $\qquad A=(A(a, b))$, where a and b are elements of $\mathfrak{G}$,

and

(3) $\qquad A(ac, bc)=A(a, b)$, where c runs over all elements of $\mathfrak{G}$.

Obviously A is regular, namely each row $A(a)$ of A contains the same number of 1's, say k. Then it holds that

(4) $\quad v=2k+1.$

Moreover A is completely determined by its first row $A(e)$, where e is the identity element of $\mathfrak{G}$. We identify $A(e)$ with its support $\mathfrak{D}$, namely the set of elements a of $\mathfrak{G}$ such that $A(e, a)=1$. So $\mathfrak{D}$ consists of k elements of $\mathfrak{G}$.

In the present note we show that the above mentioned theorem holds good for an arbitrary group $\mathfrak{G}$ of order v, provided that we choose $\mathfrak{D}$ normal in $\mathfrak{G}$, namely $\mathfrak{D}$ satisfies the condition $a^{-1}\mathfrak{D}a=\mathfrak{D}$ for every element a of $\mathfrak{G}$.

We have to leave open the case where $\mathfrak{D}$ is not normal in $\mathfrak{G}$.

§ **2.**

LEMMA 1. (i) *e doe not belong to $\mathfrak{D}$.* (ii) *For $a\neq e$ exactly one of a and a^{-1} belongs to $\mathfrak{D}$.*

PROOF. It is straightforward.

We consider the collection $\mathfrak{D}(*)$ (namely multiplicity is counted) of elements of $\mathfrak{G}$ of the form $c^{-1}d$, where both c and d belong to $\mathfrak{D}$. Let $m(a)$ denote the multiplicity of an element a of $\mathfrak{G}$ in $\mathfrak{D}(*)$. Clearly it holds that $m(e)=k$.

LEMMA 2. *A tournament A is even if and only if $m(a)$ is even for every non-identity element a of $\mathfrak{G}$.*

PROOF. $m(a)$ equals the inner product $(A(e), A(a))$.

We say that $\mathfrak{D}$ is even if $m(a)$ is even for every non-identity elemens a of $\mathfrak{G}$.

LEMMA 3. *If $\mathfrak{D}$ is even, then it holds that*

(5) $\quad k^2-k\equiv 0 \quad (mod\ 4).$

PROOF. a and a^{-1} have the same multiplicity.

By (5) we distinguish two cases: (I) k is congruent to 1 modulo 4 and (II) k is divisible by 4.

First we treat the case (I). In the proof of the next lemma we require the assumption that $\mathfrak{D}$ is normal in $\mathfrak{G}$.

LEMMA 4. *$\mathfrak{D}$ is even if and only if exactly one of a and a^2 belongs to $\mathfrak{D}$ for every non-identity element a of $\mathfrak{G}$.*

PROOF. First assume that both a and a^2 belong to $\mathfrak{D}$. Then we show that $m(a^2)$ is odd. We say that an element d of $\mathfrak{D}$ is bad if da^{-2} does not belong to $\mathfrak{D}$. Under our assumption we show that the number of bad d's is even. Under our assumption we show that the number of bad d's is even. Since k is odd in case (I), this implies that $m(a^2)$ is odd. Now since both $aa^{-2}=a^{-1}$ and $a^2a^{-2}=e$ do not belong to $\mathfrak{D}$, both a and a^2 are bad. Moreover, if b is bad and if $b\neq a$, a^2, then a^2b^{-1} is also bad, since a^2b^{-1} belongs to $\mathfrak{D}$ and $a^2b^{-1}a^{-2}$ does not belong to $\mathfrak{D}$ by the normality of $\mathfrak{D}$.

Next assume that neither a nor a^2 belongs to $\mathfrak{D}$. This time we show that $m(a^{-2})$ is odd. Since both $a^{-1}a^2=a$ and $a^{-2}a^2=e$ do not belong to $\mathfrak{D}$, both a^{-1} and a^{-2} are bad. If b is bad and if $b\neq a^{-1}$, a^{-2}, then $a^{-2}b^{-1}$ is also bad, since $a^{-2}b^{-1}$ belongs to $\mathfrak{D}$ and $a^{-2}b^{-1}a^2$ does not belong to $\mathfrak{D}$ by the normality of $\mathfrak{D}$.

Conversely we assume that exactly one of a and a^2 belongs to $\mathfrak{D}$ for every non-identity element a of $\mathfrak{G}$. We notice that every non-identity element c of $\mathfrak{G}$ may be written in the form $c=a^2$ for some element a of $\mathfrak{G}$, since $\mathfrak{G}$ has odd order. So we may proceed as above and investigate $m(a^2)$. If a belongs to $\mathfrak{D}$ and a^2 does not belong to $\mathfrak{D}$, then, since $aa^{-2}=a^{-1}$ does not belong to $\mathfrak{D}$, a is bad. Moreover, if a^{-2} is bad, then a^4 is also bad, because $a^4a^{-2}=a^2$ does not belong to $\mathfrak{D}$. If a does not belong to $\mathfrak{D}$ and a^2 belongs to $\mathfrak{D}$, then, since $a^2a^{-2}=e$ does not belong to $\mathfrak{D}$, a^2 is bad. Moreover, if a^{-1} is bad, then a^3 is also bad, because $a^3a^{-2}=a$ does not belong to $\mathfrak{D}$.

LEMMA 5. *Let $\mathfrak{D}$ be even. If an elemens a of $\mathfrak{G}$ belongs to $\mathfrak{D}$, then a^{-2} also belongs to $\mathfrak{D}$.*

PROOF. This is immediate by Lemma 4.

LEMMA 6. *If there exists a prime divisor p of v such that 2 modulo p has order divisible by 4 or odd, then there exists no even tournament of order v whose automorphism group containts a regular subgroup.*

PROOF. Assume the contrary. We use the same notation as above. Let a be an element of $\mathfrak{D}$ of order p. Using Lemma 5 repeatedly, we see that $a^{(-1)^n 2^n}$ belongs to $\mathfrak{D}$. Now assume that the order of 2 modulo p equals $4m$. Then put $n=2m$. It follows that $a^{2^{2m}}=a^{-1}$ belongs to $\mathfrak{D}$, which is a contradiction. Next assume that the order of 2 modulo p equals $2m+1$. Then put $n=2m+1$. It follows that $a^{-2^{2m+1}}=a^{-1}$ belongs to $\mathfrak{D}$, which is a contradiction.

THEOREM 1. *If the order of 2 is singly even modulo every prime divisor of v, then there exists a tournament of order v whose automorphism group contains a regular subgroup which is isomorphic to an arbitrarily given group $\mathfrak{G}$ of order v.*

PROOF. Let c and d be elements of $\mathfrak{G}$. Then we say that d is equivalent to c if and only if there exists a non-negative integer n such that $d=c^{(-2)^n}$. It is easy to see that this is a true equivalence relation.

We show that for every non-identity element a of $\mathfrak{G}$ a and a^{-1} belong to distinct equivalence classes.

Now assume that for some non-identity element a of $\mathfrak{G}$ both a and a^{-1} belong to the same equivalence class. So there exists a positive integer m such that $a^{(-2)^m}=a^{-1}$. Let p be a prime divisor of the order of a. Then p is also a prime divisor of v. Now p divides $(-2)^m+1$. If m is odd, then the order of 2 modulo p divides m against our assumption. Hence m is even and we put $m=2n$. Now let $2u$ be the order of 2 modulo p. Then, by assumption, u is odd. Thus $u\neq 2n$. If $2n$ is bigger than u, then the order of 2 modulo p divides $2n-u$. If $2n$ is less than u, then the order of 2 modulo p divides $u-2n$. Since $2n-u$ and $u-2n$ are odd, we have a contradiction.

Thus equivalence classes of non-identity elements of $\mathfrak{G}$ are paired off. So if we pick up exactly one equivalence class from each pair and form a union $\mathfrak{D}$, then $\mathfrak{D}$ is even by Lemma 4.

REMARK 1. The normality of $\mathfrak{D}$ is not needed in the proof of Theorem 1. So the following question arises. Does a new order v appear, if we put aside the normality of $\mathfrak{D}$ after all?

Secondly we treat the case (II). We notice that k is a multiple of 4 in this case.

LEMMA 7. *$\mathfrak{D}$ is even if and only if for every non-identity element a of $\mathfrak{G}$ both a and a^2 belong to $\mathfrak{D}$, or neither a nor a^2 belongs to $\mathfrak{D}$.*

PROOF. Bad elements in Lemma 4 are wanted here. The proof of Lemma 4 goes through.

LEMMA 8. *Let $\mathfrak{D}$ be even. If an element a of $\mathfrak{G}$ belongs to $\mathfrak{D}$, then a^2 also belongs to $\mathfrak{D}$.*

PROOF. This is immediate by Lemma 7.

LEMMA 9. *If there exists a prime divisor p of v such that 2 has even order modulo p, then there exists no even tournament of order v whose* *automorphism group contains a regular subgroup.*

PROOF. Let $2m$ be the order of 2 modulo p. Then 2^m+1 is divisible by p. Now assume the contrary and let a be an element of $\mathfrak{D}$ of order p. Then by Lemma 8 $a^{2^m}=a^{-1}$ belongs to $\mathfrak{D}$, which is a contradiction.

THEOREM 2. *If the order of 2 modulo every prime divisor of v is odd, then there exists an even tournament of order v whose automorphism group contains a regular subgroup which is isomorphic to an arbitrarily given group $\mathfrak{G}$ of order v.*

PROOF. Let c and d be elements of $\mathfrak{G}$. Then we say that d is equivalent to c if and only if there exists a non-negative integer n such that $d=c^{2^n}$. This is a true equivalence relation.

We show that for every non-identity a of $\mathfrak{G}$ a and a^{-1} belong to distinct equivalence classes.

Suppose that for some non-identity element a of $\mathfrak{G}$ a^{-1} is equivalent to a. Then there is a positive integer n such that $a^{2^n}=a^{-1}$. Let p be a prime divisor of the order of a. Then p is also a prime divisor of v. At any rate 2^n+1 is divisible by p. Now let u be the order of 2 modulo p. Then, by assumption, u is odd. Now $2n$ is a multiple of u. But this implies that n is a multiple of u, which is a contradiction.

Now we can complete the proof like Theorem 1.

REMARK 2. We can make the same question as in Remark 1.

References

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