Hokkaido Mathematical Journal Vol. 23 (1994) p. 319-353

On self-adjointness of Dirac operators in Boson-Fermion Fock spaces

Asao ARAI
(Received June 30, 1993)

Abstract: It is shown that a class of Dirac operators acting in the abstract Boson-Fermion Fock space, which were introduced in a previous paper (A. Arai, J. Funct. Anal. 105 (1992), 342-408), is essentially self-adjoint. The result is applied to the Wess-Zumino models of supersymmetric quantum field theory to prove the essential self-adjointness of their supercharges and Hamiltonians.

Contents

I. Introduction

In the previous papers [1, 2] (cf. also [3, 6, 9, 12]), the author developed a new theory of analysis on the abstract Boson-Fermion Fock space (BFFS), introducing operators of the de Rham and the Dirac types acting there. Some fundamental properties of these operators were investigated. In particular, index theorems for the Dirac operators have been established in terms of functional integral representations. In deriving the index theorems, however, the (essential) self-adjointness of the Dirac operators was assumed. In this paper we focus our attention on the self-adjointness problem of the Dirac operators and give a partial solution to it.

In Section II we review the fundamental framework of the theory presented in [1], summarizing properties of the de Rham type operator d_A, the free Dirac operator Q_A and the free Laplacian Δ_A acting in the abstract BFFS with A a densely defined closed linear operator playing a role of a "parameter". In Section III we first describe a perturbation of Q_A. This is defined from a perturbation of d_A, which is given in terms of a Hilbert space-valued function F on the probability space realizing the abstract Boson Fock space under consideration. We then show by an explicit construction that the perturbed Dirac operator, denoted by $Q_A(F)(Q_A(0)=Q_A)$, has at least two self-adjoint extensions if it is not essentially self-adjoint. Section IV is devoted to a detailed calculation of the perturbed Laplacian $\Delta_A(F):=\bar{Q}_A(F)^*\bar{Q}_A(F)$, where $\bar{Q}_A(F)$ is the closure of $Q_A(F)$. This is done for a class of F. In Section V we prove that, for a more restricted class of $F, \Delta_A(F)$ and $Q_A(F)$ are essentially self-adjoint on a domain included in $D(Q_A(F)^2)$. The idea of the proof is to apply a general theorem on essential self-adjointness of semibounded operators [10, 17]. In the last section we apply the abstract results in Section V to the Wess-Zumino models of supersymmetric quantum field theory (SSQFT) to prove that, for a class of interactions, the supercharges and the Hamiltonians with ultraviolet cutoffs are essentially self-adjoint on suitable domains. In Appendix we prove some elementary facts on decomposable operators in a direct integral of Hilbert spaces with a constant fibre, which are needed in the main text of the present paper.

II. Preliminaries-a review

In this section we describe the fundamental framework of the theory presented in [1] with some minor modifications in notations and definitions, and give some additional results. Throughout the present paper, we shall use the convention that the inner product $(\cdot,\cdot)_{\mathscr{K}}$ of the complex Hilbert space $\mathscr{K}$ is complex linear in the second variable.

2.1. Boson Fock space

Let $\mathscr{H}$ be a real separable Hilbert space and $\{\phi(f)|f\in\mathscr{H}\}$ be a family of Gaussian random variables on a probability space $(E, \mathscr{F}, \mu)$ such that the mapping: $f\rightarrow\phi(f)$ is linear, the Borel field $\mathscr{F}$ is generated by $\{\phi(f)|f\in\mathscr{H}\}$, and

$$\int_E e^{i\phi(f)} d\mu = e^{-\|f\|^2_{\mathscr{H}}/2}, \ f \in \mathscr{H}.$$

By abuse of notation, we also denote by ϕ an element of E.

We denote by $\mathbf{P}_n$ the set of complex-valued polynomials in n real -variables ($\mathbf{P}_0=\mathbf{C}$). For a linear operator T whose domain $D(T)$ is a subspace of $\mathscr{H}_c$, the complexification of $\mathscr{H}$, we define a subspace $\mathscr{P}_T \subset L^2(E, d\mu)$ by

$$\mathscr{P}_T = \mathscr{L}\{P_n(\phi(f_1), \cdots, \phi(f_n)) | n \geq 0, P_n \in \mathbf{P}_n, f_j \in D(T) \cap \mathscr{H}, j=1, \cdots, n\},$$

where $\mathscr{L}$ $\{\cdots\}$ denotes the subspace spanned by vectors in the set $\{\cdots\}$. If $D(T) \cap \mathscr{H}$ is dense in $\mathscr{H}_c$, then $\mathscr{P}_T$ is dense in $L^2(E, d\mu)$. For the case $D(T)=\mathscr{H}_c$, we simply write $\mathscr{P}_T = \mathscr{P}$.

Let $\mathscr{M}$ be a complex separable Hilbert space. We denote by $L^2(E, d\mu ; \mathscr{M})$ the Hilbert space of $\mathscr{M}$-valued square integrable functions on (E, μ), which is identified with $L^2(E, d\mu) \otimes \mathscr{M}$ (e.g., [19, § II. 4]). Let $T : \mathscr{H}_c \to \mathscr{M}$ be a densely defined closed linear operator such that $D(T) \cap \mathscr{H}$ is dense in $\mathscr{H}$. Then we can define an operator $T\nabla : L^2(E, d\mu) \to L^2(E, d\mu ; \mathscr{M})$ with domain $\mathscr{P}_T$ by

$$T\nabla\Psi = \sum_{j=1}^n \partial_j P_n(\phi(f_1), \cdots, \phi(f_n)) Tf_j$$

for vectors $\Psi \in \mathscr{P}_T$ of the form

$$\Psi = P_n(\phi(f_1), \cdots, \phi(f_n)) \tag{2.1}$$

and by extending it by linearity to all $\Psi \in \mathscr{P}_T$, where $\partial_j P_n$ denotes the partial derivative of the polynomial P_n in the j-th variable.

We denote by $J_{\mathscr{H}}$ the natural conjugation on $\mathscr{H}_c$ and set

$$\bar{f} = J_{\mathscr{H}} f, \qquad f \in \mathscr{H}_c.$$

LEMMA 2.1. *Let T be as above. Then $T\nabla$ is well defined.*

PROOF. For $f=f_1+if_2 \in \mathscr{H}_c (f_j \in \mathscr{H}, \ i=\sqrt{-1})$, we define $\phi(f)=\phi(f_1) + i\phi(f_2)$. It is well known or easily proven (cf. the proof of Theorem 6.3.1 in [15]) that for vectors $\Psi, \Phi \in \mathscr{P}$ of the form (2.1) and all $f \in \mathscr{H}_c$,

$$\int_E (f, \nabla\Psi)_{\mathscr{H}_c} \Phi d\mu = -\int_E \Psi (f, \nabla\Phi)_{\mathscr{H}_c} d\mu + \int_E \phi(\bar{f}) \Psi\Phi d\mu, \tag{2.2}$$

which is an integration by parts formula with respect to (w. r. t.) the measure μ. Let $\Psi_1, \Psi_2, \in \mathscr{P}$ such that $\Psi_1(\phi)=\Psi_2(\phi)$ a. e. ϕ. Then we have from (2.2) that

$$\int_E (u, T\nabla\Psi_1)_{\mathscr{M}}\Phi d\mu = \int_E (u, T\nabla\Psi_2)_{\mathscr{M}}\Phi d\mu$$

for all $u\in D(T^*)$ and $\Phi\in\mathscr{P}$. Since $\mathscr{P}$ and $D(T^*)$ are dense in $L^2(E, d\mu)$ and $\mathscr{M}$, respectively, and $\mathscr{M}$ is separable, it follows that there exists a subset N of E with $\mu(N)=0$ such that for all $u\in D(T^*)$

$$(u, T\nabla\Psi_1(\phi))_{\mathscr{M}} = (u, T\nabla\Psi_2(\phi))_{\mathscr{M}}, \qquad \phi\in E\backslash N.$$

Hecne, $T\nabla\Psi_1(\phi)=T\nabla\Psi_2(\phi)$ a. e. ϕ, which means the well-definedness of $T\nabla$ as an operator from $L^2(E, d\mu)$ to $L^2(E, d\mu; \mathscr{M})$. ■

In the case where $\mathscr{M}=\mathscr{H}_c$ and $T=I$ (identity), we simply write $I\nabla=\nabla$. For each $f\in\mathscr{H}_c$, we define an operator $\tilde{\nabla}_f$ in $L^2(E, d\mu)$ with domain $\mathscr{P}$ by

$$\tilde{\nabla}_f\Psi=(f, \nabla\Psi)_{\mathscr{H}_c}, \qquad \Psi\in\mathscr{P}.$$

Note that $\tilde{\nabla}_f$ is complex antilinear in f.

For two vector spaces $\mathscr{V}$ and $\mathscr{W}$, $\mathscr{V}\hat{\otimes}\mathscr{W}$ denotes their algebraic tensor product.

LEMMA 2.2. *Let T be as in Lemma 2.1 and $f\in\mathscr{H}_c$. Then, $\tilde{\nabla}_f$ and $T\nabla$ are closable and the following relations hold*:

$$\mathscr{P}\subset D(\tilde{\nabla}_f^*),\ \mathscr{P}\hat{\otimes}D(T^*)\subset D((T\nabla)^*), \tag{2.3}$$

$$\tilde{\nabla}_{T^*u}^*\Psi=(T\nabla)^*(\Psi u)=-\tilde{\nabla}_{J_{\mathscr{H}}T^*u}\Psi+\phi(T^*u)\Psi,\ \Psi\in\mathscr{P},\ u\in D(T^*). \tag{2.4}$$

PROOF. Using (2.2), we can prove (2.3) and (2.4). In particular, (2.3) implies that $D(\tilde{\nabla}_f^*)$ and $D((T\nabla)^*)$ are dense in $L^2(E, d\mu)$ and $L^2(E, d\mu; \mathscr{M})$, respectively. Hence $\tilde{\nabla}_f$ and $T\nabla$ are closable. ■

We denote the closures of $T\nabla$ and $\tilde{\nabla}_f$ by the same symbols, respectively.

The Hilbert space $L^2(E, d\mu)$ admits the orthogonal decomposition

$$L^2(E, d\mu)=\bigoplus_{n=0}^{\infty}\Gamma_n(\mathscr{H}),$$

where $\Gamma_0(\mathscr{H})=\mathbf{C}$ and $\Gamma_n(\mathscr{H})(n\geq 1)$ is the closed subspace generated by the Wick products: $\phi(f_1)\cdots\phi(f_n)$: $(f_j\in\mathscr{H}, j=1, \cdots, n)$ w. r. t. μ [22]. It is well known that $L^2(E, d\mu)$ is isomorphic to the Boson Fock space over $\mathscr{H}_c$ in a natural way [22].

Given a self-adjoint operator S in $\mathscr{H}$, one can define the second quantization $d\Gamma_b(S)$ of S as the self-adjoint operator in $L^2(E, d\mu)$ which is reduced by each $\Gamma_n(\mathscr{H})$ with the reduced part $d\Gamma_b^{(n)}(S)=d\Gamma_b(S)\restriction\Gamma_n(\mathscr{H})$

being of the form

$$d\Gamma_b^{(0)}(S)=0,$$
$$d\Gamma_b^{(n)}(S)\ :\ \phi(f_1)\cdots\phi(f_n):\ =\sum_{j=1}^{n}\ :\ \phi(f_1)\cdots\phi(Sf_j)\cdots\phi(f_n):,\qquad f_j\in D(S).$$

2.2. Fermion Fock space

We next recall Fermion Fock space. Let $\mathscr{K}$ be a real separable Hilbert space and $\wedge^p(\mathscr{K}_c)$ be the p-fold antisymmetric tensor product of $\mathscr{K}_c$ ($p\geq 0$, $\wedge^0(\mathscr{K}_c):=\mathbf{C}$). For $u_j\in\mathscr{K}_c$, $j=1,\cdots,p$, we define the exterior product $u_1\wedge\cdots\wedge u_p\in\wedge^p(\mathscr{K}_c)$ by

$$\begin{aligned}u_1\wedge\cdots\wedge u_p&=\frac{1}{p!}\sum_{\sigma\in\mathfrak{S}_p}\varepsilon(\sigma)u_{\sigma(1)}\otimes\cdots\otimes u_{\sigma(p)}\\&=A_p(u_1\otimes\cdots\otimes u_p)\end{aligned}$$

where $\mathfrak{S}_p$ is the symmetric group of order p, $\varepsilon(\sigma)$ the sign of the permutation σ, and $A_p=\sum_{\sigma\in\mathfrak{S}_p}\sigma\varepsilon(\sigma)/p!$ is the antisymmetrization operator on the p-fold tensor product of $\mathscr{K}_c$.

The Fermion Fock space $\wedge(\mathscr{K}_c)$ over $\mathscr{K}_c$ is defined as the Hilbert space given by

$$\wedge(\mathscr{K}_c)=\bigoplus_{p=0}^{\infty}\wedge^p(\mathscr{K}_c)$$

(e.g., [19, § II.4]).

For a self-adjoint operator T in $\mathscr{K}$, the second quantization $d\Gamma_f(T)$ of T is defined as the self-adjoint operator in $\wedge(\mathscr{K}_c)$ which is reduced by each $\wedge^p(\mathscr{K}_c)$ with the reduced part $d\Gamma_f^{(p)}(T)=d\Gamma_f(T)\restriction\wedge^p(\mathscr{K}_c)$ being of the form

$$d\Gamma_f^{(0)}(T)=0,$$
$$d\Gamma_f^{(p)}(T)=\sum_{k=1}^{p}\underbrace{I\otimes\cdots\otimes I}_{k-1\text{ times}}\otimes T\otimes\underbrace{I\otimes\cdots\otimes I}_{p-k\text{ times}}.$$

Let $b(u)$, $u\in\mathscr{K}_c$, be the annihilation operators on the Fermion Fock space $\wedge(\mathscr{K}_c)$, i.e., $b(u)$ is a bounded linear operator on $\wedge(\mathscr{K}_c)$ such that $b(u)\ :\ \wedge^p(\mathscr{K}_c)\to\wedge^{p-1}(\mathscr{K}_c)$ with

$$b(u)\restriction\wedge^0(\mathscr{K}_c)=0,$$
$$b(u)u_1\wedge\cdots\wedge u_p=\frac{1}{\sqrt{p}}\sum_{k=1}^{p}(-1)^{k-1}(u,u_k)_{\mathscr{K}_c}u_1\wedge\cdots\wedge\widehat{u}_k\wedge\cdots\wedge u_p,$$
$$p\geq 1,\ u_k\in\mathscr{K}_c,\tag{2.5}$$

where $\hat{u}_k$ indicates the omission of u_k. The adjoint $b(u)^*$, called the creation operator, maps $\wedge^p(\mathscr{K}_c)$ into $\wedge^{p+1}(\mathscr{K}_c)$ acting as

$$b(u)^* u_1 \wedge \cdots \wedge u_p = \sqrt{p+1}\, u \wedge u_1 \wedge \cdots \wedge u_p. \tag{2.6}$$

Moreover, the following canonical anticommutation relations (CARs) hold:

$$\{b(u), b(v)^*\} = (u, v)_{\mathscr{K}_c}, \qquad \{b(u), b(v)\} = 0, \qquad u, v \in \mathscr{K}_c,$$

where $\{A, B\} = AB + BA$. The CARs imply that

$$\|b(u)^\#\| = \|u\|_{\mathscr{K}_c}, \qquad u \in \mathscr{K}_c.$$

For later use, we introduce operators quadratic in $b(\cdot)^\#$ (cf. [14]). Let $\mathscr{I}_2(\mathscr{K}_c)$ be the set of Hilbert-Schmidt operators on $\mathscr{K}_c$ and $T \in \mathscr{I}_2(\mathscr{K}_c)$. Then there exist two orthonrmal systems $\{\psi_n\}_{n=1}^N$, $\{\phi_n\}_{n=1}^N$ ($N \leq +\infty$) and positive numbers λ_n such that $\sum_{n=1}^N \lambda_n^2 < \infty$ and

$$T = \sum_{n=1}^N \lambda_n (\psi_n, \cdot)_{\mathscr{K}_c} \phi_n \tag{2.7}$$

(e.g., [19, Theorem VI. 17]). We define

$$< b^* | T | b^* > = \sum_{n=1}^N \lambda_n b(\phi_n)^* b(\bar{\psi}_n)^*, \tag{2.8}$$

$$< b | T | b > = \sum_{n=1}^N \lambda_n b(\bar{\phi}_n) b(\psi_n), \tag{2.9}$$

$$< b^* | T | b > = \sum_{n=1}^N \lambda_n b(\phi_n)^* b(\psi_n). \tag{2.10}$$

One can show that, in the case $N = +\infty$, the operator on the right hand side of (2.8) [resp. (2.9), (2.10)] converges strongly on the dense subspace

$$\wedge_f(\mathscr{K}_c) := \{\Psi = \{\Psi^{(p)}\}_{p=0}^\infty \in \wedge(\mathscr{K}_c) | \Psi^{(p)} \in A_p(\mathscr{K} \hat{\otimes} \cdots \hat{\otimes} \mathscr{K}_c),\ \Psi^{(p)} = 0 \text{ for all but finitely many } p\}$$

and these limits are independent of the choice of the representation of T given by (2.7). We denote by $< b^\# | T | b^\natural >$ any of these three quadratic operators with domain $\wedge_f(\mathscr{K}_c)$.

LEMMA 2.3. *We have*

$$\wedge_f(\mathscr{K}_c) \subset D(< b^\# | T | b^\natural >^*)$$

and

$$<b^*|T|b^*>^* = <b|T^*|b>,$$
$$<b^*|T|b>^* = <b^*|T^*|b> \quad on \ \wedge_f(\mathscr{K}_c).$$

In particular, each $<b^\#|T|b^\natural>$ *is closable.*

PROOF. See [1, 2]. ■

We denote the closure of $<b^\#|T|b^\natural>$ by the same symbol.

The following lemma also will be used later.

LEMMA 2.4. *For every* $\psi \in \mathscr{K}_c \otimes \mathscr{K}_c$, *there exists a unique operator* $\Lambda(\psi) \in \mathscr{I}_2(\mathscr{K}_c)$ *such that for all* $u, v \in \mathscr{K}_c$

$$(u, \Lambda(\psi)v)_{\mathscr{K}_c} = (u \otimes \bar{v}, \psi)_{\mathscr{K}_c \otimes \mathscr{K}_c}. \tag{2.11}$$

Moreover, we have

$$\|\Lambda(\psi)\|_2 = \|\psi\|_{\mathscr{K}_c \otimes \mathscr{K}_c}, \tag{2.12}$$

where $\|T\|_2$ *denotes the Hilbert-Schmidt norm of* T. *The Hilbert space* $\mathscr{K}_c \otimes \mathscr{K}_c$ *is isomorphic to* $\mathscr{I}_2(\mathscr{K}_c)$ *under the linear transformation* $\Lambda: \mathscr{K}_c \otimes \mathscr{K}_c \to \mathscr{I}_2(\mathscr{K}_c)$.

PROOF. The existence of $\Lambda(\psi)$ can be proven by using the Riesz lemma. The uniqueness is obvious by (2.11). It is an easy exercise to prove (2.12). To prove the surjectivity of Λ, let $T \in \mathscr{I}_2(\mathscr{K}_c)$, so that it is represented as (2.7). It is easy to show that

$$\psi_T := \sum_{n=1}^{N} \lambda_n \phi_n \otimes \bar{\psi}_n, \tag{2.13}$$

strongly converges in $\mathscr{K}_c \otimes \mathscr{K}_c$ and

$$\Lambda(\psi_T) = T. \tag{2.14}$$

Thus Λ is surjective. ■

We denote by τ the transposition on $\mathscr{K}_c \otimes \mathscr{K}_c$, i.e., τ is the unitary operator on $\mathscr{K}_c \otimes \mathscr{K}_c$ such that for all $u, v \in \mathscr{K}_c$

$$\tau(u \otimes v) = v \otimes u.$$

For $\psi \in \mathscr{K}_c \otimes \mathscr{K}_c$, we set

$$\bar{\psi} = J_{\mathscr{K}} \otimes J_{\mathscr{K}} \psi.$$

LEMMA 2.5. *For all* $\psi \in \mathscr{K}_c \otimes \mathscr{K}_c$,

$$\Lambda(\psi)^* = \Lambda(\overline{\tau\psi}). \tag{2.15}$$

PROOF. Let $u, v \in \mathscr{K}_c$. Then we have

$$\begin{aligned}(u, \Lambda(\psi)^* v)_{\mathscr{K}_c} &= (\psi, v \otimes \bar{u})_{\mathscr{K}_c} \\ &= (\tau\psi, \bar{u} \otimes v)_{\mathscr{K}_c \otimes \mathscr{K}_c} \\ &= (u \otimes \bar{v}, \overline{\tau\psi})_{\mathscr{K}_c \otimes \mathscr{K}_c} \\ &= (u, \Lambda(\overline{\tau\psi}) v)_{\mathscr{K}_c}.\end{aligned}$$

Hence (2. 15) follows. ■

LEMMA 2. 6. (i) *Let* $\psi \in \mathscr{K}_c \otimes \mathscr{K}_c$ *and* $u_j \in \mathscr{K}_c$, $j=1, \cdots, p$. *Then*

$$\begin{aligned}&(< b|\Lambda(\psi)|b > u_1 \wedge \cdots \wedge u_p)^{(p-2)} \\ &= \frac{2}{\sqrt{(p-1)p}} \sum_{k<j}^{p} (-1)^{j+k} (\bar{u}_k \wedge \bar{u}_j, \psi)_{\mathscr{K}_c \otimes \mathscr{K}_c} \\ &\times u_1 \wedge \cdots \wedge \hat{u}_k \wedge \cdots \wedge \hat{u}_j \wedge \cdots \wedge u_p\end{aligned}$$

for all $p \geq 2$ *and*

$$(< b|\Lambda(\psi)|b > u_1 \wedge \cdots \wedge u_p)^{(q)} = 0, \qquad q \neq p-2.$$

(ii) *For all* $T \in \mathscr{I}_2(\mathscr{K}_c)$,

$$< b^*|T|b > = d\Gamma_f(T). \tag{2. 16}$$

PROOF. Part (i) follows from a direct computation using (2. 5), (2. 9), (2. 13) and (2. 14). As for part (ii), using (2. 5), (2. 6) and (2. 10), one first proves (2. 16) on $\wedge_f(\mathscr{K}_c)$ and then, by a limiting argument, extends it as an operator equality. ■

2. 3. BFFS, operators of the de Rham type, and free Dirac operators

The BFFS we are concerned with is defined by

$$\begin{aligned}\wedge(\mathscr{H}, \mathscr{K}) &= L^2(E, d\mu) \otimes \wedge(\mathscr{K}_c) \\ &= L^2(E, d\mu; \wedge(\mathscr{K}_c)).\end{aligned}$$

We have

$$\wedge(\mathscr{H}, \mathscr{K}) = \bigoplus_{p=0}^{\infty} \wedge^p(\mathscr{H}, \mathscr{K}),$$

where

$$\wedge^p(\mathscr{H}, \mathscr{K}) = L^2(E, d\mu; \wedge^p(\mathscr{K}_c)) = L^2(E, d\mu) \otimes \wedge^p(\mathscr{K}_c).$$

Let $A: \mathscr{H}_c \to \mathscr{K}_c$ be a densely defined closed linear operator such that the nonnegative self-adjoint operator A^*A is reduced by $\mathscr{H}$. We introduce a subspace $\mathfrak{D}_{A,p}$ of $\wedge^p(\mathscr{H}, \mathscr{K})$ by

$$\mathfrak{D}_{A,p} = \mathscr{L}\{P_n(\phi(f_1), \cdots, \phi(f_n)) u_1 \wedge \cdots \wedge u_p | n \geq 0,\ P_n \in \mathbf{P}_n,\ f_j \in D(A) \cap \mathscr{H},\ u_k \in \mathscr{K}_c,\ j=1, 2, \cdots, n;\ k=1, 2, \cdots, p\}.$$

Since $D(A^*A)$ is dense in $\mathscr{H}$ and $D(A^*A)\cap\mathscr{H}\subset D(A)\cap\mathscr{H}$, it follows that $\mathfrak{D}_{A,p}$ is dense in $\wedge^p(\mathscr{H},\mathscr{K})$.

For each $p\geq 0$, we define an operator $d_{A,p}$: $\wedge^p(\mathscr{H},\mathscr{K})\to\wedge^{p+1}(\mathscr{H},\mathscr{K})$ with domain $\mathfrak{D}_{A,p}$ by

$$d_{A,p}\Psi=\sqrt{p+1}\sum_{j=1}^{n}(\partial_j P_n)(\phi(f_1),\cdots,\phi(f_n))Af_j\wedge u_1\wedge\cdots\wedge u_p$$

for vectors $\Psi\in\mathfrak{D}_{A,p}$ of the form

$$\Psi=P_n(\phi(f_1),\cdots,\phi(f_n))u_1\wedge\cdots\wedge u_p \tag{2.17}$$

and extending it by linearity to all vectors $\Psi\in\mathfrak{D}_{A,p}$. In the same way as in the proof of Lemma 2.1, one can show that the operator d_A is well-defined (i.e., if $\Psi,\Phi\in\mathfrak{D}_{A,p}$, $\Psi=\Phi$ a.e., then $d_{A,p}\,\Psi=d_{A,p}\,\Phi$ a.e.).

LEMMA 2.7. (i) *For all* $p\geq 0$,

$$d_{A,p}\mathfrak{D}_{A,p}\subset\mathfrak{D}_{A,p+1}$$

and

$$d_{A,p+1}d_{A,p}=0.$$

Moreover, $d_{A,p}$ *is closable.*

(ii) *Let*

$$\mathfrak{D}^*_{A,p}=\mathscr{L}\{P_n(\phi(f_1),\cdots,\phi(f_n))u_1\wedge\cdots\wedge u_{p+1}|n\geq 0,\ P_n\in\mathbf{P}_n,\ f_j\in\mathscr{H},\ u_k\in D(A^*),\ j=1,2,\cdots,n;\ k=1,2,\cdots,p+1\}.$$

Then, for all $p\geq 0$,

$$\mathfrak{D}^*_{A,p}\subset D(d^*_{A,p})$$

and

$$d^*_{A,p-1}\,\Psi=\frac{1}{\sqrt{p}}\sum_{k=1}^{p}(-1)^{k-1}(\phi(A^*u_k)\tilde{P}_n-\tilde{\nabla}_{J_{\mathscr{K}}A^*u_k}\tilde{P}_n)u_1\wedge\cdots\wedge\hat{u}_k\wedge\cdots\wedge u_p$$

for vectors $\Psi\in\mathfrak{D}^*_{A,p-1}$ *of the form* (2.17), *where* $\tilde{P}_n=P_n(\phi(f_1),\cdots,\phi(f_n))$.

PROOF. See [1]. ∎

We denote the closure of $d_{A,p}$ by the same symbol. One can define from the sequence $\{d_{A,p}\}_{p=0}^{\infty}$ of operators a de Rham type operator d_A acting in $\wedge(\mathscr{H},\mathscr{K})$ by

$$D(d_A)=\{\Psi=\{\Psi^{(p)}\}_{p=0}^{\infty}\in\wedge(\mathscr{H},\mathscr{K})|\Psi^{(p)}\in D(d_{A,p}),\ \sum_{p=0}^{\infty}\|d_{A,p}\Psi^{(p)}\|^2<\infty\},$$

$$(d_A\Psi)^{(0)}=0,\ (d_A\Psi)^{(p)}=d_{A,p-1}\,\Psi^{(p-1)},\ p\geq 1,\ \Psi\in D(d_A).$$

Let

$$\mathfrak{D}_A=\{\Psi=\{\Psi^{(p)}\}_{p=0}^{\infty}\in\wedge(\mathscr{H},\mathscr{K})|\Psi^{(p)}\in\mathfrak{D}_{A,p}\cap\mathfrak{D}^*{}_{A,p-1},\ \Psi^{(p)}=0 \\ \text{for all but finitely many } p\}.$$

LEMMA 2.8. (i) *The operator d_A is densely defined with*

$$\mathfrak{D}_A\subset D(d_A)$$

and closed. Moreover

$$d_A^2=0.$$

(ii) *The adjoint d_A^* is given by*

$$D(d_A^*)=\{\Psi=\{\Psi^{(p)}\}_{p=0}^{\infty}\in\wedge(\mathscr{H},\mathscr{K})|\Psi^{(p+1)}\in D(d_{A,p}^*),\ \sum_{p=0}^{\infty}\|d_{A,p}^*\Psi^{(p+1)}\|^2<\infty\},$$

$$(d_A^*\Psi)^{(p)}=d_{A,p}^*\Psi^{(p+1)},\ p\geq 0,\ \Psi\in D(d_A^*).$$

PROOF. See [1]. ∎

We define

$$Q_A=d_A+d_A^*$$

with $D(Q_A)=D(d_A)\cap D({}_A^*)$. We call Q_A a free Dirac-Kähler operator or simply a free Dirac operator.

The free Laplacian associated with the de Rham operator d_A is defined by

$$\Delta_A=d_A^*d_A+d_Ad_A^*$$

with $D(\Delta_A)=D(d_A^*d_A)\cap D(d_Ad_A^*)$. The following theorem has been proven in [1] (cf. also [2]).

THEOREM 2.9. (i) *The free Laplacian Δ_A is a nonnegative self-adjoint operator and the operator equality*

$$\Delta_A=d\Gamma_b(A^*A)\otimes I+I\otimes d\Gamma_f(AA^*)$$

holds.

(ii) *The free Dirac operator Q_A is self-adjoint and essentially self-adjoint on every core for Δ_A. Moreover, the operator equality*

$$\Delta_A=Q_A^2$$

holds.

REMARK. Let $\{e_n\}_{n=1}^{\infty}$ be a complete orthonormal system (C.O.N.S.)

of $\mathscr{K}_c$ with $e_n \in D(A^*)$, $n \geq 1$, such that $\sum_{n=1}^{\infty} \|A^* e_n\|_{\mathscr{K}_c}^2 < \infty$. Then we can show that for all $\Psi \in \mathfrak{D}_A$

$$d_A \Psi = \sum_{n=1}^{\infty} \tilde{\nabla}_{A^* e_n} \otimes b(e_n)^* \Psi,$$
$$d_A^* \Psi = \sum_{n=1}^{\infty} \tilde{\nabla}_{A^* e_n}^* \otimes b(e_n) \Psi$$

in the sense of strong convergence. Putting

$$\Phi(f) = \frac{\tilde{\nabla}_f + \tilde{\nabla}_f^*}{\sqrt{2}}, \qquad f \in \mathscr{H}_c,$$
$$\gamma_{2n-1} = b(e_n) + b(e_n)^*, \qquad \gamma_{2n} = i(b(e_n)^* - b(e_n)),$$
$$h_{2n-1} = \frac{A^* e_n}{\sqrt{2}}, \qquad h_{2n} = \frac{i A^* e_n}{\sqrt{2}},$$

we have

$$Q_A = \sum_{n=1}^{\infty} \Phi(h_n) \otimes \gamma_n$$

on $\mathfrak{D}_A$ in the strong topology. Note that γ_n is a bounded self-adjoint operator with

$$\{\gamma_n, \gamma_m\} = 2\delta_{mn}, \qquad m, n \geq 1,$$

which means that $\{\gamma_n\}_{n=1}^{\infty}$ is a self-adjoint representation of an infinite dimensional Clifford algebra [13]. Thus Q_A is a generalization of the Dirac type operator $\not{D}_h$ given in Example 2 in Section IV of [13]; If $A^* e_n$ and $A^* e_m$ are orthogonal for all $m \neq n$, then Q_A is of the form of $\not{D}_h$ and, in this case, Theorem 4.7 in [13] gives another proof of the self-adjointness of Q_A.

The BFFS $\wedge(\mathscr{H}, \mathscr{K})$ admits the orthogonal decomposition

$$\wedge(\mathscr{H}, \mathscr{K}) = \wedge_+(\mathscr{H}, \mathscr{K}) \oplus \wedge_-(\mathscr{H}, \mathscr{K}), \tag{2.18}$$

with

$$\wedge_+(\mathscr{H}, \mathscr{K}) = \bigoplus_{p=0}^{\infty} \wedge^{2p}(\mathscr{H}, \mathscr{K}), \qquad \wedge_-(\mathscr{H}, \mathscr{K}) = \bigoplus_{p=0}^{\infty} \wedge^{2p+1}(\mathscr{H}, \mathscr{K}).$$

The spaces $\wedge_+(\mathscr{H}, \mathscr{K})$ and $\wedge_-(\mathscr{H}, \mathscr{K})$ may be regarded as a space of "even forms" and of "odd forms" on E, respectively [1, 2, 5, 6].

Let $P_\pm$ be the orthogonal projections onto $\wedge_\pm(\mathscr{H}, \mathscr{K})$, respectively, and define

$$\Gamma = P_+ - P_-.$$

Then Γ satisifies

$$\Gamma^2 = I,\ \Gamma^* = \Gamma,$$

i. e., Γ is a grading operator on $\wedge(\mathscr{H}, \mathscr{K})$. The following proposition has been proven in [1].

PROPOSITION 2. 10.

$$\Gamma D(Q_A) = D(Q_A)$$

and

$$Q_A \Gamma + \Gamma Q_A = 0 \text{ on } D(Q_A).$$

REMARKS. (i) Proposition 2. 10 and Theroem 2. 9 imply that the quadruple $\{\wedge(\mathscr{H}, \mathscr{K}), \Delta_A, Q_A, \Gamma\}$ defines a supersymmetric quantum theory [3, 4, 8, 12, 16, 23].

(ii) More detailed properties of Q_A and Δ_A are given in [1].

(iii) Decomposition theorems of the de Rham-Hodge-Kodaira type for the "de Rham complex" $\{d_{A,p}\}_{p=0}^{\infty}$ and related aspects have been discussed in [5, 11].

(iv) For a generalization of the above formalism to the case where the measure μ is not necessarily Gaussian, see [5, 6, 9].

III. Perturbation of the free Dirac operator and self-adjoint extensions

In this section we consider a perturbation of the free Dirac perator Q_A introduced in the last section. Let F be a $\mathscr{K}_c$-valued measurable function on (E, μ). Then we define an operator $\tilde{b}(F)$ acting in the BFFS with the identification

$$\wedge(\mathscr{H}, \mathscr{K}) = \int_E^{\oplus} \wedge(\mathscr{K}_c) d\mu(\phi)$$

by

$$\tilde{b}(F) = \int_E^{\oplus} b(F(\phi)) d\mu(\phi). \tag{3.1}$$

(For the notation, see Appendix.) The following lemma can be proven by applying Propositions A. 1 and A. 2 in Appendix A.

LEMMA 3. 1. *Let $F \in L^r(E, d\mu; \mathscr{K}_c)$ with some $r > 2$. Then the following* (i)-(ii) *hold*;

(i) *The operator $\tilde{b}(F)$ is densely defined with*

$$\mathfrak{D}_A \subset D(\tilde{b}(F))$$

and closed.

(ii) *We have*

$$\mathfrak{D}_A \subset D(\tilde{b}(F)^*)$$

and

$$\tilde{b}(F)^* = \int_E^{\oplus} b(F(\phi))^* d\mu(\phi). \tag{3.2}$$

In what follows, we assume that $F \in L^r(E, d\mu; \mathscr{K}_c)$ for some $r > 2$. We introduce a perturbed de Rham operator $d_A(F)$ by

$$d_A(F) = d_A + \tilde{b}(F)^*$$

with $D(d_A(F)) = D(d_A) \cap D(\tilde{b}(F)^*)$. By Lemma 3.1, we have

$$\mathfrak{D}_A \subset D(d_A(F)^*)$$

and

$$d_A(F)^* = d_A^* + \tilde{b}(F) \text{ on } D(d_A^*) \cap D(\tilde{b}(F)).$$

In particular, $d_A(F)$ is closable.

We now define a perturbed Dirac operator by

$$Q_A(F) = d_A(F) + d_A(F)^*$$

with $D(Q_A(F)) = D(Q_A) \cap D(\tilde{b}(F)) \cap D(\tilde{b}(F)^*)$. It is obvious that $Q_A(F)$ is a symmetric operator. We denote the closure of $Q_A(F)$ by $\bar{Q}_A(F)$. We are concerned with the self-adjointness of $\bar{Q}_A(F)$. We first show that $\bar{Q}_A(F)$ has self-adjoint extensions. A key fact for that is given by the following proposition.

PROPOSITION 3.2.

$$\Gamma D(\bar{Q}_A(F)) = D(\bar{Q}_A(F))$$

and

$$\{\bar{Q}_A(F), \Gamma\} = 0 \text{ on } D(\bar{Q}_A(F)).$$

PROOF. It is sufficient to prove these facts for $\bar{Q}_A(F)$ replaced by $Q_A(F)$. Let $\Psi = \{\Psi^{(p)}\}_{p=0}^{\infty} \in D(\tilde{b}(F)^*)$. Then we have

$$(\Gamma\Psi)^{(p)} = (-1)^p \Psi^{(p)}, \qquad p \geq 0. \tag{3.3}$$

Hence

$$\|b(F(\phi))^*(\Gamma\Psi)(\phi)\|^2_{\wedge(\mathscr{K}_c)}=\|b(F(\phi))^*\Psi(\phi)\|^2_{\wedge(\mathscr{K}_c)},$$

which implies that $\Gamma\Psi\in D(\tilde{b}(F)^*)$. Thus $\Gamma D(\tilde{b}(F)^*)\subset D(\tilde{b}(F)^*)$. Since $\Gamma^2=I$, it follows that $D(\tilde{b}(F)^*)\subset\Gamma D(\tilde{b}(F)^*)$. Thus we obtain

$$\Gamma D(\tilde{b}(F)^*)=D(\tilde{b}(F)^*). \tag{3.4}$$

Moreover, we have by (3.3) and (2.6)

$$(\tilde{b}(F)^*\Gamma\Psi)^{(p)}(\phi)=(-1)^{p-1}\sqrt{p}\,F(\phi)\wedge\Psi^{(p-1)}(\phi),$$
$$(\Gamma\tilde{b}(F)^*\Psi)^{(p)}(\phi)=(-1)^{p}\sqrt{p}\,F(\phi)\wedge\Psi^{(p-1)}(\phi).$$

Hence

$$\{\tilde{b}(F)^*,\Gamma\}=0 \text{ on } D(\tilde{b}(F)^*). \tag{3.5}$$

Similarly we can show that

$$\Gamma D(\tilde{b}(F))=D(\tilde{b}(F)) \tag{3.6}$$

and

$$\{\tilde{b}(F),\Gamma\}=0 \text{ on } D(\tilde{b}(F)). \tag{3.7}$$

Proposition 2.10 and (3.4)-(3.7) imply the desired result. ■

Jorgensen [18] introduecd a notion of abstract Dirac operator: Let $\mathscr{X}$ be a complex Hilbert space and γ be a grading operator on $\mathscr{X}$, i.e., γ is a self-adjoint operator on $\mathscr{X}$ such that $\gamma^2=I$ $(\gamma\neq\pm I)$. A closed symmetric operator T in $\mathscr{X}$ is called an abstract Dirac operator w.r.t. γ if γ leaves $D(T)$ invariant and

$$\{\gamma, T\}=0 \text{ on } D(T).$$

In terms of this notion, Proposition 3.2 is rephrased as follows: The operator $\bar{Q}_A(F)$ is an abstract Dirac operator w.r.t. Γ.

It has been shown in [18] that every abstract Dirac operator has a self-adjoint extension which is also an abstract Dirac operator. Here we explicitly construct two self-adjoint extensions of $\bar{Q}_A(F)$ by employing an idea used in [6, 7].

By (2.18), every $\Psi\in\wedge(\mathscr{H},\mathscr{K})$ can be represented as

$$\Psi=\begin{pmatrix}\Psi_+\\ \Psi_-\end{pmatrix},\qquad \Psi_\pm\in\wedge_\pm(\mathscr{H},\mathscr{K}).$$

Then every linear operator in $\wedge(\mathscr{H},\mathscr{K})$ is given by a 2×2 matrix with entries being linear operators. For example, we have

$$\Gamma=\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{3.8}$$

It follows from Proposition 3.2 that there exists a unique densely defined closed linear operator $Q_A(F)_+$(resp. $Q_A(F)_-$) from $\wedge_+(\mathscr{H},\mathscr{K})$(resp. $\wedge_-(\mathscr{H},\mathscr{K})$) to $\wedge_-(\mathscr{H},\mathscr{K})$ (resp. $\wedge_+(\mathscr{H},\mathscr{K})$) such that

$$\bar{Q}_A(F)=\begin{pmatrix} 0 & Q_A(F)_- \\ Q_A(F)_+ & 0 \end{pmatrix}. \tag{3.9}$$

The symmetricity of $\bar{Q}_A(F)$ implies that

$$Q_A(F)_+\subset Q_A(F)_-^*. \tag{3.10}$$

Let

$$Q_A^{(1)}(F)=\begin{pmatrix} 0 & Q_A(F)_+^* \\ Q_A(F)_+ & 0 \end{pmatrix}. \tag{3.11}$$

and

$$Q_A^{(2)}(F)=\begin{pmatrix} 0 & Q_A(F)_- \\ Q_A(F)_-^* & 0 \end{pmatrix}. \tag{3.12}$$

PROPOSITION 3.3. *Each $Q_A^{(j)}(F)$ is a self-adjoint extension of $\bar{Q}_A(F)$ and an abstract Dirac operator w. r. t.* Γ.

PROOF. It is easy to see that $Q_A^{(j)}(F)$ is self-adjoint. By (3.9) and (3.10), $Q_A^{(j)}(F)$ is an extension of $\bar{Q}_A(F)$. Using (3.8), one can easily check that $Q_A^{(j)}$ is an abstract Dirac operator w. r. t. Γ. ■

IV. The Laplacian associated with the perturbed Dirac operator

By von Neumann's theorem (e.g., [20, Theorem X.25]), the operator

$$\Delta_A(F):=\bar{Q}_A(F)^*\bar{Q}_A(F)$$

is self-adjoint and nonnegative. We call this operator the Laplacian associated with the perturbed Dirac operator $\bar{Q}_A(F)$ or simply the perturbed Laplacian. The essential self-adjointness of $Q_A(F)$ is closely related to that of $\Delta_A(F)$ as is shown in the following lemma.

LEMMA 4.1. *Let $\mathfrak{D}$ be a core for $\Delta_A(F)$ such that $\mathfrak{D}\subset D(Q_A(F)^2)$. Then $Q_A(F)$ is essentially self-adjoino on $\mathfrak{D}$.*

Lemma 4.1 follows from an application of a general fact given in the following lemma.

LEMMA 4.2. *Let T be a symmetric operator in a Hilbert space $\mathscr{X}$. Suppose that there exists a dense subspace D such that $D \subset D(T^2)$ and T^2 is essentially self-adjoint on D. Then T is essentially self-adjoint on D.*

PROOF. The fact stated in this lemma should be more or less well-known (cf. [20, Chapter X, Problem 28]), but, for the sake of completeness, we give a proof. Let $\bar{T}$ be the closure of T. Then, by von Neumann's theorem, the operator

$$L=(\bar{T})^* \bar{T}=T^* \bar{T}$$

is self-adjoint and nonnegative. Since T is symmetric, we have $T^2 \subset L$. Hence it follows that D is a core for L. We can easily show that for all $f \in D$,

$$\|Tf\| \leq \frac{1}{\sqrt{2}} \|(L+1)f\|.$$

Moreover, for all $f \in D$,

$$0=|(Tf, (L+1)f)-((L+1)f, Tf)| \leq \|(L+1)^{1/2} f\|^2.$$

Thus, by applying a variant of Nelson's commutator theorem (e.g., [20, Theorem X.37]), we obtain the desired result. ■

PROOF OF LEMMA 4.1.
Apply Lemma 4.2 with $\mathscr{X}=\wedge(\mathscr{H}, \mathscr{K})$ and $T=Q_A(F)$. ■

Employing Lemma 4.1, we shall prove, for a class of F, the essential self-adjointness of $Q_A(F)$ by proving that of $\Delta_A(F)$ on a domain included in $D(Q_A(F)^2)$. For this purpose, we need to know an explicit form of $Q_A(F)^2$ on a suitable domain. The rest of this section is devoted to the computaion of $Q_A(F)^2$.

We introduce a class of $\mathscr{K}_c$-valued measurable functions on E. Let $1 \leq r, s < \infty$ and define a norm $\|\cdot\|_{r,s}$ on $\mathscr{P}_A \hat{\otimes} D(A^*)$ by

$$\begin{aligned}\|\Phi\|_{r,s}=\|\Phi\|_{L^r(E,d\mu;\, \mathscr{K}_c)}+\|A\nabla \otimes I\Phi\|_{L^s(E,d\mu;\, \mathscr{K}_c \otimes \mathscr{K}_c)} \\ +\|\bar{A}\nabla \otimes I\Phi\|_{L^s(E,d\mu;\, \mathscr{K}_c \otimes \mathscr{K}_c)},\end{aligned}$$

where

$$\bar{A}=J_{\mathscr{K}} A J_{\mathscr{K}}.$$

We denote by $\mathscr{W}_A^{r,s}(\mathscr{K}_c)$ the completion of $\mathscr{P}_A \hat{\otimes} D(A^*)$ in the norm $\|\cdot\|_{r,s}$.

DEFINITION 4.3. We say that a $\mathscr{K}_c$-valued function Φ on E is in the set $\mathbf{F}_A^{r,s}$ if $\Phi \in D((A\nabla)^*) \cap \mathscr{W}_A^{r,s}(\mathscr{K}_c)$ and, for all $f \in D(A) \cap \mathscr{H}$, $(\Phi, Af)_{\mathscr{K}_c}$ is a real-valued function on E.

Let G be a $\mathscr{K}_c \otimes \mathscr{K}_c$-valued measurable function on E. Then, for a.e. $\phi \in E$, we can define three kinds of quadraic operators $< b^\# | \Lambda(G(\phi)) | b^\natural >$ (see Section 2.2). These operators can be extended to operators in $\wedge(\mathscr{H}, \mathscr{K})$ as decomposable operators:

$$< \tilde{b}^\# | \Lambda(G) | \tilde{b}^\natural > := \int_E^\oplus < b^\# | \Lambda(G(\phi)) | b^\natural > d\mu(\phi).$$

We introduce

$$\mathfrak{D}_A^{(2)} = \mathscr{L}\{P_n(\phi(f_1), \cdots, \phi(f_n)) u_1 \wedge \cdots \wedge u_p | n, p \geq 0,\ P_n \in \mathbf{P}_n, f_j \in D(A^*A) \cap \mathscr{H}, \\ u_k \in D(AA^*),\ j = 1, 2, \cdots, n;\ k = 1, 2, \cdots, p\},$$

which is dense in $\wedge(\mathscr{H}, \mathscr{K})$. We denote the closures of the operators $A\nabla$, $\bar{A}\nabla$: $L^2(E, d\mu) \to L^2(E, d\mu; \mathscr{K}_c)$ with domain $\mathscr{P}_A$ by the same symbols. The main result of this section is the following.

THEOREM 4.4. *Let $F \in \mathbf{F}_A^{r,s}$ with $r > 4$ and $s > 2$. Then, $\mathfrak{D}_A^{(2)} \subset D(Q_A(F)^2)$ and*

$$\begin{aligned} \Delta_A(F) &= Q_A(F)^2 \\ &= \Delta_A + (A\nabla)^* F + \|F\|_{\mathscr{K}_c}^2 + < \tilde{b}^* | \Lambda(\overline{\bar{A}\nabla \otimes IF}) | \tilde{b} > \\ &\quad + < \tilde{b}^* | \Lambda(\overline{\bar{A}\nabla \otimes IF})^* | \tilde{b} > + < \tilde{b}^* | \Lambda(A\nabla \otimes IF) | \tilde{b}^* > \\ &\quad + < \tilde{b} | \Lambda(A\nabla \otimes IF)^* | \tilde{b} > \end{aligned} \tag{4.1}$$

on $\mathfrak{D}_A^{(2)}$.

Formula (4.1) has been derived in [1] with a different notation for the quadratic operators in $\tilde{b}^\#(\cdot)$ and under slightly different conditions, although the details of the derivation of (4.1) was not given there. The present formulation gives a refinement of the corresponding result in [1] and may be a most general one to obtain such an explicit form of $\Delta_A(F)$ as (4.1). For these reasons and for the sake of completeness, here we present, by a series of lemmas, a detailed proof of (4.1).

In what follows, we often omit the subscript $\mathscr{H}$ in the inner product $(\cdot,\cdot)_{\mathscr{H}}$ of the Hilbert space $\mathscr{H}$ if there is no danger of confusion.

Let T be a densely defined closed linear operator from $\mathscr{H}_c$ to $\mathscr{K}_c$ such that $D(T) \cap \mathscr{H}$ is dense in $\mathscr{H}$.

LEMMA 4.5. *Let $G \in L^s(E, d\mu; \mathscr{K}_c) \cap D((T\nabla)^*)$ with some $s > 2$. Suppose that there exist sequences $\{G_n\}_{n=1}^\infty \subset \mathscr{P}$ and $\{e_n\}_{n=1}^\infty \subset D(T^*)$ such* *that* $\sum_{n=1}^{N} G_n e_n \to G$ *in* $L^s(E, d\mu; \mathscr{K}_c)$ *as* $N\to\infty$. *Then, for all* $\Psi\in\mathscr{P}_T$,

$$((T\nabla)^* G, \Psi)_{L^2(E,d\mu)} = \sum_{n=1}^{\infty}(-\tilde{\nabla}_{J_{\mathscr{H}}T^*e_n} G_n + \phi(T^* e_n)G_n, \Psi)_{L^2(E,d\mu)}. \quad (4.2)$$

PROOF. Let $\Psi\in\mathscr{P}_T$. Then we have

$$\begin{aligned}((T\nabla)^*G, \Psi) &= (G, T\nabla\Psi)\\ &= \sum_{n=1}^{\infty}(G_n, \tilde{\nabla}_{T^*e_n}\Psi)\\ &= \sum_{n=1}^{\infty}(\tilde{\nabla}_{T^*e_n} G_n, \Psi).\end{aligned}$$

Then, using Lemma 2.2, we obtain (4.2). ■

For each function $G\in L^s(E, d\mu; \mathscr{K}_c)$ with $s>2$, we define an operator $\tilde{\nabla}_{T,G}$ in $L^2(E, d\mu)$ by

$$\begin{aligned}&D(\tilde{\nabla}_{T,G}) = \mathscr{P}_T,\\ &\tilde{\nabla}_{T,G}\Psi = (G, T\nabla\Psi)_{\mathscr{K}_c}, \qquad \Psi\in\mathscr{P}_T.\end{aligned}$$

By Hölder's inequality, one can easily show that $\tilde{\nabla}_{T,G}\Psi\in L^2(E, d\mu)$.

LEMMA 4.6. *Let G be as in Lemma* 4.5. *Then,* $\tilde{\nabla}_{T,G}$ *is closable with*

$$D(\tilde{\nabla}_{T,G}^*)\supset\mathscr{P}_T \quad (4.3)$$

and

$$\tilde{\nabla}_{T,G}^*\Psi = \Psi(T\nabla)^*G - \tilde{\nabla}_{\bar{T},\bar{G}}\Psi, \quad \Psi\in\mathscr{P}_T. \quad (4.4)$$

PROOF. Let $\Psi, \Phi\in\mathscr{P}_T$. Then

$$\begin{aligned}(\tilde{\nabla}_{T,G}\Phi, \Psi) &= \int_E (T\nabla\Phi(\phi), G(\phi))_{\mathscr{K}_c}\Psi(\phi)d\mu\\ &= \sum_{n=1}^{\infty}(\tilde{\nabla}_{T^*e_n}\Phi, G_n\Psi)\\ &= \sum_{n=1}^{\infty}(\Phi, \tilde{\nabla}_{T^*e_n}(G_n\Psi)).\end{aligned}$$

Then, using Lemma 2.2 and 4.5, we see that (4.3) and (4.4) hold. ■

LEMMA 4.7. *Let* $F\in\mathscr{W}_A^{r,s}$ *with* $r>2$ *and* $s>2$. *Then the following* (i)-(iv) *hold*;

(i) $\mathfrak{D}_A\subset D(d_A\tilde{b}(F)^*)\cap D(\tilde{b}(F)^* d_A)$ *and*

$$\{d_A, \tilde{b}(F)^*\} = \langle\tilde{b}^*|\Lambda(A\otimes IF)|\tilde{b}^*\rangle \quad (4.5)$$

on $\mathfrak{D}_A$.

(ii) $\mathfrak{D}_A \subset D(d_A^* b(F)) \cap D(b(F) d_A^*)$ *and*

$$\{d_A^*, b(F)\} = < \tilde{b} | \Lambda(A \otimes IF)^* | \tilde{b} > \tag{4.6}$$

on $\mathfrak{D}_A$.

(iii) $\mathfrak{D}_A \subset D(d_A \tilde{b}(F)) \cap D(\tilde{b}(F) d_A)$ *and*

$$\{d_A, \tilde{b}(F)\} = < \tilde{b}^* | \Lambda(\overline{\bar{A}\nabla \otimes IF}) | \tilde{b} > + \tilde{\nabla}_{A,F} \tag{4.7}$$

on $\mathfrak{D}_A$.

(iv) $\mathfrak{D}_A \subset D({}_A^* \tilde{b}(F)^*) \cap D(\tilde{b}(F)^* d_A^*)$ *and*

$$\{d_A^*, \tilde{b}(F)^*\} = < \tilde{b}^* | \Lambda(\overline{\bar{A}\nabla \otimes IF})^* | \tilde{b} > - \tilde{\nabla}_{\bar{A}, \bar{F}} + (A\nabla)^* F \tag{4.8}$$

on $\mathfrak{D}_A$.

PROOF. It is easy to see that $\mathfrak{D}_A \subset D(\tilde{b}(F)^{\#})$. It is sufficient to prove (4.5)-(4.8) for vectors of the form

$$\Psi = \{0, \cdots, 0, \Psi^{(p)}, 0, \cdots\} \in \mathfrak{D}_A \tag{4.9}$$

with

$$\Psi^{(p)} = P_m(\phi(f_1), \cdots, \phi(f_m)) u_1 \wedge \cdots \wedge u_p,$$
$$f_j \in D(A) \cap \mathscr{H},\ u_k \in D(A^*), \quad j = 1, \cdots, m,\ k = 1, \cdots, p.$$

Throughout the proof, we set

$$\tilde{P}_m = P_m(\phi(f_1), \cdots, \phi\ (f_m)).$$

(i) For Ψ given by (4.9), $(\tilde{b}(F)^* \Psi)^{(k)} = 0$ for $k \neq p+1$ and

$$(\tilde{b}(F)^* \Psi)^{(p+1)}(\phi) = \sqrt{p+1}\, \tilde{P}_m F(\phi) \wedge u_1 \wedge \cdots \wedge u_p. \tag{4.10}$$

By the assumption on F, there exist sequences $\{F_n\}_{n=1}^{\infty} \subset \mathscr{P}_A$ and $\{e_n\}_{n=1}^{\infty} \subset D(A^*)$ such that $F^{(N)} := \sum_{n=1}^{N} F_n e_n$ satisfies

$$\|F^{(N)} - F\|_{L^r(E, d\mu;\, \mathscr{K}_c)} \to 0, \tag{4.11}$$
$$\|A\nabla \otimes IF^{(N)} - A\nabla \otimes IF\|_{L^s(E, d\mu;\, \mathscr{K}_c) \otimes \mathscr{K}_c} \to 0, \tag{4.12}$$
$$\|\bar{A}\nabla \otimes IF^{(N)} - \bar{A}\nabla \otimes IF\|_{L^s(E, d\mu;\, \mathscr{K}_c) \otimes \mathscr{K}_c} \to 0, \tag{4.13}$$

as $N \to \infty$. By (4.10) and Hölder's inequality, we have

$$\begin{aligned} \|\tilde{b}(F^{(N)})^* \Psi - \tilde{b}(F)^* \Psi\|^2 &= \|\tilde{b}(F - F^{(N)})^* \Psi\|^2 \\ &\leq C \int_E d\mu |\tilde{P}_m|^2\ \|F - F^{(N)}\|_{\mathscr{K}_c}^2 \\ &\leq C \Big(\int_E d\mu |\tilde{P}_m|^q \Big)^{2/q} \Big(\int_E d\mu \|F^{(N)} - F\|_{\mathscr{K}_c}^r \Big)^{2/r} \end{aligned}$$

with a constant $C>0$, where $1/q+1/r=1/2$. Hence, by (4.11), we obtain

$$\tilde{b}(F^{(N)})^*\Psi \to \tilde{b}(F)^*\Psi$$

as $N\to\infty$. Since $F_n\in\mathscr{P}_A$, it follows that $\tilde{b}(F^{(N)})^*\Psi\in D(d_A)$ and

$$(d_A\tilde{b}(F^{(N)})^*\Psi^{(p+2)}=\sqrt{(p+1)(p+2)}\,\{\eta_N^{(1)}+\eta_N{}^{(2)}\},$$

where

$$\eta_N^{(1)}=\sum_{j=1}^m \widetilde{\partial_j P_m Af_j\wedge F^{(N)}\wedge u_1\wedge\cdots\wedge u_p}$$

$$\begin{aligned}\eta_N^{(2)}&=\sum_{n=1}^N \tilde{P}_m A\nabla F_n\wedge e_n\wedge u_1\wedge\cdots\wedge u_p\\ &=\tilde{P}_m A_{p+2}((A\nabla\otimes IF^{(N)})\otimes u_1\otimes\cdots\otimes u_p).\end{aligned}$$

All the other components of $d_A\tilde{b}(F^{(N)})^*\Psi$ are zero. As in the preceding case, we have

$$\eta_N^{(1)}=\sum_{j=1}^m \widetilde{\partial_j P_m Af_j\wedge F^{(N)}\wedge u_1\wedge\cdots\wedge u_p}.$$

Simlarly, using (4.12), we have

$$\eta_N^{(2)} \to \tilde{P}_m A_{p+2}((A\nabla\otimes IF)\otimes u_1\otimes\cdots\otimes u_p).$$

Since d_A is closed, we conclude that $\tilde{b}(F)^*\Psi\in D(d_A)$ and

$$\begin{aligned}(d_A\tilde{b}(F)^*(\Psi)^{(p+2)}=\sqrt{(p+1)(p+2)}\Big\{&\sum_{j=1}^m \widetilde{\partial_j P_m Af_j\wedge F\wedge u_1\wedge\cdots\wedge u_p}\\ &+\tilde{P}_m A_{p+1}((A\nabla\otimes IF)\otimes u_1\otimes\cdots\otimes u_p)\Big\}.\end{aligned}$$

On the other hand, it is easy to see that $\Psi\in D(\tilde{b}(F)^*d_A)$ and

$$(\tilde{b}(F)^*d_A\Psi)^{(p+1)}=\sqrt{(p+1)(p+2)}\sum_{j=1}^m \widetilde{\partial_j P_m F\wedge Af_j\wedge u_1\wedge\cdots\wedge u_p}.$$

Using the antisymmetry of wedge product, we obtain

$$(\{d_A,\ \tilde{b}(F)^*\}\Psi)^{(p+2)}=\sqrt{(p+1)(p+2)}\,\tilde{P}_m A_{p+2}(A\nabla\otimes IF\otimes u_1\otimes\cdots\otimes u_p).$$

We have

$$\sum_{n=1}^N b(A\nabla F_n)^* b(e_n)^* u_1\wedge\cdots\wedge u_p$$

$$=\sqrt{(p+1)(p+2)}\, A_{p+2}((A\nabla\otimes IF^{(N)})\otimes u_1\otimes\cdots\otimes u_p).$$

Hence

$$(\{d_A,\ \tilde{b}(F)^*\}\Psi)^{(p+2)}=\lim_{N\to\infty}\sum_{n=1}^{N} b(A\nabla F_n)^* b(e_n)^* u_1\wedge\cdots\wedge u_p \tag{4.14}$$

in $\wedge^{p+2}(\mathscr{H},\mathscr{K})$. There exists a subsequene $\{N_k\}$ such that (4.14) with $N=N_k$ holds a.e. in the strong topology of $\wedge(\mathscr{K}_c)$. Hence we have for all $v_k\in\mathscr{K}_c$, $k=1,\cdots,p+2$,

$$\begin{aligned}
&(v_1\wedge\cdots\wedge v_{p+2},\ (\{d_A,\ \tilde{b}(F)^*\}\Psi)^{(p+2)})\\
&=\lim_{h\to\infty}\sum_{n=1}^{N_k}(\mathrm{b}(e_n)b(A\nabla F_n)v_1\wedge\cdots\wedge v_{p+2},\ \Psi^{(p)})\\
&=\sum_{i<j}^{p+2}\frac{1}{\sqrt{(p+1)(p+2)}}(-1)^{i+j}\\
&\times\lim_{k\to\infty}\sum_{n=1}^{N_k}((\bar{v}_i\wedge\bar{v}_j,\ \bar{e}_n\otimes\overline{A\nabla F_n})v_1\wedge\cdots\wedge\widehat{v}_i\wedge\cdots\wedge\widehat{v}_j\wedge\cdots\wedge v_{p+2},\ \Psi^{(p)})\\
&=(<\tilde{b}|\Lambda(A\nabla\otimes IF)^*|\tilde{b}>v_1\wedge\cdots\wedge v_{p+2},\ \Psi^{(p)})\\
&=(v_1\wedge\cdots\wedge v_{p+2},\ (<\tilde{b}^*|\Lambda(A\nabla\otimes IF)|\tilde{b}^*>\Psi)^{(p+2)}),
\end{aligned}$$

where, in the third equality, we have used Lemmas 2.5 and 2.6. Therefore we obtain

$$(\{d_A,\ \tilde{b}(F)^*\}\Psi)^{(p+2)}=(<b^*|\Lambda(A\nabla\otimes IF)|b^*>\Psi)^{(p+2)},\ \text{a.e.}$$

Thus (4.5) follows.

(ii) We need only to show that $\Psi\in D(d_A^*\tilde{b}(F))\cap D(\tilde{b}(F)d_A^*)$. Then (4.6) follows from taking the adjoint of (4.5) on $\mathfrak{D}_A$. It is easy to see that $\Psi\in D(\tilde{b}(F)d_A^*)$. We have

$$(\tilde{b}(F)\Psi)^{(p-1)}=\frac{1}{\sqrt{p}}\sum_{j=1}^{p}(-1)^{j-1}\tilde{P}_m(F,\ u_j)u_1\wedge\cdots\wedge\widehat{u}_j\wedge\cdots\wedge u_p.$$

Hence, in the same way as in the proof of part (i), we can show that

$$\tilde{b}(F^{(N)})\Psi\ \to\ \tilde{b}(F)\Psi$$

in $\wedge(\mathscr{H},\mathscr{K})$ as $N\to\infty$. Obviously $\tilde{b}(F^{(N)})\Psi\in D(d_A^*)$. Moreover, we have

$$\begin{aligned}
&(d_A^*\tilde{b}(F^{(N)})\Psi)^{(p-2)}\\
&=-\frac{1}{\sqrt{p(p-1)}}\sum_{j=1}^{p}(-1)^{j-1}\sum_{k<j}(-1)^{k-1}\{\nabla_{J_{\mathscr{H}}A^*u_k}\tilde{P}_m(F^{(N)},\ u_j)\\
&-\nabla_{J_{\mathscr{H}}A^*u_j}\tilde{P}_m(F^{(N)},\ u_k)\\
&+\tilde{P}_m(A\nabla F^{(N)},\ u_k\otimes u_j)-\tilde{P}_m(A\nabla F^{(N)},\ u_j\otimes u_k)
\end{aligned}$$

$$-\phi(A^*u_k)\tilde{P}_m(F^{(N)}, u_j)+\phi(A^*u_j)\tilde{P}_m(F^{(N)}, u_k)\}$$
$$\times u_1\wedge\cdots\wedge\hat{u}_k\wedge\cdots\hat{u}_j\wedge\cdots u_p.$$

By (4.11) and (4.12), the right hand side converges in $\wedge^{p-2}(\mathscr{H},\mathscr{K})$ as $N\to\infty$. Thus $\tilde{b}(F)\Psi\in D(d_A^*)$, i. e., $\Psi\in D(d_A^*\tilde{b}(F))$.

(iii) We have

$$(d_A\tilde{b}(F^{(N)})\Psi)^{(p)}=\sum_{j=1}^{p}(-1)^{j-1}\{(F^{(N)}, u_j)(A\nabla\tilde{P}_m)\wedge u_1\wedge\cdots\hat{u}_j\wedge\cdots\wedge u_p$$
$$+\sum_{n=1}^{N}\tilde{P}_m(e_n, u_j)(A\nabla F_n^*)\wedge u_1\wedge\cdots\hat{u}_j\wedge\cdots\wedge u_p\}. \quad (4.15)$$

In general, we have for all u_n, u_n, $u\in\mathscr{K}_c$ and $N\geq 1$

$$\|\sum_{n=1}^{N}(u_n, u)v_n\|\leq\|\sum_{n=1}^{N}v_n\otimes\bar{u}_n\|\|u\|.$$

Hence

$$\|\sum_{n=1}^{N}(e_n, u_j)A\nabla F_n^*\|_{\mathscr{K}_c}\leq\|\bar{A}\nabla\otimes IF^{(N)}\|_{\mathscr{K}_c\otimes\mathscr{K}_c}\|u_j\|.$$

Using this estimate and (4.13), we see that the right hand side of (4.15) converges in $\wedge^p(\mathscr{H},\mathscr{K})$. Thus $\Psi\in D(d_A\tilde{b}(F))$. As in part (ii), we can show that

$$(d_A\tilde{b}(F)\Psi)^{(p)}$$
$$=\sum_{j=1}^{p}(-1)^{j-1}(F, u_j)(A\nabla\tilde{P}_m)\wedge u_1\wedge\cdots\hat{u}_j\wedge\cdots\wedge u_p$$
$$+\tilde{P}_m(<\tilde{b}^*|\Lambda(\overline{\bar{A}\nabla\otimes IF})|\tilde{b}>u_1\wedge\cdots\hat{u}_j\wedge\cdots\wedge u_p)^{(p)}.$$

On the other hand, it is easy to see that $\Psi\in D(\tilde{b}(F)d_A)$ and

$$(\tilde{b}(F)d_A\Psi)^{(p)}=(F, A\nabla\tilde{P}_m)u_1\wedge\cdots\wedge u_p$$
$$-\sum_{j=1}^{p}(-1)^{j-1}(F, u_j)(A\nabla\tilde{P}_m)\wedge u_1\wedge\cdots\hat{u}_j\wedge\cdots\wedge u_p.$$

Thus (4.7) follows.

(iv) In the same way as in the preceding cases, we can show that $\Psi\in D(d_A^*b(F)^*)\cap D(\tilde{b}(F)^*d_A^*)$. Then, taking the adjoint of (4.7) on $\mathfrak{D}_A$, we obtain (4.8). ■

LEMMA 4.8. *Let $F\in L^r(E, d\mu\,;\,\mathscr{K}_c)$ with $r>4$. Then $\mathfrak{D}_I\subset D(\tilde{b}(F)^{\#}\tilde{b}(F)^{\#})$ and*

$$\{\tilde{b}(F), \tilde{b}(F)^*\}=\|F\|_{\mathscr{K}_c}^2,\ \{\tilde{b}(F), \tilde{b}(F)\}=0,\ \{\tilde{b}(F)^*, \tilde{b}(F)^*\}=0 \quad (4.16)$$

on $\mathfrak{D}_I$.

PROOF. Let $\Psi\in\mathfrak{D}_I$. Then, by Hölder's inequality, we have

$$\begin{aligned}\|\tilde{b}(F)^{\#}\tilde{b}(F)^{\#}\Psi\|^2 &\leq \int_E \|F(\phi)\|_{\mathscr{K}_c}^4 \|\Psi(\phi)\|_{\wedge(\mathscr{K}_c)}^2 d\mu(\phi)\\ &\leq \left(\int_E \|F(\phi)\|_{\mathscr{K}_c}^r d\mu(\phi)\right)^{4/r}\left(\int_E \|\Psi(\phi)\|_{\wedge(\mathscr{K}_c)}^q d\mu(\phi)\right)^{2/q}\\ &<\infty,\end{aligned}$$

where $4/r+2/q=1$. Hence $\mathfrak{D}_I\subset D(\tilde{b}(F)^{\#}\tilde{b}(F)^{\#})$. Formula (4.16) follows from the CARs of $b^{\#}(\cdot)$. ■

PROOF OF THEOREM 4.4.

We have $\mathfrak{D}_A^{(2)}\subset D(d_A^* d_A)\cap D(d_A d_A^*)\subset\mathfrak{D}_A$. Hence the domain properties stated in Lemmas 4.7 and 4.8 imply that $\mathfrak{D}_A^{(2)}\subset D(Q_A(F)^2)$ and we have

$$\begin{aligned}Q_A(F)^2=Q_A^2&+\{d_A, \tilde{b}(F)\}+\{d_A, \tilde{b}(F)^*\}+\{d_A^*, \tilde{b}(F)\}+(d_A^*, \tilde{b}(F)^*\}\\ &+(\tilde{b}(F)+\tilde{b}(F)^*)^2\end{aligned}$$

on $\mathfrak{D}_A^{(2)}$. Then (4.5)-(4.8) and (4.16) yield (4.1). ■

As a corollary of Lemma 4.7, we have

PROPOSITION 4.9. *Let $F\in\mathscr{W}_A^{r,s}$ with $r>2$ and $s>2$. Then*

$$d_A(F)^2=0 \quad \text{on } \mathfrak{D}_A$$

if and only if

$$A\nabla\otimes IF(\phi)\in\wedge^2(\mathscr{K}_c)^{\perp}\ \text{a.e.}\phi. \tag{4.17}$$

PROOF. By part (i) of Lemma 4.7, we have

$$d_A(F)^2=\{d_A, \tilde{b}(F)^*\}=<\tilde{b}^*|\Lambda(A\nabla\otimes IF)|\tilde{b}^*>$$

on $\mathfrak{D}_A$. By the proof of part (i) of Lemma 4.7, $<\tilde{b}^*|\Lambda(A\nabla\otimes IF)|\tilde{b}^*>=0$ on $\mathfrak{D}_A$ if and only if (4.17) holds. Thus the desired result follows. ■

Proposition 4.9 gives a necessary and sufficient condition for $d_A(F)$ to be nilpotent on $\mathfrak{D}_A$. Under condition (4.17), the last two terms on the right hand side of (4.1) vanish, so that the form of $\Delta_A(F)$ becomes simpler.

V. Essential self-adjointness of the perturbed Laplacian and the perturbed Dirac operator

In this section we prove that for a class of F, $\Delta_A(F)$ and $Q_A(F)$ are essentially self-adjoint on a suitable domain.

Let

$$\mathfrak{D}_A^\infty=\mathscr{L}\{P_n(\phi(f_1),\cdots,\phi(f_n))u_1\wedge\cdots\wedge u_p|n,p\geq 0, P_n\in\mathbf{P}_n, f_j\in C^\infty(A^*A), u_k\in C^\infty(AA^*), j=1,\cdots,n; k=1,\cdots,p\},$$

where $C^\infty(T):=\bigcap_{n=1}^\infty D(T^n)$($T$: a linear operator in a Hilbert space). We denote by N_b the number operator on the Boson Fock space $L^2(E,d\mu)$; $N_b=d\Gamma_b(I)$, i.e., N_b is the self-adjoint operator such that $N_b\restriction\Gamma_n(\mathscr{H})=n$. We prove the following theorem.

THEOREM 5.1. *Let $F\in\mathbf{F}_A^{r,s}$ with $r>4$ and $s>2$. Suppose that $F\in\bigoplus_{n=0}^M\Gamma_n(\mathscr{H})\otimes\mathscr{K}_c$ for some $M<\infty$ and there exists constant $C>0$ such that*

$$\|\|F\|_{\mathscr{K}_c}^2\Psi\|_{L^2(E,d\mu)}\leq C\|(N_b+1)^2\Psi\|_{L^2(E,d\mu)}, \tag{F.1}$$
$$\|(A\nabla)^*F\Psi\|_{L^2(E,d\mu)}\leq C\|(N_b+1)^2\Psi\|_{L^2(E,d\mu)}, \tag{F.2}$$
$$\|\|A\nabla\otimes IF\|_{\mathscr{K}_c\otimes\mathscr{K}_c}^2\Psi\|_{L^2(E,d\mu)}\leq C\|(N_b+1)^2\Psi\|_{L^2(E,d\mu)}, \tag{F.3}$$
$$\|\|\bar{A}\nabla\otimes IF\|_{\mathscr{K}_c\otimes\mathscr{K}_c}^2\Psi\|_{L^2(E,d\mu)}\leq C\|(N_b+1)^2\Psi\|_{L^2(E,d\mu)}, \tag{F.4}$$

for all $\Psi\in D(N_b^2)$. Then $\Delta_A(F)$ is essentially self-adjoint on $\mathfrak{D}_A^\infty$.

As a corollary of Theorem 5.1, we have the following result.

THEOREM 5.2. *Under the assumption of Theorem* 5.1, *$Q_A(F)$ is essentially self-adjoint on $\mathfrak{D}_A^\infty$.*

PROOF. This follows from Theorems 4.4, 5.1, and an application of Lemma 4.1 with $\mathfrak{D}=\mathfrak{D}_A^\infty\subset\mathfrak{D}_A^{(2)}$. ■

The rest of this section is devoted to the proof of Theorem 5.1. The basic idea for that is to employ the following theorem.

THEOREM 5.3 ([10], cf.also [17]). *Let $\mathscr{M}_n$, $n\geq 0$, be Hilbert spaces and*

$$\mathscr{M}=\bigoplus_{n=0}^\infty\mathscr{M}_n$$

be the infinite dircet sum of $\{\mathscr{M}_n\}_{n=0}^\infty$. Let

$$\mathscr{D}_0=\{f=\{f^{(n)}\}_{n=0}^\infty\in\mathscr{M}|f^{(n)}=0 \text{ for all but finitely many } n\}.$$

Let $\hat{N}$ be the self-adjoint operator such that $\hat{N}\restriction\mathscr{M}_n=n$ (the degree operator on $\mathscr{M}$). Let T be a self-adjoint operator in $\mathscr{M}$ which is reduced by each $\mathscr{M}_n$ and S be a symmetric operator in $\mathscr{M}$ which satisfies the following conditions (i) *and* (ii):

(i) *$\mathscr{D}_0\subset D(S)$ and there exists a constant $C>0$ such that*

$$\|Sf\| \leq C\|(\hat{N}+1)^2 f\|, \quad f \in \mathscr{D}_0.$$

(ii) *There exists an integer* $p \geq 0$ *such that for all* $f \in \mathscr{D}_0$ *and for* $|m-n| > p+1$,

$$(f^{(m)}, Sf^{(n)}) = 0.$$

Suppose that $T+S$ *is bounded from below on* $\mathscr{D}_0 \cap D(T)$. *Then* $T+S$ *is essentially self-adjoint on* $\mathscr{D}_0 \cap D(T)$.

REMARK. Condition (i) is a special case of the condition (B1) of Theorem 2.1 in [10], i.e., the case $L=I$ with the notation there.

We prepare some lemmas. Let

$$\mathscr{P}_n = \mathscr{L}\{: \phi(f_1)\cdots\phi(f_n)) : | f_j \in \mathscr{H}, j=1, \cdots, n,\} \subset \Gamma_n(\mathscr{H}), \ n \geq 0.$$

Note that $\mathscr{P}_n \perp \mathscr{P}_m$ for $n \neq m$.

LEMMA 5.4. *Let* $G \in \bigoplus_{j=0}^{M} \mathscr{P}_j$ *for some* $M < \infty$ *and* $\Phi_m \in \mathscr{P}_m$ *and* $\Psi_n \in \mathscr{P}_n$. *Then the following* (i) *and* (ii) *hold*:

(i) *If* $|m-n| > M$, *then*

$$(\Phi_m, G\Psi_n)_{L^2(E,d\mu)} = 0.$$

(ii) *If* $|m-n| > M-1$, *then*

$$(\Phi m, (\nabla_f G)\Psi_n)_{L^2(E,d\mu)} = 0$$

for all $f \in \mathscr{H}_c$.

PROOF. We need only to note that $G\Psi_n \in \bigoplus_{k=0}^{M+n} \mathscr{P}_k$ and $(\nabla_f G)\Psi_n \in \bigoplus_{k=0}^{M-1+n} \mathscr{P}_k$. ■

We can write

$$\wedge(\mathscr{H}, \mathscr{K}) = \bigoplus_{n=0}^{\infty} \mathscr{F}_n$$

with

$$\mathscr{F}_n = \bigoplus_{m+p=n} \Gamma_m(\mathscr{H}) \otimes \wedge^p(\mathscr{K}_c).$$

Each element $\Psi^{(n)} \in \mathscr{F}_n$ is written as

$$\Psi^{(n)} = \sum_{m+p=n} \Psi_n^{(m,p)}$$

with $\Psi_n^{(m,p)}\in\Gamma_m(\mathscr{H})\otimes\wedge^p(\mathscr{K}_c)$. Let

$$\mathscr{P}_{A,n}^{\infty}=\mathscr{L}\{:\phi(f_1)\cdots\phi(f_n)):|f_j\in C^{\infty}(A^*A),j=1,\cdots,n,\},\quad n\geq 0.$$

Then the subspace

$$\mathscr{D}_n:=\bigoplus_{m+p=n}\mathscr{P}_{A,m}^{\infty}\hat{\otimes}A_p(C^{\infty}(AA^*)\hat{\otimes}\cdots\hat{\otimes}C^{\infty}(AA^*))$$

is dense in $\mathscr{F}_n$.

LEMMA 5.5. *Let $F\in\mathbf{F}_A^{r,s}\cap[\bigoplus_{m=1}^{M}\Gamma_m(\mathscr{H})\otimes\mathscr{K}_c]$ with $r>4, s>2$ and $M<\infty$. Let $\Phi^{(n)}\in\mathscr{D}_n, \Psi^{(m)}\in\mathscr{D}_m$. Then the following* (i)-(v) *hold*:

(i) *If $|n-m|>2M$, then*

$$(\Phi^{(n)},\|F\|_{\mathscr{K}_c}^2\Psi^{(m)})_{\wedge(\mathscr{H},\mathscr{K})}=0. \tag{5.1}$$

(ii) *If $|n-m|>M+1$, then*

$$(\Phi^{(n)},(A\nabla)^*F\Psi^{(m)})_{\wedge(\mathscr{H},\mathscr{K})}=0.$$

(iii) *If $|n-m|>M-1$, then*

$$(\Phi^{(n)},<\tilde{b}^*|\Lambda(\overline{\bar{A}\nabla\otimes IF})^{\#}|\tilde{b}>\Psi^{(m)})_{\wedge(\mathscr{H},\mathscr{K})}=0. \tag{5.2}$$

(iv) *If $|n-m-2|>M-1$, then*

$$(\Phi^{(n)},<\tilde{b}^*|\Lambda(A\nabla\otimes IF)|\tilde{b}^*>\Psi^{(m)})_{\wedge(\mathscr{H},\mathscr{K})}=0. \tag{5.3}$$

(v) *If $|n-m+2|>M-1$, then*

$$(\Phi^{(n)},<\tilde{b}|\Lambda(A\nabla\otimes IF)^*|\tilde{b}>\Psi^{(m)})_{\wedge(\mathscr{H},\mathscr{K})}=0.$$

PROOF. (i) Let $F^{(N)}$ be as in the proof of Lemma 4.7. Then we have

$$\begin{aligned}(\Phi^{(n)},\|F\|_{\mathscr{K}_c}^2\Psi^{(m)})&=\lim_{N\to\infty}(\Phi^{(n)},\|F^{(N)}\|_{\mathscr{K}_c}^2\Psi^{(m)})\\&=\lim_{N\to\infty}\sum_{k,l=1}^{N}(e_k,e_l)(\Phi^{(n)},F_k^*F_l\Psi^{(m)}).\\&=\lim_{N\to\infty}\sum_{k,l=1}^{N}(e_k,e_l)\sum_{s+p=n}\sum_{r+p=m}(\Phi_n^{(s,p)},F_k^*F_l\Psi_m^{(r,p)}).\end{aligned}$$

Since $F_k\in\bigoplus_{m=0}^{M}\Gamma_m(\mathscr{H})$ for all k, it follows that $F_k^*F_l\in\bigoplus_{m=0}^{2M}\Gamma(\mathscr{H})$. Hence, by Lemma 5.4(i),

$$(\Phi_n^{(s,p)},F_k^*F_l\Psi_m^{(r,p)})=0$$

if $|s-r|=|n-m|>2M$. Thus (5.1) follows.

(ii) It is sufficient to show that, if $|s-r|>M+1$, then

$$(P_s, (A\nabla)^*F)Q_r)=0 \tag{5.4}$$

for all $P_s \in \mathscr{P}^{\infty}_{A,s}$, $Q_r \in \mathscr{P}^{\infty}_{A,r}$. We have by Lemma 4.5

$$(P_s, (A\nabla)^*FQ_r)=\lim_{N\to\infty}\sum_{k=1}^{N}(P_s, (-\tilde{\nabla}_{J_{\mathscr{H}}A^*e_k}F_k+\phi(A^*j_k)F_k)Q_r),$$

which, together with Lemma 5.4 (ii), implies (5.4).

(iii) By the proof of Lemma 4.7 (iii), we have

$$(\Phi^{(n)}, <\tilde{b}^*|\Lambda(\overline{\bar{A}\nabla\otimes IF})|\tilde{b}>\Psi^{(m)})=\lim_{N\to\infty}\sum_{k=1}^{N}(\Phi^{(n)}, b(AJ_{\mathscr{H}}\nabla F_k)^*b(e_n)\Psi^{(m)}).$$

Since $b(u)^*b(v)$ maps $\wedge^p(\mathscr{K}_c)$ into itself, it turns out that we need only to show that, if $|s-r|>M-1$, then

$$(P_s, (u, AJ_{\mathscr{H}}\nabla F_k)_{\mathscr{K}_c}Q_r)=0$$

for all $P_s \in \mathscr{P}^{\infty}_{A,s}$, $Q_r \in \mathscr{P}^{\infty}_{A,r}$, $u \in \mathscr{K}_c$ and $k\geq 1$. This follows from Lemma 5.4 (ii). Thus (5.2) follows.

(iv) By the proof of Lemma 4.7 (ii), we have

$$(\Phi^{(n)}, <\tilde{b}^*|\Lambda(A\nabla\otimes IF)|\tilde{b}^*>\Psi^{(m)})=\lim_{N\to\infty}\sum_{k=1}^{N}(\Phi^{(n)}, b(A\nabla F_k)^*b(e_n)^*\Psi^{(m)}).$$

Since $b(u)^*b(v)^*$ maps $\wedge^p(\mathscr{K}_c)$ into $\wedge^{p+2}$, it turns out that we need only to show that, if $|s-r|>M-1$, then

$$(P_s, (u, A\nabla F_k)_{\mathscr{K}_c}Q_r)=0$$

for all $P_s \in \mathscr{P}^{\infty}_{A,s}$, $Q_r \in \mathscr{P}^{\infty}_{A,r}$ $u \in \mathscr{K}_c$ and $k\geq 1$. This follows from Lemma 5.4 (ii). Thus (5.3) follows.

(v) Similar to the proof of (iv) or consider the adjoint relation of (5.3). ∎

We denote by N_f the number operator in $\wedge(\mathscr{K}_c)$: $N_f=d\Gamma_f(I)$, i.e., N_f is the selfadjoint operator in $\wedge(\mathscr{K}_c)$ such that $N_f \upharpoonright \wedge^p(\mathscr{K}_c)=p$.

LEMMA 5.6. *Let $T\in\mathscr{I}_2(\mathscr{K}_c)$. Then, for all $\Psi\in D(N_f^{1/2})$,*

$$\|<b^*|T|b>\Psi\|\leq\|T\|_2\|N_f^{1/2}\Psi\|, \tag{5.5}$$
$$\|<b|T|b>\Psi\|\leq\|T\|_2\|N_f^{1/2}\Psi\|, \tag{5.6}$$
$$\|<b^*|T|b^*>\Psi\|\leq\|T\|_2\|(N_f+2)^{1/2}\Psi\|. \tag{5.7}$$

PROOF. Let T be as in (2.7). Let $\Phi\in\wedge(\mathscr{K}_c)$ and $\Psi\in\wedge_f(\mathscr{K}_c)$. Then

$$\begin{aligned}|(\Phi, <b^*|T|b>\Psi)| &\le \sum_{n=1}^{N}|\lambda_n|\ |(b(\phi_n)\Phi, b(\psi_n)\Psi)| \\ &\le \sum_{n=1}^{N}|\lambda_n|\ \|\Phi\|\|b(\psi_n)\Psi)\| \\ &\le \left(\sum_{n=1}^{N}|\lambda_n|^2\right)^{1/2}\left(\sum_{n=1}^{N}\|b(\psi_n)\Psi\|^2\right)^{1/2}\|\Phi\|.\end{aligned}$$

We have

$$\sum_{n=1}^{N}|\lambda_n|^2=\|T\|_2^2$$

and

$$\begin{aligned}\sum_{n=1}^{N}\|b(\psi_n)\Psi\|^2 &= \sum_{n=1}^{\infty}(\Psi, b(\psi_n)^*b(\psi_n)\Psi) \\ &\le(\Psi, N_f\Psi) \\ &=\|N_f^{1/2}\Psi\|^2.\end{aligned}$$

Hence we obtain

$$|(\Phi, <b^*|T|b>\Psi)|\le\|T\|_2\|N_f^{1/2}\Psi\|\|\Phi\|,$$

which implies (5. 5) with $\Psi\in\wedge_f(\mathscr{K}_c)$. Since $\wedge_f(\mathscr{K}_c)$, is a core for N_f, a limiting argument gives (5. 5) with $\Psi\in D(N_f^{1/2})$. Similarly we can prove (5. 6).

To prove (5. 7), we recall that, if $T\in\mathscr{I}_2(\mathscr{K}_c)$, then $T^*\in\mathscr{I}_2(\mathscr{K}_c)$ and $\|T\|_2=\|T^*\|_2$ (e. g., [18, § VI. 6]) and note that (5. 6) implies that $K:=<b|T^*|b>(N_f+\epsilon)^{-1/2}$ is bounded for all $\epsilon>0$ with $\|K\|\le\|T^*\|_2=\|T\|_2$. Hence its adjoint is bounded with $\|K^*\|\le\|T\|_2$. On the other hand,

$$K^*=(N_f+\epsilon)^{-1/2}<b^*|T|b^*> \text{ on } \wedge_f(\mathscr{K}_c).$$

Hence, for all $\Psi\in\wedge_f(\mathscr{K}_c)$,

$$\|(N_f+\epsilon)^{-1/2}\ <b^*|T|b^*>\Psi\|\le\|T\|_2\|\Psi\|.$$

Note that

$$(N_f+\epsilon)^{-1/2}\ <b^*|T|b^*>\Psi=<b^*|T|b^*>(N_f+2+\epsilon)^{-1/2}\ \Psi.$$

Thus we obtain

$$\|<b^*|T|b^*>\Psi\|\le\|T\|_2\|(N_f+2+\epsilon)^{1/2}\ \Psi\|.$$

Taking the limit $\epsilon\to 0$, we have

$$\|<b^*|T|b^*>\Psi\|\le\|T\|_2\|(N_f+2)^{1/2}\ \Psi\|.$$

Thus (5. 7) follows. ■

We are now ready to prove Theorem 5. 1

PROOF OF THEOREM 5. 1.
Let

$$H_1=\|F\|^2_{\mathscr{K}_c},\quad H_2=(A\nabla)^*F,$$
$$H_3=<\tilde{b}^*|\Lambda(\overline{\bar{A}\nabla\otimes IF})|\tilde{b}>,\quad H_4=<\tilde{b}^*|(\Lambda(\overline{\bar{A}\nabla\otimes IF})^*|\tilde{b}>,$$
$$H_5=<\tilde{b}^*|\Lambda(A\nabla\otimes IF)|\tilde{b}^*>,\quad H_6=<\tilde{b}|(\Lambda(A\nabla\otimes IF)^*|\tilde{b}>,$$

and

$$U_F=\sum_{j=1}^6 H_j,$$

so that, by Theorem 4. 4,

$$\Delta_A(F)=Q_A(F)^2=\Delta_A+U_F \text{ on } \mathfrak{D}_A^{(2)}.$$

Let

$$N=N_b\otimes I+I\otimes N_f.$$

Then we have

$$N\upharpoonright\mathscr{F}_n=n.$$

By Lemma 5. 6 and (2. 12), we have

$$\|H_3\Psi\|^2,\ \|H_4\Psi\|^2\leq\int_E\|\bar{A}J\nabla\otimes IF(\phi)\|^2\|N_f^{1/2}\Psi(\phi)\|^2_{\wedge(\mathscr{K}_c)}\,d\mu(\phi),$$
$$\|H_5\Psi\|^2,\ \|H_6\Psi\|^2\leq\int_E\|A\nabla\otimes IF(\phi)\|^2\|(N_f+2)^{1/2}\ \Psi(\phi)\|^2_{\wedge(\mathscr{K}_c)}d\mu(\phi),$$

for all Ψ such that the right hand sides are finite. Let G be a function on E such that

$$\|G^2\Psi\|_{L^2(E,d\mu)}\leq C\|(N_b+1)^2\Psi\|_{L^2(E,d\mu)}.$$

Then, for all $c>0$ and $\epsilon>0$, we have

$$\begin{aligned}
&\int_E|G(\phi)|^2\|(N_f+c)^{1/2}\ \Psi(\phi)\|^2_{\wedge(\mathscr{K}_c)}d\mu(\phi)\\
&\leq\int_E|G(\phi)|^2\|\Psi(\phi)\|_{\wedge(\mathscr{K}_c)}\|(N_f+c)\Psi(\phi)\|_{\wedge(\mathscr{K}_c)}d\mu(\phi)\\
&\leq\epsilon\int_E\|(N_f+c)\Psi(\phi)\|^2_{\wedge(\mathscr{K}_c)}d\mu(\phi)+\frac{1}{4\epsilon}\int_E|G(\phi)|^4\|\Psi(\phi)\|^2_{\wedge(\mathscr{K}_c)}d\mu(\phi)\\
&=\epsilon\|(N_f\otimes I+c)\Psi\|^2_{\wedge(\mathscr{H},\mathscr{K})}+\frac{1}{4\epsilon}\||G|^2\otimes I\Psi\|^2_{\wedge(\mathscr{H},\mathscr{K})}\\
&\leq C_1\|(N+1)^2\Psi\|^2_{\wedge(\mathscr{H},\mathscr{K})}
\end{aligned}$$

with a constant $C_1>0$. Hence, using the conditions (F. 1)-(F. 4), we obtain

$$\|H_j\Psi\|_{\wedge(\mathscr{H},\mathscr{K})}\le D\|(N+1)^2\Psi\|_{\wedge(\mathscr{H},\mathscr{K})},\quad \Psi\in D(N^2), j=1,\cdots,6, \tag{5.8}$$

with a constant $D>0$. Lemma 5.5 implies that, for all $\Phi^{(n)}\in\mathscr{D}_n$, $\Psi^{(m)}\in\mathscr{D}_m$ with $|n-m|>\max\{2M, M+1\}$,

$$(\Phi^{(n)}, H_j\Psi^{(m)})_{\wedge(\mathscr{H},\mathscr{K})}=0,\quad j=1,\cdots,6. \tag{5.9}$$

Since $\mathscr{D}_n$ is a core for N^2 and we have (5.8), we can extend (5.9) to all $\Phi^{(n)}\in\mathscr{F}_n$ and $\Psi^{(m)}\in\mathscr{F}_m$. Hence if follows that

$$(\Phi^{(n)}, U_F\Psi^{(m)})_{\wedge(\mathscr{H},\mathscr{K})}=0,\quad \Phi^{(n)}\in\mathscr{F}_n, \Psi^{(m)}\in\mathscr{F}_m.$$

We also have from (5.8)

$$\|Q_A(F)^2\Psi\|\le C\|(\Delta_A+N^2+I)\Psi\|,\quad \Psi\in\mathfrak{D}_A^\infty, \tag{5.10}$$

with a constant $C>0$. It is not so difficult to show that $\mathfrak{D}_A^\infty$ is a core for Δ_A+N^2 (e.g., apply Theorem VIII. 11 in [19]). Hence we can extend (5.10) to all $\Psi\in D(\Delta_A)\cap D(N^2)$, at the same time, obtaining

$$D(\Delta_A)\cap D(N^2)\subset D(\bar{Q}_A(F)^2)$$

and

$$\Delta_A(F)=\bar{Q}_A(F)^2=\Delta_A+U_F \text{ on } D(\Delta_A)\cap D(N^2).$$

By these results, we can can apply Theorem 5.3 to the present case with

$$\mathscr{M}=\wedge(\mathscr{H},\mathscr{K}), \mathscr{M}_n=\mathscr{F}_n, \hat{N}=N, T=\Delta_A, S=U_F$$

to conclude that $\Delta_A(F)=\bar{Q}_A(F)^2$ is essentially self-adjoint on $D(\Delta_A)\cap\mathfrak{D}_0$ with $\mathfrak{D}_0=\{\Psi=\{\Psi^{(n)}\}_{n=0}^\infty\in\wedge(\mathscr{H},\mathscr{K})|\Psi^{(n)}\in\mathscr{F}_n, \Psi^{(n)}=0$ for all but finitely many $n\}$.

Finally we show that $\mathfrak{D}_A^\infty$ is a core for $\Delta_A(F)=\bar{Q}_A(F)^2$. Let $\Psi=\{\Psi^{(m)}\}_{m=0}^\infty\in D(\Delta_A)\cap\mathfrak{D}_0$. Then there exists an $M<\infty$ such that $\Psi^{(m)}=0$ for all $m>M$. Let $\Psi^{(m)}=\sum_{s+p=m}\Psi_m^{(s,p)}$ with $\Psi_m^{(s,p)}\in\Gamma_s(\mathscr{H})\otimes\wedge^p(\mathscr{K}_c)$. Since Δ_A is reduced by each

$$\mathscr{F}_{s,p}:=\Gamma_s(\mathscr{H})\otimes\wedge^p(\mathscr{K}_c)$$

and

$$\mathscr{D}_{s,p}:=\mathscr{S}_{A,s}^\infty\otimes A_p(C^\infty(AA^*)\otimes\cdots\hat{\otimes}C^\infty(AA^*))\subset\mathfrak{D}_A^\infty$$

is a core for $\Delta_A\restriction\mathscr{F}_{s,p}$, there exists a sequence $\{\Psi_m^{s,p}(n)\}_{n=1}^\infty\subset\mathscr{D}_{s,p}$ such that

$$\Psi_m^{(s,p)}(n)\to\Psi_m^{(s,p)},$$
$$\Delta_A\Psi_m^{(s,p)}(n)\to\Delta_A\Psi_m^{(s,p)},$$

as $n \to \infty$. Hence, putting

$$\Psi(n)=\sum_{n=0}^{M}\sum_{s+p=m}\Psi_m^{(s,p)}(n)\in\mathfrak{D}_A^{\infty},$$

we have

$$\begin{aligned}&\Psi(n)\to\Psi,\\ &\Delta_A\Psi(n)\to\Delta_A\Psi,\end{aligned}\tag{5.11}$$

as $n \to \infty$. Since $N^2\Psi_m^{(s,p)}=m^2\Psi_m^{s,p}$, it follows that

$$N^2\Psi(n)\to N^2\Psi$$

as $n \to \infty$. Hence, by (5.8),

$$U_F\Psi(n)\to U_F\Psi$$

as $n \to \infty$, which, together with (5.11), gives

$$\Delta_A(F)\Psi(n)\to\Delta_A(F)\Psi$$

as $n \to \infty$. Thus $D(\Delta_A)\cap\mathfrak{D}_0$ is included in the domain of the closure of $\Delta_A(F)$ restricted to $\mathfrak{D}_A^{\infty}$. This result and the essential self-adjointness of $\Delta_A(F)$ on $D(\Delta_A)\cap\mathfrak{D}_0$ imply that $\Delta_A(F)$ is essentially self-adjoint on $\mathfrak{D}_A^{\infty}$. ■

VI. Application to models of SSQFT

In [1] the author showed that some models of SSQFT are given as concrete realizations of the abstract theory described in Sections II-IV. In each of those models, the Dirac operator $Q_A(F)$ and the Laplacian $\Delta_A(F)$ correspond to a supercharge and the supersymmetric Hamiltonian, respectively. Hence we can apply the results obtained in the present paper to those SSQFT models to prove the essential self-adjointness of their supercharges and supersymmetric Hamiltonians. Here we only state the results on the $N=1$ and the $N=2$ Wess-Zumino (WZ) models. For the details of these models, see [1, 16]. We follow the notations of Section VII in [1].

6.1. The $N=1$ WZ model

The Hilbert space of state vectors of this model is the BFFS $L^2(\mathscr{D}(T_l^1)', d\mu_0)\otimes\wedge(L^2(T_l^1))$(the case where $E=\mathscr{D}(T_l^1)'$, $\mathscr{K}_c=L^2(T_l^1)$) and a supercharge of the model is given by

$$Q_\kappa=Q_0+\frac{1}{\sqrt{2}}\int_{T_l^1}(\psi_+(x)+\psi_-(x))(a:\phi_\kappa(x)^2:+b\phi_\kappa(x))\ dx,$$

where a and b are real constants. Applying Theorems 5.1 and 5.2, we can show that Q_κ and Q_κ^2 (the supersymmetric Hamiltonian of the model) are essentially self-adjoint on the subspace

$$\mathscr{L}\{P_n(\phi(f_1), \cdots, \phi(f_n))u_1\wedge\cdots\wedge u_p|P\in\mathbf{P}_n, f_j, u_k\in C^\infty(T_l^1), j=1, \cdots, n, \\ k=1, \cdots, p;\ n, p\geq 0\}.$$

6.2. The $N=2$ WZ model

The BFFS for this model in $L^2(\mathscr{D}(T_l^1)'\times \mathscr{D}(\mathrm{T}_l^1)', d\mu_0\otimes d\mu_0)\otimes \wedge(L^2(T_l^1)\oplus L^2(T_l^1))$. A supercharge of the model is given by

$$\tilde{Q}_\kappa=Q_0-\frac{i}{\sqrt{2}}\int_{T_l^1}\{\psi_1(x)P(\Phi_\kappa(x))+\psi_2(x)P(\Phi_\kappa(x))^*+\psi_1(x)^*P(\Phi_\kappa(x))^* \\ +\psi_2(x)^*P(\Phi_\kappa(x))\}\ dx,$$

where

$$P(z)=\lambda z^2+\mu z,\quad z\in\mathbf{C},$$

with constants $\mu, \lambda\in\mathbf{C}$. Applying Theorems 5.1 and 5.2, we can prove that $\tilde{Q}_\kappa$ and $\tilde{Q}_\kappa^2$ are essentially self-adjoint on the subspace

$$\mathscr{L}\{P_n(\phi(f_1\oplus g_1), \cdots, \phi(f_n\oplus g_n))(u_1\oplus v_1)\wedge\cdots\wedge(u_p\oplus v_p)|\mathbf{P}_n\in\mathbf{P}_n, \\ f_j, g_j, u_k, v_k\in C^\infty(T_l^1), j=1, \cdots, k=1, \cdots, p;\ n, p\geq 0\}.$$

REMARK. The above results can be extended to the case where the one-torus T_l^1 is replaced by $\mathbf{R}$ and a space-cutoff function with suitale regularities is introduced in the interaction term of Q_κ (resp. $\tilde{Q}_\kappa$).

Appendix. Some facts on decomposable operators

Let (M, μ) be a measure space and $\mathscr{H}$ be a separable Hilbert space. We say that an operator A in the Hilbert space

$$L^2(M, d\mu\,;\,\mathscr{H})=\int_M^\oplus \mathscr{H}\,d\mu(m)$$

is decomposable if for a.e. $m\in M$, there exists an operator $A(m)$ in $\mathscr{H}$ such that

$D(A)=\{f\in L^2(M, d\mu\,;\,\mathscr{H})|f(m)\in D(A(m))$ a.e. $m\in M$, $A(m)f(m)$ is measurable,

$$\int_M\|A(m)f(m)\|_{\mathscr{H}}^2 d\mu<\infty\},$$

and, for all $f\in D(A)$,

$$(Af)(m)=A(m)f(m) \quad \text{a. e. } m.$$

In this case we write

$$A=\int_M^{\oplus} A(m)d\mu(m).$$

The $A(m)$ are called fibres of A (cf. [21, § XIII. 16]).

PROPOSITION A. 1. *Let A be a decomposable operator in L^2 $(M, d\mu;$ $\mathscr{H})$ such that each fibre $A(m)$ is closed. Then A is closed.*

PROOF. Let $f_n \in D(A)$ such that $Af_n \to g \in L^2$ $(M, d\mu;$ $\mathscr{H})$ and $f_n \to f \in L^2$ $(M, d\mu;$ $\mathscr{H})$ as $n \to \infty$. Then it follows that there exists a subsequence $\{n_k\}_{k=1}^{\infty}$ such that for a. e. m

$$f_{n_k}(m) \to f(m), \quad A(m)f_{n_k}(m) \to g(m),$$

as $k \to \infty$ in the norm of $\mathscr{H}$. The closedness of $A(m)$ implies that $f(m) \in D(A(m))$ and $g(m)=A(m)f(m)$. Hence $f \in D(A)$ and $Af=g$. Thus A is closed. ∎

We denote by $L^p(M, d\mu)\hat{\otimes}\mathscr{H}$ the algebraic tensor product of $L^p(M, d\mu)$ and $\mathscr{H}$.

PROPOSITION A. 2. *Let (M, μ) be a probability measure space. Let A be a decomposable operator in $L^2(M, d\mu;$ $\mathscr{H})$ such that each fibre $A(m)$ is bounded on $\mathscr{H}$ and $\|A(\cdot)\| \in L^p(M, d\mu)$ for some $p>2$. Let $r=2p/(p-2)$. Then A is densely defined with*

$$D(A) \supset L^r(M, d\mu)\hat{\otimes}\mathscr{H} \tag{A. 1}$$

and A^* is given by

$$A^*=\int_M^{\oplus} A(m)^* d\mu(m). \tag{A. 2}$$

PROOF. Since $A(m)$ is bounded, $D(A(m))=\mathscr{H}$ so that for all $u \in L^2(M, d\mu)$ and $f \in \mathscr{H}$, we have $u(m)f \in D(A(m))$. Let $u \in L^r(M, d\mu)$. Then, by Hölder's inequality, we have

$$\int_M \|A(m)u(m)f\|_{\mathscr{H}}^2 d\mu \leq \int |u(m)|^2 \|A(m)\|^2 \|f\|_{\mathscr{H}}^2 d\mu$$
$$\leq \left(\int_M |u(m)|^r d\mu\right)^{r/2} \left(\int_M \|A(m)\|^p d\mu\right)^{p/2} \|f\|_{\mathscr{H}}^2 < \infty.$$

Hence $u(\cdot)f \in D(A)$. Thus (A. 1) follows. Since the subspace $L^r(M, d\mu)$ $\hat{\otimes}\mathscr{H}$ is dense in $L^2(M, d\mu;$ $\mathscr{H})$, it follows that $D(A)$ is dense in $L^2(M, d\mu;$

$\mathscr{H}$).

Let $f\in D(A^*)$ and $A^*f=g$. Then, for all $h\in\mathscr{H}$ and $u\in L^r(M, d\mu)$, we have

$$\int_M u(m)^*\eta(m)d\mu=0\,, \tag{A. 3}$$

where $\eta(m)=(A(m)h, f(m))_{\mathscr{H}}-(h, g(m))_{\mathscr{H}}$. Let $s=2p/(p+2)$, so that $1/r+1/s=1$. Then, by Hölder's inequality, we have

$$\int_M(\|A(m)\|\|f(m)\|_{\mathscr{H}})^s d\mu\leq\left(\int_M\|A(m)\|^p d\mu\right)^{2/(p+2)} \times\left(\int_M\|f(m)\|_{\mathscr{H}}^2 d\mu\right)^{p/(p+2)}<\infty.$$

Hence $(A(\cdot)h, f(\cdot))_{\mathscr{H}}\in L^s(M, d\mu)$. Similarly we have $(f, g(\cdot))_{\mathscr{H}}\in L^s(M, d\mu)$. Therefore $\eta\in L^s(M, d\mu)$. Since $L^s(M, d\mu)$ is the dual space of $L^r(M, d\mu)$, (A. 3) implies that $\eta(m)=0$ a. e. m. Hence $(A(m)h, f(m))_{\mathscr{H}}=(h, g(m))_{\mathscr{H}}$ a. e. m. Since $\mathscr{H}$ is separable, it follows that $A(m)^*f(m)=g(m)$ a. e. m. Hence we obtain

$$D(\mathrm{A}^*)\subset D\left(\int_M^{\oplus}A(m)^*d\mu(m)\right). \tag{A. 4}$$

It is easy to see that the converse inclusion relation of (A. 4) holds. Thus (A. 2) follows. ∎

References

[1] A. Arai, A general class of infinite dimensional Dirac operators and path integral representation of their index, J. Funct. Anal. 105 (1992), 342-408.

[2] A. Arai, Path integral representation of the index of Kähler-Dirac operators on an infinite dimensional manifold, J. Funct. Anal. 82 (1989), 330-369.

[3] A. Arai, Supersymmetric embedding of a model of a quantum harmonic oscillator interacting with infinitely many bosons, J. Math. Phys. 30 (1989), 512-520.

[4] A. Arai, Supersymmetry and singular perturbations, J. Funct. Anal. 60 (1985), 378-393.

[5] A. Arai and I. Mitoma, De Rham-Hodge-Kodaira decomposition in ∞-dimensions, Math. Ann. 291 (1991), 51-73.

[6] A. Arai, Supersymmetric extension of quantum scalar field theories, in "Quantum and Non-Commutative Analysis," Ed. H. Araki et al, Kluwer Academic Publishers, Dordrecht, 1993, pp. 73-90.

[7] A. Arai and O. Ogurisu, Meromorphic N=2 Wess Zumino supersymmetric quantum mechanics, J. Math. Phys. 32 (1991), 2427-2434.

[8] A. Arai, Fock-space representation of the relativistic supersymmetry algebra in the two-dimensional space-time, Hokkaido University Preprint Series in Math. 123 (1991).

[9] A. Arai, A general class of infinite dimensional Dirac operators and related aspects, in "Functional Analysis & Related Topics," Ed. S. Koshi, World Scientifie, Sin-

gapore, 1991.
[10] A. ARAI, A theorem on essential self-adjointness with application to Hamiltonians in non-relativistic quantum field theory, J. Math. Phys. 32 (1991), 2082-2088.
[11] A. ARAI and I. MITOMA, Comparison and nuclearity of spaces of differential forms on topological vector spaces, J. Funct. Anal. 111 (1993), 278-294.
[12] A. ARAI, Dirac operators in Boson-Fermion Fock spaces and supersymmetric quantum field theory, J. Geometry and Physics 11 (1993), 465-490.
[13] A. ARAI, Characterization of anticommutativity of self-adjoint operators in connection with Clifford algebra and applications, Integr. Equat. Oper. Th. 17 (1993), 451-463.
[14] F. A. BEREZIN, "The Method of Second Quantization," Academic, New York, 1996.
[15] J. GLIMM and A. JAFFE, "Quantum Physics," 2nd ed., Springer, New York, 1987.
[16] A. JAFFE and A. LESNIEWSKI, Supersymmetric quantum fields and infinite dimensional analysis, "Nonperturbative Quantum Field Theory," Ed. G.'t Hooft et al, Plenum, New York, 1988.
[17] P. E. T. JORGENSEN, Essential self-adjointness of semi-bounded operators, Math. Ann. 237 (1978), 187-192.
[18] P. E. T. JORGENSEN, Spectral theory for self-adjoint operator extensions associated with Clifford algebras, Contemporary Math. 148 (1993), 131-150.
[19] M. REED and B. SIMON, "Methods of Modern Mathematical Physics Vol. I: Functional Analysis," Academic Press, New York, 1972.
[20] M. REED and B. SIMON, "Methods of Modern Mathematical Physics Vol. II: Fourier Analysis, Self-adjointness," Academic Press, New York, 1975.
[21] M. REED and B. SIMON, "Methods of Modern Mathematical Physics Vol. IV: Analysis of Operators," Academic Press, New York, 1978.
[22] B. SIMON, "The $P(\phi)_2$-Euclidean (Quantum) Field Theory," Princeton University, Princeton, New Jersey, 1974.
[23] E. WITTEN, Supersymmetry and Morse theory, J. Differential Geom. 17 (1982), 661-692.

Department of Mathematics,
Hokkaido University
Sapporo 060, Japan