Hokkaido Mathematical Journal Vol. 23 (1994) p. 185-212

Motion of a graph by convexified energy

Yoshikazu GIGA

In the memory of Moéko

(Received April 12, 1993)

1. Introduction

This paper is concerned with evolution equations of hypersurfaces Γ_t in $\boldsymbol{R}^n$. We consider

$$V=\frac{-1}{\beta(\vec{n})}\left(\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\frac{\partial\gamma}{\partial p_i}(\vec{n})\right)+c\right) \quad \text{on } \Gamma_t. \tag{1.1}$$

Here $\vec{n}$ represents the unit normal vector (field) of Γ_t and V represents the normal velocity of Γ_t. The function $\gamma=\gamma(p_1, \cdots, p_n)$ is assumed to be positively homogeneous of degree one and its restriction on the unit sphere S^{n-1} is often called the interface energy density. The function $\beta: S^{n-1}\rightarrow \boldsymbol{R}$ is assumed to be positive and continuous; c is a constant. The sign in front of $1/\beta$ is taken so that the equation (1.1) becomes the mean curvature fiow equation if $\gamma(p)=|p|$, $\beta\equiv 1$ and $c=0$. The equation (1.1) is considered as a mathematical model for the dynamics of surfaces of a melting solid when the effect outside the surface is negligible. We refer to a paper [AG 1] of Angenent and Gurtin for its derivation from the second law in the thermodynamics and the force balances.

Usually, γ is assumed to be convex and C^2 outside the origin. However, in physics there is also the possibility that γ is not convex as studied in [AG 1,2]. If γ is not convex, the equation (1.1) is not well-posed even locally because it is backward parabolic in some direction of $\vec{n}$. To track the evolution of the hypersurface it seems to be natural to consider the convexification $\tilde{\gamma}$ of γ when γ is not convex. We are interested in the evolution of the hypersurface by (1.1) where γ is replaced by $\tilde{\gamma}$.

In this paper we consider the evolution when the hypersurface Γ_t is a curve represented by the graph of a function on $\boldsymbol{R}$. Even in this simple case there arise several problems. First, solution Γ_t may develop singularities in a finite time because $\tilde{\gamma}$ may not be strictly convex. Second, $\tilde{\gamma}$ may not be C^2 even if γ is smooth, so the interpretation of (1.1) is not clear. Instead of considering general convexified $\tilde{\gamma}$ we restrict ourselves to handle typical one by assuming that $\tilde{\gamma}$ is not C^2 at most in finitely many directions and the gradient of $\tilde{\gamma}$ is locally Lipschitz outside zero.

For this $\tilde{\gamma}$ our equation (1. 1) is still degenerate parabolic. We thus adapt the theory of viscosity solutions to show the unique global-in-time existence of solutions Γ_t for given initial data Γ_0 when Γ_0 is represented as the graph of a function growing at most linearly at the space infinity. The equation (1. 1) is of the form

$$u_t - a(u_x)u_{xx} - b(u_x) = 0 \tag{1. 2}$$

when Γ_t is represented by the graph of a function $u(t, x)$, where a and b are determined by γ, β, c. A comparison principle plays an important role to prove the unique existence of viscosity solutions. Sine a may have jump discontinuities more than two, our unique existence theorem in not included in the literature although basic strategy is close to [CGG 1] and [GGIS]. The theory in [CGG 1, GGIS] applies to the case when a is bounded and nonnegative and has only one jump. Our theory generalized those in [GGIS] when the space dimension is one. For existence, different from [CCG 1, GGIS] we do not assume that the equation is geometric [CGG 1], although our equation is not fully nonlinear but quasilinear.

If γ is not convex, $\tilde{\gamma}$ may not be strictly convex. For (1. 2) we observe that a may be on finitely many union of some intervales; here assumptions on $\tilde{\gamma}$ is invoked. The set of x where $u_x(t, x)$ belongs to these intervals is called nonparabolic region (for precise definition see § 6). In this region diffusion effect is not observed. Moreover we give a condition on β such that the portion of Γ_t corresponding to nonparabolic region moves just by a translation except near the boundary of the portion.

We are interested in the behavior of nonparabolic region by assuming that a in (1. 2) vanishes only on an interval $[p_1, p_2]$ and the infimum of a outside $[p_1, p_2]$ is positive. Under the above mentioned conditions on a we conclude that nonparabolic region is decreasing in time at least for convex initial data $u(0, x)$ if $u_x(0, x) < p_1$, $u_x(0, x') > p_2$ for some x and x'. If $b \geq 0$ near p_1 and p_2 we conclude that the nonparabolic region disappears in a finite time. We do not know whether this restriction is technical or not. In this situation we prove that $u(t, x)$ becomes singular in a finite time even if $u(0, x)$ is smooth (Theorem 6. 15). For special a with $b=0$, (1. 2) becomes the Stefan problem by taking $U = u_x$ as a new variable [AG 1]. Some of our results may be proved by applying the theory of the Stefan problem [M] since U is a generalized solution of the Stefan problem at least formally. However, we directly analyze viscosity solutions of (1. 2) without using the theory of the Stefan problem. As a bonus our method applies to the case $b \not\equiv 0$.

The equation (1.1) with anisotropic γ attracts many mathematicians as well as the mean curvature flow problems. In [CGG 1] Chen, Goto and the author constructed a global (unique) generalized solution by a level set approach provided that Γ_0 is compact and that γ is convex and C^2; see also [GG 1]. (The paper [CGG 3] includes corrections of technical errors in [CGG 1]). Nearly at the same time the similar idea is applied to the mean curvature flow problem by Evans and Spruck [ES]. Properties of solutions Γ_t for anisotropic γ studied by Soner [S], [CGG 2] and [GG 2]. If γ is strictly convex and $c=0$, extension problem of the solution Γ_t after singularities is also studied by Angenent [A] when Γ_t is a closed curve by a different method. For the corvexified $\tilde{\gamma}$ Ohnuma and Sato [OhS] extended the theory of [CGG 1] when $n=2$ under the same assumptions on $\tilde{\gamma}$ as ours and constructed a unique global generalized solution for general compact initial data Γ_0 without assuming that Γ_0 is represented as a graph. Angenent and Gurtin [AG 2] solved (1.1) with $\gamma=\tilde{\gamma}$ for $n=2$ at least locally if no nonparabolic region appears for the initial curve. There in a nice review article by Taylor, Cahn and Handwerker [TCH] for various mathematical approaches to the motion by anisotropic γ.

In Section 2 we derive (1.2) form (1.1) and explain how related to other notations in the literature. We also state our main results in § 3 and § 4 applied to (1.1). In Section 3 we establish fundamental comparison theorem to (1.2). The existence is proved in Section 4. Section 5 is devoted to the condition on β so that nonparabolic region moves essentially by a translation. In Section 6 we studied the behavior of nonparabolic region directly.

During this work is prepared the author learned that Gurtin, Soner and Souganidis [GSS] also studied generalized evolution Γ_t by (1.1) with the convexified energy when Γ_0 is a closed curve. They in particular obtained a similar comparison theorem obtained in [OhS]. They also proved their solution is consistent with the one studied in [AG 2], where β is taken as in Section 5. After this work was completed, the author learned that results in Ohnuma and Sato [OhS] are extenced to $n\geq 3$ by H. Ishii.

The author is grateful to Professor Morton Gurtin who brought this problem to his attention. This work is partly supported by the Inamori Foundation. This paper is dedicated to my first daughter Moêko who completed her 150 days life on May 23, 1992.

2. Evolution equations for graphs

If the hypersurface Γ_t is given as the graph of a function $u(t, x)$, $x \in \boldsymbol{R}^{n-1}$, the upward unit normal vector field $\vec{n}$ is of the form

$$\vec{n}=\left(\frac{-\nabla u}{\sqrt{1+|\nabla u|^2}}, \frac{1}{\sqrt{1+|\nabla u|^2}}\right),$$

where $\nabla=\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n-1}}\right)$. The normal velocity V is given by

$$V=\frac{u_t}{\sqrt{1+|\nabla u|^2}}.$$

The equation (1.1) becomes the equation for u

$$u_t=\sqrt{1+|\nabla u|^2}\frac{1}{\bar{\beta}|\nabla u|^2}\left(\sum_{i=1}^{n-1}\frac{\partial}{\partial x_i}\lambda_i(\nabla u)-c\right) \tag{2.1}$$

with $\lambda_i(p')=-\frac{\partial}{\partial p_i}\gamma(-p', 1)$, $p'=(p_1, \dots, p_{n-1})$,

$$\bar{\beta}(p')=\beta\left(-\frac{p'}{\sqrt{1+|p'|^2}}, \frac{1}{\sqrt{1+|p'|^2}}\right).$$

In this computation we have used the fact that $\partial\gamma/\partial p_i$ is positively homogeneous of degree zero.

If $n=2$, *i.e.*, Γ_t is a curve, (2.1) becomes

$$u_t=\sqrt{1+u_x^2}\frac{1}{\bar{\beta}(u_x)}(\lambda(u_x)_x-c), \lambda=\lambda_1. \tag{2.2}$$

The convexity of γ near $(-p_0', 1) \in \boldsymbol{R}\times\boldsymbol{R}$ implies the nondecreasing property of λ near p_0'. So if γ is convex on $\boldsymbol{R}\times\{1\}$, (2.2) is a degenerate parabolic equation.

We may rewrite (2.2) as

$$u_t-A(u_x)_x-b(u_x)=0 \tag{2.3}$$

with

$$A(q)=\int_0^q\frac{\sqrt{1+p^2}}{\bar{\beta}(p)}\frac{d\lambda}{dp}(p)dp$$
$$b(p)=-c\frac{\sqrt{1+p^2}}{\bar{\beta}(p)},$$

or

$$u_t-a(u_x)u_{xx}-b(u_x)=0 \tag{2.4}$$

with

$$a(p)=dA/dp=\sqrt{1+p^2}\ \bar{\beta}(p)^{-1}d\lambda/dp.$$

We are interested in the case that γ is not convex which is equivalent to the nonconvexity of the set $\gamma\leq 1$ in $\boldsymbol{R}^2$. For $\vec{m}\in S^1$ let θ denote its argument, *i.e.*, $\vec{m}=(\cos\theta, \sin\theta)$. As in [AG 1] we set

$$f(\theta)=\gamma(\vec{m}(\theta)). \tag{2.5}$$

Then the level set $\gamma=1$ agrees with the polar diagram of $1/f$ (or Frank diagram of f) *i.e.* the locus of $\vec{m}(\theta)/f(\theta)$, $0\leq\theta<2\pi$, since $\gamma(\vec{m}(\theta)/f(\theta))=f(\theta)^{-1}\gamma(\vec{m}(\theta))=1$. Suppose that the polar diagram of $1/f(\theta)$ is a closed curve. If it has negative curvature only on one open interval $\alpha_1<\theta<\alpha_2$, then the convexification $\tilde{\gamma}$ of γ is linear in the direction $\vec{m}(\theta)$ for $\theta_1\leq\theta\leq\theta_2$ with some θ_1, θ_2 satisfying $\theta_1\leq\alpha_1<\alpha_2\leq\theta_2$. Note that $\tilde{\gamma}$ may not be C^2 at $\vec{m}(\theta_1)$, $\vec{m}(\theta_2)$ even if γ is smooth. The following assumptions on $\tilde{\gamma}$ include the above mentioned example.

2.1. Assumptions on $\tilde{\gamma}$. The set $\tilde{\gamma}=1$ is piecewise C^2 except finitely many points. The curvature of $\tilde{\gamma}=1$ is bounded and nonnegative.

If we write (1.1) with $\gamma=\tilde{\gamma}$ as (2.4), these assumptions imply that a is (locally) bounded, nonnegative and continuous except finitely many points. If $\tilde{\gamma}$ is linear in the direction $\vec{m}(\theta)$, $\theta_1\leq\theta\leq\theta_2$, we observe that $a(p')=0$ provided that the argument of $(-p', 1)$ is in between θ_1 and θ_2. Unique existence of solutions to (2.4) presented in § 4 immediately yields:

2.2. THEOREM. *Assume 2.1 on $\tilde{\gamma}$. Suppose that β is continuous and positive on S^{n-1}. Let u_0 be Lipschitz on $\boldsymbol{R}$. Then there is a unique viscosity solution u of (2.2) (with $\gamma=\tilde{\gamma}$) continuous on $[0, \infty)\times\boldsymbol{R}$ such that $u(0, x)=u_0(x)$ and that for evemy $T>0$*

$$|u(t, x)|\leq K(|x|+1),\ x\in\boldsymbol{R},\ 0\leq t\leq T$$

with some $K>0$.

2.3. Capillary force. We conclude this section by studying the relation of derivatives of γ and the capillary force

$$\begin{aligned}\vec{C}(\theta)&=-f(\theta)^2\frac{d}{d\theta}\frac{\vec{m}(\theta)}{f(\theta)}\\&=f(\theta)(\sin\theta, -\cos\theta)+f'(\theta)(\cos\theta, \sin\theta)\end{aligned}$$

introduced in [AG 1]. By (2.5) we see

$$f'(\theta)=\partial_1\gamma(\vec{m})(-\sin\theta)+\partial_2\gamma(\vec{m})(\cos\theta),\ \partial_i\gamma=\partial\gamma/\partial p_i,\ i=1, 2$$

which yields

$$\vec{C}(\theta)=(\gamma\sin\theta-\partial_1\gamma\sin\theta\cos\theta+\partial_2\gamma\cos^2\theta,\\ -\gamma\cos\theta-\partial_1\gamma\sin^2\theta+\partial_2\gamma\cos\theta\sin\theta)$$

where γ, $\partial_1\gamma$, $\partial_2\gamma$ are evaluated at $\vec{m}=(\cos\theta, \sin\theta)$. Using the homogeneity of γ:

$$\gamma=\partial_1\gamma\cos\theta+\partial_2\gamma\sin\theta,$$

we observe that

$$\vec{C}(\theta)=(\partial_2\gamma(\sin^2\theta+\cos^2\theta),\ -\partial_1\gamma(\cos^2\theta+\sin^2\theta))\\ =(\partial_2\gamma(\vec{m}),\ -\partial_1\gamma(\vec{m})).$$

Thus the capillary force equals the minus curl of γ in two dimensional space, *i.e.*,

$$\vec{C}(\theta)=-(\nabla^{\perp}\gamma)(\cos\theta, \sin\theta),\ \nabla^{\perp}\gamma=(-\partial_2\gamma, \partial_1\gamma).$$

3. Fundamental properties of viscosity solutions

We consider a degenerate parabolic equation:

$$u_t-a(u_x)u_{xx}-b(u_x)=0 \quad \text{in} \quad (0,\infty)\times\boldsymbol{R} \tag{3.1}$$

when a is not necessarily continuous. Here a is a nonnegative function defined except on a finite subset $\Sigma=\{p\}_{i=1}^{\ell}$. The function a is assumed to be continuous except at Σ and bounded on every compact set of $\boldsymbol{R}$. The function b is assumed to be continuous on $\boldsymbol{R}$.

3.1. COMPARISON THEOREM. *Suppose that u and v are viscosity sub- and supersolutions of* (3.1), *respectively. Suppose that u and v are upper and lower semicontinuous on $[0,\infty)\times\boldsymbol{R}$, respectively. Suppose that u and v are continuous up to $t=0$. Suppose furthermore that for each $T>0$*

$$|u(t,x)-u(t,y)|\le L_T|x-y|,\ 0\le t<T\\ |v(t,x)-v(t,y)|\le L_T|x-y|,\ 0\le t<T$$

with L_T independent of x and y. If $u(0,x)\le v(0,x)$, then $u(t,x)\le v(t,x)$ for $t\ge0$, $x\in\boldsymbol{R}$.

We shall prove a more general comparison theorem. To state the general version we recall the notion of lower and upper semicontinuous envelope (relaxation) of functions.

3.2. Semicontinuous envelope. Suppose that h is a real-valued function from a subset L of $\boldsymbol{R}^d$. The lower semicontinuous envelope (relaxation) h_* of h is defined by

$$h_*(x)=\liminf_{\varepsilon\downarrow 0}\{h(y)\,;\ |y-x|<\varepsilon,\ y\in L\}$$

for $x\in\bar{L}$, where $\bar{L}$ denotes the closure of L. For example, suppose that

$$F(p, X)=-a(p)X-b(p) \tag{3.2}$$

is defined on $(\boldsymbol{R}\backslash\Sigma)\times\boldsymbol{R}$, where a and b are as in (3.1). Then it is easy to see

$$F_*(p_i, X)=\begin{cases} -a^*(p_i)X-b(p_i) & \text{for } X\geq 0 \\ -a_*(p_i)X-b(p_i) & \text{for } X<0 \end{cases},\ p_i\in\Sigma. \tag{3.3}$$

The upper semicontinuous envelope h^* is defined by

$$h^*=-(-h)_*.$$

For the above F we have

$$F^*(p_i, X)=\begin{cases} -a_*(p_i)X-b(p_i) & \text{for } X\geq 0 \\ -a^*(p_i)X-b(p_i) & \text{for } X<0 \end{cases},\ p_i\in\Sigma. \tag{3.4}$$

We consider an equation of form

$$u_t+F(u_x, u_{xx})=0 \quad \text{in} \quad (0, T)\times\boldsymbol{R}. \tag{3.5}$$

We list assumptions on F to state a comparison result including Theorem 2.1. Let $\Sigma=\{p_i\}_{i=1}^{\ell}$ be a given finite subset of $\boldsymbol{R}$.

$F:(\boldsymbol{R}\backslash\Sigma)\times\boldsymbol{R}\to\boldsymbol{R}$ is continouns. (F1)
F is degenerate elliptic, *i.e.*, $F(p, X)\leq F(p, Y)$ for $X\geq Y$. (F2)
$-\infty<F_*(p_i, 0)=F^*(p_i, 0)<\infty$, $p_i\in\Sigma$. (F3)
For each $M>0$ the value (F4)
$c_M=\sup\{|F(p, X)|\,;\,|p|\leq M, |X|\leq M, p\notin\Sigma\}$ is finite.

3.3. General Comparison Theorem. *Assume* (F1)-(F4). *Suppose that* u *and* v *are viscosity sub-and supersolutions of* (3.5), *respectively. Suppose that there is a constant* $K>0$ *such that*

$$u(t, x)\leq K(|x|+1),\ v(t, x)\geq -K(|x|+1) \ \text{on}\ (0, T)\times\boldsymbol{R},$$
$$u_0(x)-v_0(y)\leq K(|x-y|+1),$$

where $u_0(x)=u^*(0, x)$, $v_0(x)=v_*(0, x)$. *Suppose furthermore that there is a modulus* $\mathscr{M}_0$ *such that*

$$u_0(x)-v_0(y)\leq\mathscr{M}_0(|x-y|).$$

Then there is a modulus $\mathscr{M}$ *such that*

$$u^*(t, x) - v_*(x, y) \leq \mathscr{M}(|x-y|) \text{ on } (0, T)\times \boldsymbol{R}. \tag{3.6}$$

In particular $u^* \leq v_*$ *on* $(0, T)\times \boldsymbol{R}$.

Here $\mathscr{M}: [0, \infty) \to [0, \infty)$ is called a *modulus* if $\mathscr{M}$ is continuous and nondecreasing with $\mathscr{M}(0)=0$. If the set Σ consists of a single point, Theorem 3.3 easily follows from [GGIS, Theorem 2.1], where the initial boundary value problem is also discussed. Theorem 3.3 can be also entended to the intial boundary value problem; see § 3.5.

Theorem 3.1 follows from Theorem 3.3. Indeed, if we set F by (3.2), (F1) is trivial; (F2) follows from $a \geq 0$. The property (F4) comes from (3.3)-(3.4); (F4) follows from local boundedness of a. The Lipschitz property of u in Theorem 3.1 implies the growth conditions on u and v. It also implies

$$u_0(x) - v_0(y) \leq L_T|x-y|$$

since $u_0 \leq v_0$ on $\boldsymbol{R}$. We apply Theorem 3.3 on an arbitrary interval $(0, T)$ to get Theorem 3.1.

Recently, Ohnuma and Sato [OhS] obtained a comparison theorem when $F(p, X)$ has singularities other than $p=0$. They assumed that $F(p, X)$ is singular when $p \in \boldsymbol{R}^n$ belongs to finitely many half lines starting from the origin. Their theory does not apply to the one dimensional problem so Theorem 3.1 is not included in their results.

3.4. Remarks on viscosity solutions. For later convenience we recall a definition of viscosity subsolution of (3.5). A function $u: (0, T)\times \boldsymbol{R} \to \boldsymbol{R}$ is called a viscosity sub-(super)solution of (3.5) if $u^* < \infty$ (resp. $u^* > -\infty$) on $[0, T]\times \boldsymbol{R}$ and

$$\tau + F_*(p, X) \leq 0 \text{ for all } (\tau, p, X) \in \mathscr{P}^{2,+}u^*(t, x), (t, x) \in (0, T)\times \boldsymbol{R}.$$
$$(\text{resp. } \tau + F^*(p, X) \geq 0 \text{ for all } (\tau, p, X) \in \mathscr{P}^{2,-}u_*(t, x), (t, x) \in (0, T)\times \boldsymbol{R}).$$

Here $\mathscr{P}^{2,+}$ denotes the *parabolic super* 2-*jet* in $(0, T)\times \boldsymbol{R}$, *i.e.* $\mathscr{P}^{2,+}u(t, x)$ is the set of $(\tau, p, X) \in \boldsymbol{R}\times\boldsymbol{R}\times\boldsymbol{R}$ such that

$$u(s, y) \leq u(t, x) + \tau(s-t) + p(y-x) + \frac{1}{2}X(y-x)^2$$
$$+ o(|s-t| + |y-x|^2) \text{ as } (s, y) \to (t, x);$$

similarly $\mathscr{P}^{2,-}u = -\mathscr{P}^{2,+}(-u)$. The correspondisg definition of viscosity supersolutions is easy to imagine so is omitted (cf. [GGIS]).

According to the sketch of the proof of Theorem 3.3 presented below, to get a comparison renult one may replace F_* and F^* in the definition of viscosity sub- and supersolutions by any lower and upper semicontinuous function $F_\#$ and $F^\#$ respectively, provided that

$$\begin{aligned}&F_\#(p, X)=F^\#(p, X)=F(p, X) \quad \text{for } p\notin\Sigma\\ &-\infty<F_\#(p_i. \ 0)=F^\#(p_i, 0)<\infty \quad \text{for } p_i\in\Sigma.\end{aligned}$$

This alternative definition is sometimes useful especially to get a solution by an approximation argument. This type of definition is first appeared in Evans and Spruck [ES] for the level set equation to the mean unvature flow problem (cf. [CGG2]).

3.5. SKETCH OF THE PROOF OF THEOREM 3.3. The proof parallels that of [GGIS]. By adding a linear function to u and v we may assume that $0=p_1<p_2<\cdots<p_l$ for $p_i\in\Sigma$. As in [GGIS, Proposition 2.3], (F1), (F4) and growth assumptions on u, v yields an estimate:

$$u(t, x)-v(t, y)\leq K'|x-y|+M(1+T) \text{ on } (0, T)\times\boldsymbol{R}, \tag{3.7}$$

where $K'>K$ is arbitrary and M depends on K'. Here and hereafter we drop $*$ of u and v. To find a good super 2-jet of

$$w(t, x, y)=u(t, x)-v(t, y),$$

we introduce a test function

$$\begin{aligned}&\Psi(t, x, y)=f_\varepsilon(x-y)+B(t, x, y)\\ &B(t, x, y)=\delta(x^2+y^2)+\gamma/(T-t)\end{aligned}$$

with $\varepsilon, \delta, \gamma>0$. The choice of f_ε is crucial. Let f be a smooth nondecreasing function on $[0, \infty)$ such that

$$\begin{aligned}&f(0)=f'(0)=f''(0)=0\\ &f'(x)=p_i \quad \text{implies} \quad f''(x)=0, 1\leq i\leq\ell\\ &f(x)=x^4 \quad \text{for sufficiently large } x;\end{aligned}$$

such a function, of couse, exists. We set

$$f_\varepsilon(x)=\varepsilon^{1/3}f(|x|/\varepsilon^{1/3}), \ x\in\boldsymbol{R}.$$

Note that if Σ consists of a single point $p_1=0$ then one may take $f_\varepsilon(x)=x^4/\varepsilon$. In this case the choice of Ψ is exactly the same as [GGIS, (2.11)]. The following argument works even for initial boundary value problems.

Suppose that (3.6) were false. Then

$$\alpha=\lim_{\theta\downarrow 0}\sup\{w(t, x, y); \ |x-y|<\theta, \ (t, x, y)\in[0, T)\times\boldsymbol{R}\times\boldsymbol{R}\}>0.$$

By (3.7) we see $\alpha<\infty$. We shall discuss the maximum of

$$\Phi(t, x, y)=w(t, x, y)-\Psi(t, x, y).$$

By the definition of α for each $\varepsilon>0$ we easily observe that there are $\delta_0(\varepsilon)$, $\gamma_0(\varepsilon)>0$ such that

$$\sup_{[0,T)\times\boldsymbol{R}^2}\Phi(t, x, y)>\alpha/2.$$

This is [GGIS, Proposition 2.4] where dependence of ε is not explicitly written. Let $(\hat{t}, \hat{x}, \hat{y})$ be a maximizer of Φ over $[0, T]\times\boldsymbol{R}^2$. The existence follows from growth conditions of u and v, and the barrier B. By the choice of f_ε, as in [GGIS, Proposition 2.5], we observe that $|\hat{x}-\hat{y}|\to 0$ as $\varepsilon\to 0$ (uniform in $0<\delta<\delta_0(\varepsilon)$, $0<\gamma<\gamma_0(\varepsilon)$) and that $\delta\hat{x}\to 0$, $\delta\hat{y}\to 0$ as $\delta\to 0$ (uniform in ε and γ). Since B plays a barrier near space infinity and $t=T$, as in [GGIS, Proposition 2.6] there is $\varepsilon_0>0$ such that for $0<\varepsilon<\varepsilon_0$, $0<\delta<\delta_0(\varepsilon)$, $0<\gamma<\gamma_0(\varepsilon)$, $(\hat{t}, \hat{x}, \hat{y})$ is interior point of $(0, T)\times\boldsymbol{R}^2$. We shall fix ε (and γ from now on. (Since our problem is the Cauchy problem we may take $\varepsilon=1$.)

Since $(\Psi_t, \Psi_x, \Psi_y, \nabla^2\Psi)(\hat{t}, \hat{x}, \hat{y})\in\mathscr{P}^{2,+}w(\hat{t}, \hat{x}, \hat{y})$, applying the Crandall-Ishii lemma [CI, Theorem 6] with (F4) we observe that for $\lambda>0$ there is $(\tau_1, X)\in\boldsymbol{R}\times\boldsymbol{R}$, $(\tau_2, Y)\in\boldsymbol{R}\times\boldsymbol{R}$ such that

$$\begin{aligned}&(\tau_1, \hat{\Psi}_x, X)\in\overline{\mathscr{P}}^{2,+}u(\hat{t}, \hat{x})\\&(-\tau_2, -\hat{\Psi}_y, -Y)\in\overline{\mathscr{P}}^{2,-}v(\hat{t}, \hat{y})\\&\hat{\Psi}_t=\tau_1+\tau_2\\&-\left(\frac{1}{\lambda}+|A|\right)I\leq\begin{pmatrix}X & 0\\ 0 & Y\end{pmatrix}\leq A+\lambda A^2 \quad\text{with}\quad A=\nabla^2\Psi(\hat{t}, \hat{x}, \hat{y}),\end{aligned}\tag{3.8}$$

where $\wedge$ denotes the value at $(\hat{\tau}, \hat{x}, \hat{y})$ and $\nabla^2\Psi$ is the Hessian matrix

$$\nabla^2\Psi=\begin{pmatrix}\Psi_{xx} & \Psi_{xy}\\ \Psi_{yx} & \Psi_{yy}\end{pmatrix}.$$

The bar over $\mathscr{P}^{2,\pm}$ denotes the closure. Since u and v are sub- and supersolutions, respectively and since F_* and F^* are semicontinuous, we obtain

$$\tau_1+F_*(\hat{\Psi}_x, X)\leq 0, \quad -\tau_2+F^*(-\hat{\Psi}_y, -Y)\geq 0$$

which yields

$$0\geq\gamma T^2+F_*(\hat{\Psi}_x, X)-F^*(-\hat{\Psi}_y, -Y), \tag{3.9}$$

since $\hat{\Psi}_t=\tau_1+\tau_2\geq\gamma/T^2$. We fix $\lambda=1$ and divide the situation into two cases

Case 1. $f'_\varepsilon(\hat{x}-\hat{y}) \to p_i \in \Sigma$ for some subsequence $\delta_k \to 0$.

Case 2. $f'_\varepsilon(\hat{x}-\hat{y}) \to p \notin \Sigma$ for some subsequence $\delta_k \to 0$.

Case 1. From the choice of f it follows that $f''_\varepsilon(\hat{x}-\hat{y}) \to 0$ an $\delta_k \to 0$. Since

$$\hat{\Psi}_x = f'_\varepsilon(\hat{x}-\hat{y}) + 2\delta\hat{x}, \quad \hat{\Psi}_y = -f'_\varepsilon(\hat{x}-\hat{y}) - 2\delta\hat{y}$$
$$A = f''_\varepsilon(\hat{x}-\hat{y}) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + 2\delta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

we see

$$A \le o(1) \quad \text{as} \quad \delta_k \to 0$$

which yields $X \le o(1)$, $-Y \ge o(1)$ by (3.8). By the degenerate ellipticity (F2) we see

$$F_*(\hat{\Psi}_x, X) \ge F_*(\hat{\Psi}_x, o(1))$$
$$F^*(-\hat{\Psi}_y, -Y) \le F^*(-\hat{\Psi}_y, o(1)).$$

Since $\delta\hat{x} \to 0$, $\delta\hat{y} \to 0$ as $\delta_k \to 0$, letting $\delta_k \to 0$ in (3.9) yields

$$0 \ge \gamma/T^2 + F_*(p_i, 0) - F^*(p_i, 0).$$

By (F3) this contradicts $\gamma > 0$. As alluded in § 3.4 we may replace F_* and F^* by $F_\#$ and $F^\#$ in the last paragraph of § 3.4.

Case 2. This is more standard. The estimate (3.8) yields

$$X + Y \le o(1) \quad \text{as} \quad \delta_k \to 0.$$

The estimate (3.8) also says that X, Y are compact an $\delta_k \to 0$. Letting $\delta_k \to 0$ in (3.9) and using (F2) yields a contradiction to $\gamma > 0$.

3.6. Lipschitz Preserving Theorem. *Suppose that F satisfies $(F1)$-$(F4)$. Let u be a viscosity solution of* (3.5) *which is continuous on* $[0, T] \times \boldsymbol{R}$. *Assume that*

$$|u(t, x)| \le K(|x|+1) \quad \textit{on} \quad [0, T] \times \boldsymbol{R}$$

with $K > 0$ independent of t and x. Assume that $u(0, x)$ is Lipschitz, i.e.,

$$|u(0, x) - u(0, y)| \le L|x-y|$$

with $L > 0$ independent of x, y. Then

$$|u(t, x) - u(t, y)| \le L|x-y| \quad \textit{on} \quad [0, T] \times \boldsymbol{R}^2.$$

This is an easy Corollary of Theorem 3.3 (cf. [GGIS, Corollary 2.11]).

We only need to compare

$$v(t, x)=u(t, x+h)+L|h|$$

with u so that

$$u(t, x)-u(t, x+h)\leq L|h|, \quad h\in \boldsymbol{R}.$$

The left hand side is dominated by $-L|h|$ from below using a similar comparison. Note that v solves (3. 5).

3. 7. CONCAVITY PRESERVING THEOREM. *Assume the same hypotheses of Theorem* 3. 6. *Suppose that* $X\mapsto F(p, X)$ *is convex for all* $p\notin \Sigma$, $X\in \boldsymbol{R}$. *If* $u(0, x)$ *is concave, then* $x\mapsto u(t, x)$ *is concave for all* $t\in[0, T]$.

We take f as in § 3. 5 with a further requirement $f''\geq 0$. This is of course possible. Since $f_k(x)=k^{1/3}f(|x|/k^{1/3})$ intersects $y=x$ at one point for $x>0$, there is a function $g(k)$, $k>0$ such that

$$x\leq f_k(x)+g(k) \quad \text{for} \quad x\geq 0$$

and that the equality holds at the only one point $x=x(k)$. The proof parallels that of [GGIS, Theorem 3. 1] if we set

$$f(\xi)=f_k(x+y-2z)+g(k), \xi=(x, y, z)$$

in [GGIS, (3. 9)]. We omit the detail.

4. Existence of solutions

We construct a viscosity solution to (3. 1) when intial data is Lipschitz. We present two methods - Perron's method and approximation method. Ishii [I] first adapted Perron's method for viscosity solutions of the Hamilton-Jacobi equations.

4. 1. EXISTENCE THEOREM. *Let* u_0 *be Lispschitz on* $\boldsymbol{R}$. *Then there is a unique viscosity solution* u *of* (3. 1) *continuous on* $[0, \infty)\times \boldsymbol{R}$ *such that* $u(0, x)=u_0(x)$ *and that for every* $T>0$

$$|u(t, x)|\leq K(|x|+1), \ x\in \boldsymbol{R}, \ 0\leq t\leq T$$

with some $K>0$.

4. 2. Construction of a subsolution. Let L be the Lispschitz constant of u_0. We set

$$M=\sup_{|p|\leq L} a(p), \ N=\sup_{|p|\leq L} |b(p)|.$$

Let v be the unique solution of a linear equation:

$$\begin{aligned} &v_t - Mv_{xx} + N = 0 &&\text{in } (0, \infty)\times\boldsymbol{R} \\ &v(0, x) = -L|x| &&\text{on } \boldsymbol{R}. \end{aligned}$$

By differentiating the equation it follows from the weak maximum principle that $v_{xx}\le 0$ and $|v_x|\le L$ for $t\ge 0$ and $x\in\boldsymbol{R}$. We thus observe that

$$v_t - a(v_x)v_{xx} - b(v_x) \le v_t - Mv_{xx} + N = 0$$

which implies v is a subsolution of (3.1). Since a supremum of subsolutions is still a subsolution the function

$$u_-(t, x) = \sup_{\xi\in\boldsymbol{R}}(v(t, x-\xi) + u_0(\xi))$$

is a (viscosity) subsolution of (3.1). Since u_0 is Lipschitz with L we see $u_-(0, x) = u_0(x)$. Since $v_t\le 0$, the definition of u_-implies

$$u_-(t, x) \le u_0(x).$$

On the other hand

$$u_-(t, x) \ge v(t, x) + u_0 \ge (0) \ge -L|x| + v(t, 0) + u_0(0).$$

We thus observe that for each $T>0$

$$|u_-(t, x)| \le K(|x|+1),\ x\in\boldsymbol{R},\ 0\le t\le T$$

with $K>0$. We now obtain the next lemma; the supersolution u_+ can be constructed in the similar way.

4.3. LEMMA. *Suppose that u_0 is Lipschitz continuous. There is a lower (upper) semicontinuous viscosity sub-(super) solution* u_-(resp. u_+) *of* (3.1) *such that*

$$\begin{aligned} &u_-(t, x) \le u_0(x) \le u_+(t, x), && x\in\boldsymbol{R},\ t\ge 0 \\ &u_\pm(0, x) = u_0(x) && \\ &|u_\pm(t, x)| \le K_T(|x|+1), && 0\le t\le T \end{aligned}$$

with some $K_T>0$ independent of x and t.

4.4. Perron's method. According to this method the viscosity solution of (3.1) with initial data $u_0(x)$ is given by

$$\begin{aligned} u(t, x) = \sup\{w(t, x);\ &w(t, x) \text{ is a viscosity subsolution} \\ &\text{such that } u_- \le w \le u_+ \text{ on } [0, \infty)\times\boldsymbol{R}\}. \end{aligned}$$

By the growth condition in Lemma 4.3 it is easy to see Theorem 3.3 is applicable to get $u^* \leq u_*$ which implies the continuity on $[0, \infty) \times \boldsymbol{R}$. By Lipschitz Preserving Theorem $x \mapsto u(t, x)$ is Lipschitz continuous.

4.5. Approximation. There is another way to construct a solution. We just indicate this method. We approximate u_0 by smooth global Lipschitz function $u_{0\delta}$ with bounded second derivatives. Also approximate a by a_ε so that $c/\varepsilon \geq a_\varepsilon \geq \varepsilon > 0$ with some c and that $a_\varepsilon \to a$ uniformly on every compact set in $\boldsymbol{R} \backslash \Sigma$. Since the approximate equation

$$u_t - a_\varepsilon(u_x) u_{xx} - b(u_x) = 0$$

is uniformly parabolic, there is a unique solution $u^{\varepsilon,\delta}$[LSU].

By the choice of $u_{0\delta}$, $|\nabla u^{\varepsilon,\delta}|$, $|u_t^{\varepsilon,\delta}|$ is bounded as $\varepsilon \to 0$. Ascoli-Arzela's theorem yields a convergent subsequence of $u^{\varepsilon,\delta}$ as $\varepsilon \to 0$ (uniformly on every compact set in $[0, \infty) \times \boldsymbol{R}$). By the stability theorem (see e.g. [CGG 1, Proposition 2.4]) we see the limit is the viscosity solution of (3.1) with $u(0, x) = u_{0\delta}(x)$. Note that the definition using $F_\#$ and $F^\#$ is easier to check that the limit is the viscosity solution.

It remains to take $\delta \downarrow 0$. We should observe that

$$\sup_x |u(t, x) - v(t, x)| \leq \sup_x |u(0, x) - v(0, x)| = A$$

for viscosity solutions. (This is obtained by comparing v with $u \pm A$ by Theorem 3.3). Then the solution u with $u(0, x) = u_0(x)$ is given as the uniform limit of the Cauchy sequence $u^\delta(t, x)$ (with intital data $u_{0\delta}$.)

In [ES] Evans and Spruck constructed a solution to the level set equation

$$u_t - |\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = 0$$

for the mean curvature flow by an approximation method.

5. Effects of lower order terms

We study a role of $\beta(\vec{n})$ in (1.1) when the curvature term involving γ does not effect in the direction of $\vec{n}$. For $\vec{m} = (\cos\theta, \sin\theta)$ we set

$$g(\theta) = \beta(\vec{m}(\theta)) \quad (n=2).$$

For $0 \leq \theta < 2\pi$ let Γ^θ be an oriented line containing the origin such that its unit normal equals $\vec{m}(\theta)$. Let Γ_t^θ be a solution of (1.1) with intial data Γ^θ, where $c \neq 0$.

5.1. Lemma on motion of lines. *Suppose that* $0 < \theta_2 - \theta_1 < \pi$. *Then the following two statements are equivalent.*

(i) *The polar diagram of g consists of a straight line for $\theta_1 \leq \theta \leq \theta_2$.*
(ii) *There is a (unique) $\vec{R} \in \boldsymbol{R}^2$ such that $\Gamma_t^\theta = \Gamma^\theta + \vec{R}t$.*

PROOF. Since Γ^θ is a line we may assume $\gamma \equiv 0$. We may also assume $c = -1$ without the loss of generality. Since the speed of Γ_t^θ is constant we may assume $t = 1$ in (ii). The proof is based on elementary geometry.
Let $\vec{R}$ be the intersection of $\Gamma_1^{\theta_1}$ and $\Gamma_1^{\theta_2}$; the length of $\vec{R}$ denotes R. Let $\vec{q}(\theta) = q(\theta)\vec{m}(\theta)$ be a vector such that $\vec{q}(\theta) - \vec{R}$ is orthogonal to $\vec{R}$. Comparing the similarity ratio of triangles we observe that

$$R/q(\theta_i) = R^{-1}/g(\theta_i) \quad (i = 1, 2).$$

Similarly Γ_1^θ intersects with $\vec{R}$ if and only if

$$R/q(\theta) = R^{-1}/g(\theta).$$

We thus observe that (i) and (ii) are equivalent. □

5.2. REMARK. Although we restrict ourselves to the case $n = 2$, Lemma 5.1 can be generalized to the case $n = 3$ with obvious modifications.

5.3. THEOREM. *Suppose that the polar diagram of $1/f$ ($f(\theta) = \gamma(\vec{m}(\theta))$) consists of a straight line for $\theta_1 \leq \theta \leq \theta_2$, $0 < \theta_2 - \theta_1 < \pi$. Then there is a (unique) $\vec{R}$ (determined by g and c) such that the right hand side of (1.1) becomes $\vec{R} \cdot \vec{n}$ with $\vec{n} = \vec{n}(\theta)$ for $\theta_1 \leq \theta \leq \theta_2$ if and only if the polar diagram of g consists of a straight line for $\theta_1 \leq \theta \leq \theta_2$. Here θ in $\vec{n}(\theta)$ denotes the argument of $\vec{n}$.*

This follows immediately from Lemma 5.1 since the curvature term in (1.1) vanishes for $\theta_1 < \theta < \theta_2$. Theorem 5.3 gives a condition for rigid motion where diffusion plays no effect. We apply Theorem 5.3 when Γ_t is represented as a graph of a function $u(t, x)$. We set

$$v(t, x) = u(t, x + R_1 t) - R_2 t, \quad \vec{R} = (R_1, R_2). \tag{5.1}$$

If u solves (2.2), then v solves

$$u_t - A(v_x)_x - c(v_x) = 0 \tag{5.2}$$

with $c(p) = b(p) + R_1 p - R_2$, where A and b are defined in (2.3).

5.4. COROLLARY. *Assume the hypotheses of Theorem 5.3 concerning γ and B so that $\vec{R}$ is defined. Suppose that $0 < \theta_1 < \theta_2 < \pi$. Then $a(p) = c(p) = 0$ if $\theta_1 < \mathrm{Arctan}\, p + \pi/2 < \theta_2$, where $a = A'$.*

6. Motion of nonparabolic regions

We are interested in the case that γ in (1.1) is not convex although it is at least C^2 outside the origin. We consider the same example mentioned just before Assumption 2.1. In other words suppose that $\gamma\geq 0$ and the second derivative of γ has negative eigenvalues only in the direction $\vec{m}(\theta)$ for some $\alpha_1<\theta<\alpha_2$. Let $\tilde{\gamma}$ be the convexification of γ. Then $\tilde{\gamma}$ is linear in the direction $\vec{m}(\theta)$ for $\theta_1<\theta<\theta_2$ with some θ_1, θ_2 satisfying $\theta_1\leq\alpha_1<\alpha_2\leq\theta_2$.

We further assume that the set $\tilde{\gamma}=1$ has positive curvature in the direction $\vec{m}(\theta)$ with $\theta\leq\theta_1$, $\theta_2\leq\theta$. (This in particular implies $\theta_i\neq\alpha_i(i=1,2)$.) To fix the idea we assume $0<\theta_1<\theta_2<\pi$. We rewrite above mentioned assumptions on $\tilde{\gamma}$ which of coure implies Assumption 2.1. Note that the curve $\tilde{\gamma}=1$ is the polar diagram of $1/\tilde{\gamma}(\vec{m}(\theta))$ parametrized by θ.

6.1. Assumptions on $\tilde{\gamma}$. The curve $\tilde{\gamma}=1$ is closed C^1 curve and C^2 except at $\theta=\theta_i(i=1,2)$, where $0<\theta_1<\theta_2<\pi$. Its curvature is positive for $\theta<\theta_1$, $\theta_2<\theta$ and can be continuously extended to $\theta=\theta_i(i=1,2)$ with positive values. For $\theta_1\leq\theta\leq\theta_2$ the curve $\tilde{\gamma}=1$ is a straight line.

If the curve Γ_t moved by (1.1) is given as the graph of a function $u(t,x)$, it solves (2.4). If β is assumed to be continuous, Assumption 6.1 yields the following properties of a in (2.4); actually both are equivalent.

$$a \text{ is continuous on } \boldsymbol{R} \text{ except } p_1 \text{ and } p_2 \quad (p_1<p_2) \text{ and is left(right) continuous at } p=p_1 \quad (p=p_2). \tag{6.1}$$

$$a\equiv 0 \text{ on } (p_1, p_2). \tag{6.2}$$

$$\inf\{a(p)\,;\ p\leq p_1,\ p_2\leq p,\ |p|\leq L\}>0 \text{ for each } L>0. \tag{6.3}$$

Here $p_i=\tan(\theta_i-\pi/2)$ $(i=1,2)$.

6.2. Assumptions on β. The polar diagram of $\beta(\vec{m}(\theta))$ is a Lipschitz closed curve and is a straight line for $\theta_1\leq\theta\leq\theta_2$ where θ_1, θ_2 are as in Assumption 6.1.

By Corollary 5.4 we may shift u as in (5.1) to get (5.2) with

$$c(p)\equiv 0 \quad \text{for} \quad p_1\leq p\leq p_2.$$

We shall thus study (3.1)

$$u_t-a(u_x)u_{xx}-b(u_x)=0$$

under the assumptions of (6.1)-(6.3) with

$$b(p)\equiv 0 \quad \text{for} \quad p_1\leq p\leq p_2 \tag{6.4}$$

g is locally Lipschitz cotinuous. (6. 5)

Of course Comparison Theorem 3.1 and Existence Theorem 4. 1 apply to our problem. We suppress the word viscosity from now on.

6. 3. LEMMA ON SPACIAL SOLUTIONS. *Assume that* (6. 1)-(6. 5) *for* (3. 1).

(i) *Suppose that* $u_0(x)$ *is Lipschitz with* $p_1 \leq u_{0x}(x) \leq p_2$ *for a. e. x. Then* $u_0(x)$ *itself solves* (3. 1). *In other words* u_0 *is a stationary solution of* (3. 1).

(ii) *Let* u *be the solution of* (3. 1) *with globally Lipschitz initial data* $u_0(x)$. *If* $u_{0x} \leq p_1$ *a. e. x, then* $u_x \leq p_1$ *a. e. x for all* $t \geq 0$. *Moreover,* u *solves*

$$u_t - \bar{a}(u_x)u_{xx} - b(u_x) = 0$$

with $\bar{a}(p) = a(p_1 - 0)$ *for* $p \geq p_1$ *and* $\bar{a}(p) = a(p)$ *for* $p < p_1$, *where* $a(p_1 - 0)$ *denotes the left limit at* p_1.

(iii) *If, in addition to* (ii), $u_0(x) = qx$ *for* $x < 0$ *and* $u_0(x) = p_1 x$ *for* $x \geq 0$ *with some* $q < p_1$, *then* $u(t, x) > p_1 x$ *for all* x, t. *Moreover* $\lim_{t \to \infty} u(t, x) = \infty$ *provided that* $b \geq 0$ *on* $[q, p_1]$.

PROOF. (i) If

$$(\tau, p, X) \in \mathscr{P}^{2,\pm} u(t, x)$$

with $u(t, x) = u_0(x)$, then $\tau = 0$, $p_1 \leq p \leq p_2$. Since $a(p) = b(p) = 0$ on (p_1, p_2), b is continuous and $a \geq 0$, we observe by (3. 3), that $u_0(x)$ is a solution of (3. 1).

(ii) As in the proof of Lipschitz Preserving Theorem 3. 6, we have $u_x \leq p_1$ for all $t \geq 0$. Therefore

$$(\tau, p, X) \in \mathscr{P}^{2,\pm} u(t, x)$$

implies $p \leq p_1$. We thus observe that u solves

$$u_t - \bar{a}(u_x)u_{xx} - b(u_x) = 0.$$

(iii) We may assume $p_1 = 0$ by adding a linear function to u. As in the proof of Lipschitz Preserving Theorem 3. 6 we observe that $q \leq u_x \leq 0$ (a. e. x). By Theorem 3. 7 $u(t, x)$ is convex in x. Thus, by (ii), u is a supersolution of

$$v_t - a_0 v_{xx} - b(v_x) = 0 \tag{6. 6}$$

with $a_0 = \inf_{q \leq p \leq 0} a(p)$, which is positive by (6. 3). By comparison it suffices to prove (iii) for solution v of (6.6) with initial data $u_0(x)$. Since b is Lipschitz and $b(0)=0$, v solves

$$v_t - a_0 v_{xx} - B(t,x)v_x = 0$$

with bounded coefficient B. Applying the strong maximum principle, we observe that $v(t,x)>0$ for all $t>0$ since $u_0\geq 0$.

It remains to prove that $\lim_{t\to\infty} v(t,x)=\infty$. Since also $q\leq v_x\leq 0$ holds, v satisfies

$$v_t - a_0 v_{xx} \geq 0$$

provided that $b\geq 0$ on $[q, 0]$. We consider

$$w_t - a_0 w_{xx} = 0$$

with $w(0,x)=u_0(x)$. It suffices to prove that $\lim_{t\to\infty} w(t,x)=\infty$ since $v\geq w$ by comparison. We may assume that $a_0=1$. The solution w is expressed as

$$w(t,x)=\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{0} e^{-|x-y|^2/4t} qy dy.$$

An elementary calculation shows

$$w(t,x)=\sqrt{\frac{t}{\pi}}e^{-x^2/4t}(-q)+qx\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{0} e^{-|x-y|^2/4t}dy.$$

Since the second term is bounded in t, this shows $\lim_{t\to\infty} w(t,x)=\infty$. □

6.4. Local supersolutions. We study the initial boundary value problem for a semilinear heat equation

$$\begin{aligned} &u_t - a_0 u_{xx} - b(u_x) = 0 \quad \text{on } (0,\infty)\times(-\infty,\xi) \\ &u|_{t=0}=u_0 \quad \text{on } (-\infty,\xi) \\ &u(t,\xi)=u_0(\xi) \quad \text{for } t>0, \end{aligned} \tag{6.7}$$

where a_0 is a positive constant and b is bounded and continuous. A standard parabolic theory [LSU] guarantees that the global existence of unique solutions in Sobolev spaces at least when u_0 is smooth up to $x=\xi$ and grows at most linearly as $x\to-\infty$. One can also prove, by Sobolev's embeddings, that u_x is Hölder continuous of any exponent (since b is bounded) in t and x even near the corner point $(t,x)=(0,\xi)$. Moreover, such Hölder norm depends on b only through its bound, so the following stability is easily obtained by Ascoli-Arzela's theorem.

Suppose that u_n is a solution of (6.7) with b replaced by b_n. Then near $(0,\xi)$, u_{nx} converges uniformly to u_x provided that b_n converges to b

pointwise with uniform bound for b_n in n. Note that such stability results hold near any point (t_0, x_0), $x_0<\xi$, $t_0\geq 0$. All results mentioned above are standard and by no means optimal. We refer to [LSU] for the comprehensive theory of parabolic equations.

6.4. A. LEMMA. *For* (6.7) *suppose that* $\xi>0$, $a_0>0$ *and that* b *is continuous and bounded. Suppose that* b *vanishes on* $[0, \infty)$. *Suppose that* u_0 *is smooth up to* $x=\xi$ *with* $u_{0x}(\xi)>0$, *globally Lipschitz and convex on* $(-\infty, \xi)$. *Assume that* $0\leq u_{0x}(x)\leq 1$ *if and only if* $0\leq x\leq\xi$. *Let* u *be the solution of* (6.7). *Then*

(i) $\dfrac{\partial u}{\partial x}(t, \xi)\leq 1$ *for all* $t\geq 0$.

(ii) *There are* $T>0$ *and* $\delta_0>0$ *such that*

$$\frac{\partial u}{\partial x}(t, \xi)\geq\delta_0>0 \text{ for } 0\leq t<T$$

(iii) *The mapping* $x\mapsto u(t, x)$ *is convex for* $0\leq t<T$.

PROOF. (i) By the stability of solution u with respect to b we may assume that b is smooth and vanishes in a neighborhood of $[0, \infty)$. Differentiating the equation in x yields the equation for $w=u_x$:

$$w_t-a_0w_{xx}-b'(u_x)w_x=0 \quad \text{in } (0, \infty)\times(-\infty, \xi). \tag{6.8}$$

Let T_0 denote

$$T_0=\sup\{t\,;\,u_x(t, \xi)\leq 1\}.$$

If $T_0<\infty$, we have a small interval $(T_0, T_0+\xi)$ such that $u_x(t, \delta)>1$ for $T_0<t<T_0+\delta$. Since

$$\begin{aligned} a_0w_x(t, \xi)&=u_t(t, \xi)-b(u_x(t, \xi)) \\ &=0-0 \quad \text{for } T_0<t<T_0+\delta, \end{aligned}$$

w solves (6.8) with

$$\begin{aligned} &w_x(t, \xi)=0 \quad \text{for } T_0<t<T_0+\delta \\ &w=w(T_0, x) \quad \text{at } t=T_0. \end{aligned}$$

By the maximum principle we see $w(t, x)\leq 1$ for $T_0<t<T_0+\delta$. This contradicts $T_0<\infty$.

(ii) This is just by continuity of u_x and $u_{0x}(\xi)>0$.

(iii) We may assume that (ii) holds with T, δ_0 independent of any approximate solutions u_n of (6.7) with b replaced by b_n. Recall (6.8) for $w=u_x$. If we extend w by

$$w(t,x)=w(t,2\xi-x) \text{ for } x>\xi,$$

$w_x=0$ at $x=\xi$ for $0\le t\le T$ since $\delta_0\le u_x(t,\xi)\le 1$. Thus w solves

$$w_t-a_0w_{xx}-B(t,x)w_x=0 \ (0,\infty)\times(-\infty,\infty) \tag{6.9}$$

with $B=b'(u_x(t,x))$ for $x\le\xi$, $-b'(u_x(t,2\xi-x))$ for $x>\xi$. Since $\delta_0\le u_x(t,\xi)\le 1$ for $0\le t\le T$, we see $B=0$ near $x=\xi$ so B is continuous. Since constants are solutions of (6.9), the number of local maxima cannot increase. Since $w(0,x)$ takes its only maximum at $\xi=0$ and monotone in $x>\xi$ and $x<\xi$, $w(t,x)$ must take its only maximum at $\xi=0$. Then w is nondecreasing in $x<\xi$ which means that u is convex in x.

6.4. B. LEMMA ON LOCAL SUPERSOLUTIONS. *Assume* (6.1)-(6.5) *for* (3.1). *Suppose that* u_0 *is globally Lipschitz and that* u_{0x} *is nondecreasing in* $(-\infty,z]$, *where* $-\infty<z<\infty$. *Suppose that*

$$x_0=\inf\{x\,;\,u_{0x}(x+0)>p_1\}<z$$

and that $u_{0x}(x-0)\le p_2$ *for* $x<z$. *For each* $\varepsilon>0$ *there is* $T>0$ *and a continuous supersolution of* (3.1) *on* $(0,T)\times(-\infty,z)$ *such that*

$$\bar u=u_0 \text{ on } [0,T)\times(x_0+\varepsilon,z),$$
$$\bar u_{t=0}\ge u_0.$$

PROOF. We may assume $p_1=0$, $p_2=1$ by adding a linear function and multiplying a constant with u. We may also assume $x_0=0$. For $\xi=\varepsilon$ let u be the solution of the initial-boundary value problem of (6.7) with

$$a_0=\sup\{a(p)\,;-L\le p\le 0\},$$

where L is a Lipschitz constant of u_0. We then set

$$\bar u(t,x)=\begin{cases}u(t,x) & x\le\xi\\ u_0(x) & \xi\le x<z\end{cases}$$

for $t\ge 0$.

By the form of $\bar u$ it suffices to prove that $\bar u$ is a supersolution on $(0,T)\times(-\infty,z)$ for some $T>0$. Let T be as in Lemma 6.4.A. For $x<\xi$ since u is convex in x by Lemma 6.4.A, we observe that

$$\begin{aligned}&\bar u_t-a(\bar u_x)\bar u_{xx}-b(\bar u_x)\\ &\quad=u_t-a_0u_{xx}-b(u_x)+(a_0-a(u_x))u_{xx}=(a_0-a(u_x))u_{xx}\ge 0.\end{aligned}$$

At $x=\xi$ we observe that

$$(\tau,p,X)\in\mathscr{P}^{2,-}\bar u(t,\xi)$$

implies that $\tau=0$, $0<p\leq 1$ by Lemma 6.4.A. Thus $\bar{u}$ is a supersolution up to $x=\xi$. For $x>\xi$ since $u_{0x}(x-0)\leq 1$, $u_{0x}(x+0)\geq 0$, $\bar{u}=u_0$ is a supersolution (cf. Lemma 6.3(i).) We thus observed that $\bar{u}$ is a supersolution of (3.1) on $(0, T)\times(-\infty, z)$. $\square$

6.5. Nonparabolic region. Suppose that g is convex on $(-\infty, z)$ and that $g_x(z-0)<p_2$. We set

$$x=\inf\{y\,;\,g_x(y+0)>p_1\}$$

and call the interval $[x, z)$ *a nonparabolic region.* If we set

$$x_*=\inf\{y\,;\,g_x(y+0)\geq p_1\},$$

the interval $[x_*, z)$ is called *a weakly nonparabolic region.* Roughly speaking (x, z) corresponds to the region where $p_1<g_x<p_2$ while $[x_*, z)$ corresponds to the region where $p_1\leq g_x<p_2$. We are interested in the motion of the nonparabolic region for solution $u(t, \cdot)$ of (3.1).

Assume (6.1)-(6.5) for (3.1). Suppose that u is a solution of (3.1) with globally Lipshitz initial data u_0. Suppose that u_0 is convex on $(-\infty, z)$ and $[x_0, z)$ is its nonparabolic region where $x_0<z$. Let $[x(t), z)$ denote the nonparabolic region of $u(t, \cdot)$ if it exists.

6.6. DECREASING LEMMA ON NONPARABOLIC REGION. (i) *The nonparabolic region $[x(t), z)$ exists for $0\leq t<T$ with some T, $0<T\leq\infty$. If $T<\infty$, then one of following holds*:

$$\lim_{t\to T} x(t)=z,\ u_x(T, z-0)=p_2,\ u_x(T, z+0)=p_1.$$

(ii) *$u(t, x)=u_0(x)$ for $x\in[x(t), z)$.*
(iii) *$x(t)$ is a continuous, increasing function on $[0, T)$.*
(iv) *For $t>0$ the nonparabolic region $[x(t), z)$ agrees with the weakly nonparabolic region.*
(v) *For $t>0$, $u_x(t, x-0)<p_1$ with $x<x(t)$.*
(vi) *If $b\geq 0$ on some $[q_0, p_1](q_0<p_1)$, then $T<\infty$ for $z<\infty$ and $\lim_{t\to\infty} x(t)=\infty$ for $z=\infty$.*

We shall prove this lemma in several steps. We may assume that $p_1=0$, $p_2=1$ in this lemma and following statements. Suppose that $z=\infty$ so that $u(t, \cdot)$ is globally Lipschitz and convex on $\boldsymbol{R}$ (by Theorems 3.6 and 3.7.) For $t_0\geq 0$ suppose that $x(t_0)>-\infty$ is well defined. We set

$$v_0(x)=\begin{cases}-\delta(x-x(t_0))+u(t_0, x(t_0)) & \text{for } x<x(t_0)-\varepsilon\\ u(t_0, x(t_0)) & \text{for } x\geq x(t_0)-\varepsilon.\end{cases}$$

Here ε, $\delta>0$ is taken so that $u(t_0, x)\geq v_0(x)$. Since we have assumed $p_1=0$, $p_2=1$, the solution v of (3.1) with $v(t_0, x)=v_0(x)$ solves a uniform parabolic equation as proved in Lemma 6.3 (ii). By the proof of Lemma 6.3 (iii) we observe $v(t, x)>u(t_0, x(t_0))$ $(t>t_0)$ by the strong maximum principle. On the other hand the comparison principle for (3.1) yields $u(t, x)\geq v(t, x)$. This implies that $u(t, x(t_0))>u(t_0, x(t_0))$.

We shall show that $u(t, x)\geq u(t_0, x)$ for $x(t_0)\leq x$. We set $w_0(x)=u(t_0, x(t_0))$ for $x\leq x(t_0)$ and $w_0(x)=u(t_0, x)$ for $x\geq x(t_0)$. Then by Lemma 6.3 (i), $w_0(x)$ is a stationary solution of (3.1). Since $w_0(x)\leq u(t_0, x)$, we have $u(t, x)\geq w_0(x)$ $(t\geq t_0)$ by comparison. We have proved:

6.7. LEMMA. *Under the assumptions of Lemma* 6.6 *with* $z=\infty$
(i) $u(t, x)\geq u(t_0, x)$ *for* $t\geq t_0\geq 0$, $x\geq x(t_0)$,
(ii) $u(t, x(t_0))>u(t_0, x(t_0))$ *for* $t>t_0$.

We shall study the behavior of

$$\eta(t)=\inf\{x\, ;\, u(t, x)=u_0(x)\} \text{ for } t>0.$$

6.8. LEMMA. *Under the assumptions of Lemma* 6.7 *we have*
(i) $\eta(t)$ *is a nondecreasing continuous function on* $(0, \infty)$.
(ii) $\eta(t)>\eta(t_0)$ *for* $t>t_0$ *provided that* $\eta(t_0)=x(t_0)$.
(iii) $\lim_{t\downarrow 0}\eta(t)=x(0)$.

PROOF. (i) By Lemma 6.7 (ii) we observe $x(0)<\eta(t_1)$ for $t_1>0$. Suppose that $\eta(t_2)<\eta(t_1)$ for some $t_2>t_1$. Then there would exist x such that $x(0)\leq x<\eta(t_1)$ and $\eta(t_2)<x$. Lemma 6.7 (i) yields $u_0(x)\leq u(t_1, x)\leq u(t_2, x)$. Since $\eta(t_2)<x$, we would have $u(t_2, x)=u_0(x)$ so that $u(t_1, x)=u_0(x)$ which leads a contradiction $x\geq\eta(t_1)$. We have thus proved that $\eta(t)$ is nondecreasing.

Since the set of (t, x) such that $u(t, x)=u_0(x)$ is closed in $[0, \infty)\times\boldsymbol{R}$, $\eta(t)$ is lower semicontinuous. Since $\eta(t)$ is nondecreasing, this implies that η is left continuous. By comparing local supersolutions of Lemma 6.4.B we observe that $\eta(t)$ is right continuous and that $\eta(t)$ takes finite values for all $t>0$.

(ii) By Lemma 6.7 (ii) it is clear $\eta(t)>x(t_0)=\eta(t_0)$ for $t>t_0$.
(iii) This can be proved by comparison with local supersolutions as in the proof of (i). We omit the detail. □

6.9. LEMMA. *Assume the same hypotheses of Lemma* 6.7. *Then* $x(t)=\eta(t)$ *for* $t\geq 0$ *by setting* $\eta(0)=x(0)$.

PROOF. We first observe that $x(t)$ is left upper semicontinuous, *i. e.*

$$\overline{\lim_{t\uparrow r}} x(t)\leq x(r)$$

for $r>0$. Indeed, suppose that this were false. Then there would exist a sequence $t_j\uparrow r$ such that $\lim_{j\to\infty}x(t_j)=x_0>x(r)$ and $x(t_j)>x(r)$. We may assume $p_0=0$. By the definition of $x(t)$ we observe that

$$u(t_j, x(t_j))=\int_{x(r)}^{x(t_j)} u_x(t, y)dy+u(t_j, x(r))\leq 0+u(t_j, x(r)).$$

Since u is continuous, letting $j\to\infty$ yields

$$u(r, x(r))\geq u(r, x_0).$$

Since $u_x\geq 0$ on $(x(r), x_0)$, this implies $u_x=0$ on $(x(r), x_0)$. However, this contradicts the definition of $x(r)$.

Suppose that $x(t_0)<\eta(t_0)$ for some $t_0>0$. We set

$$s_0=\inf\{s\,; x(t)<\eta(t) \text{ for all } t,\ s\leq t\leq t_0\}.$$

Since η is continuous by Lemma 6.8, the left upper semicontinuity of x implies $s_0<t_0$ and $\eta(s_0)=x(s_0)$. We regard $u(t_1, x)$ $(s_0<t_1<t_0)$ as initial data. By Lemma 6.7(i) and comparison with local supersolutions we observe that $u(t_1, x)=u(t,x)$ for t, x $(t_1\leq t<t_1+\delta,\ x(t_1)+\delta<x)$ with some $\delta>0$. In particular, $\eta(t)=\eta(t_1)$ for $t_1\leq t<t_1+\delta$. This implies that $\eta(t)$ is constant on $[s_0, t_0]$ by the continuity of η. By Lemma 6.8(ii) (iii), $\eta(t)>\eta(s_0)$ for $t>s_0$ since $\eta(s_0)=x(s_0)$. This leads a contradiction so we have proved $x(t)\geq\eta(t)$. The opposite inequality is trivial. □

We need a kind of strong maximum principle for viscosity solutions to prove Lemma 6.6(iv). The next lemma is well known for classical solutions for equations with less regular coefficients ([N] see also [PW]). The version presented here is by no means optimal.

6.10. LEMMA. *Suppose that u is continuous viscosity subsolution of*

$$Lu=u_t-a_0u_{xx}-b_0u_x=0$$

in $K=\{(t, x)\,; (x-x_1)^2+(t-t_1)^2<R^2,\ t\leq t_1\}$ *for a given* (t_1, x_1) *and* $R>0$. *Here* $a_0\geq 0$ *and* b_0 *are constants. Suppose that* $u<M$ *on* K *for* $t<t_1$. *Then* $u(t_1, x_1)<M$.

PROOF. The idea of the proof is almost the same as [PW, Chap. 3, Lemma 3] except the last step. We use a barrier function

$$u=e^{-((x-x_1)^2+\alpha(t-t_1))}-1$$

so that $Lv \leq -c_0$ in K, $t \leq t_1$ with some positive constant c_0 by taking a large. Let D denote

$$D=\{(t, x) \in K ; (x-x_1)^2+a(t-t_1)<0\}.$$

For sufficiently small $\varepsilon>0$ we can arrange that $w=u+\varepsilon v$ satisfies $w \leq M$ on ∂D. Using comparison principle for viscosity solution of $Lw=0$ in D, we observe that $w \leq M$ in D. Here we need some regularity of a_0 and b_0. Note that usual proof for comparison principle for degenerate parabolic equation is given for cylindical domains but the proof works for this case [CIL].

Suppose that w takes the value M at (t_1, x_1). Since $w \leq M$ in D, $w-\psi$ with $\psi=M+N(x-x_1)^4$ takes its maximum 0 over K at (t_1, x_1) for sufficiently large N. By a definition of viscosity subsolution [CIL], $L\psi(t_1, x_1) \leq -\varepsilon c_0$ since w is a subsolution of $Lw=-\varepsilon c_0$. However since $L\psi=LM=0$ at (t_1, x_1), this lead a contradiction. So we have proved $u(t_1, x_1)=w(t_1, x_1)<M$. □

PROOF OF LEMMA 6.6. We first assume $z=\infty$. By Lemmas 6.8 and 6.9 we obtain (i), (ii), (iii) with $T=\infty$.

If $z<\infty$, we take a continuous function on $\boldsymbol{R}$ such that $U_0=u_0$ for $x \leq z$ and that $U_0'(x)=u_0'(z-0)$ for all $x \geq z$. Let U be the solution of (3.1) with initial data U_0. Let $X(t)$ be the infimum of the nonparabolic region of U. If non of equalities in (i) holds for $T=T_0$, $x(t)=X(t)$ and u, we can prove that $u=U$ for $0 \leq t \leq T_0$, $x \leq z$, since $u_0(x)$ is a solution of (3.1) near z. (cf. Lemma 6.3). This yields (i), (ii), (iii) for $z<\infty$.

To prove (iv) we may assume $z=\infty$ and $p_1=0$. Suppose that $\alpha=x_*(t_1)<x(t_1)=\beta$ for some $t_1>0$ so that $M=u(t_1, x)=u_0(\beta)$ for $\alpha \leq x \leq \beta$. For $t<t_1$, we set $m(t)=\sup\{x ; u(t, x) \geq M\}$. Since u is convex, $u>M$ for $x<m(t)$ and $u<M$ for $x>m(t)$ if $t<t_1$.

Case 1. $m_0=\overline{\lim_{t \uparrow t_1}} m(t)<\beta$.

It is not difficult to see that u is a solution of

$$u_t-\bar{a}(u_x)u_{xx}-b(u_x)=0 \quad \text{for } x<x(t) \tag{6.10}$$

with $\bar{a}$ defined in Lemma 6.3. Since u is convex in x, this implies

$$u_t-Au_{xx}-b(u_x) \leq 0 \quad \text{for } x<x(t),$$

where $A=\sup\{a(p) ; |p| \leq L\}$ with $L \geq |u_{0x}|$. We recall that b is Lipschitz and $b(0)=0$ and that $u_x \leq 0$ for $x \leq x(t)$. Thus

$$u_t-Au_{xx}-Bu_x \leq 0 \quad \text{for } x<x(t)$$

with some constant B. Since $m_0<\beta$, there is x_1, $m_0<x_1<\beta$ such that $u<M$ on K with $t<t_1$ if R is sufficiently small; K is defined in Lemma 6.10. We may assume

$$K\subset\bigcup_{0<t\leq t_1}\{t\}\times(-\infty, x(t))=U$$

by taking R smaller. By Localization lemma [CGG3, 8] u is a viscosity subsolution of

$$u_t-Au_{xx}-Bu_x=0$$

on K. We now apply Lemma 6.10 and conclude that $u(t_1, x_1)<M$. This lead a contradiction.

Case 2. $m_0=\beta$

Since u_{xx}, u_x has a definite sign, from (6.10) we also observe that u is a viscosity supersolution of

$$u_t-a_0u_{xx}-b_0u_x=0 \quad \text{for } x<x(t)$$

with $a_0=\inf\{a(p);-L\leq p<0\}$ and some $b_0\in\boldsymbol{R}$. If the limit

$$\lim_{t\uparrow t_1}m(t)=\tilde{m}_0$$

exists, there is x_1, $\alpha<x_1<m_0=\tilde{m}_0$ such that $u>M$ on K with $t<t_1$ for small R so that $K\subset U$. Applying Lemma 6.10 for $-u$ yields a contradiction as in Case 1.

It remains to prove that $\tilde{m}_0$ exists. Since a viscosity solution of (6.10) (with given initial and boundary data) is obtained as a local uniform limit of solutions of a more regular equation approximating (6.10), u solves

$$u_t-a_0u_{xx}-b_0u_x\geq 0,\ x<x(t) \tag{6.11}$$

in the sense of distribution. Let u^ε be a mollified function obtained by convolution with a space-time mollifier. Then u^ε solves (6.11) classically for $x<x(t)-\sigma_1(\varepsilon)$, $t>\sigma_2(\varepsilon)$ where $\sigma_j(\varepsilon)\to 0$ as $\varepsilon\to 0$. We may assume that u^ε is convex and decreasing there. For $\sigma_2(\varepsilon)<t<t_1-\sigma_3(\varepsilon)$ one can uniquely define $m^\varepsilon(t)$ by $u^\varepsilon(t, m^\varepsilon(t))=M$ with some σ_3. Since

$$\frac{d}{dt}u^\varepsilon(t, m(t))=0 \text{ yields } m^\varepsilon(t)'=-u_t^\varepsilon/u_x^\varepsilon,$$

and $u_{xx}^\varepsilon\geq 0$, the inequality (6.11) yields $u_t^\varepsilon\geq b_0u_x^\varepsilon$. Since $u_x\leq 0$, this yields $m^\varepsilon(t)'\geq -b_0$. Letting $\varepsilon\downarrow 0$ yields

$$m(s)-m(t)\geq -b_0(s-t),\ 0<\tau<s<t_1.$$

Thus $m(t)$ does not oscillate so that the limit $\tilde{m}_0$ exists.

To prove(v) suppose that $u_x(t, x-0) \geq p_1$. Then the monotonicity of u_x implies $u_x(t, x+0) \geq p_1$ so that x belongs to the weakly parabolic region. By (iii) this implies $x \geq x(t)$ which leads a contradiction.

The last assertion (iv) can be proved by comparing u with a solution which agrees with a special solution in Lemma 6.3(iii) up to translations and additive constants. □

6.11. REMARK. If we know that $x(t)$ is Lipschitz on $[0, T']$ for each $T' < T$, then

$$u_x(t, x(t)-0) = p_1, \ 0 < t < T.$$

Indeed, if not, for some t we see

$$u_x(t, x(t)-0) = q < p_1$$

by Lemma 6.6(v). Since $x(t)$ is Lipschitz continuous, for each q_1, $q < q_1 < p_1$ there is τ such that for every $X \geq 0$ it holds

$$u(s, x) - u(t, x(t)) \geq \tau(s-t) + q_1(x - x(t)) + 2^{-1}X(x - x(t))^2 \\ + o(|x - x(t)|^2 + |s-t|) \text{ as } x \to x(t), \ s \uparrow t$$

for $s \leq t$, x near $x(t)$. By Localization lemma [CGG3, 8(i)] we see (τ, q_1, X) must satisfy

$$\tau - a(q_1)X \geq 0.$$

Since $a(q_1) \geq a_0 > 0$ and X is arbitrary, this inequality is contradictory. We thus proved $u_x(t, x(t)-0) = p_1$.

We do not know whether $x(t)$ is Lipschitz in time in this generality.

As in § 6.5 for a convex function g on $\boldsymbol{R}$ we set

$$x_1 = \inf\{y \, ; \, g_x(y+0) > p_1\}, \ x_2 = \sup\{y \, ; \, g_x(y-0) < p_2\}$$

and call $[x_1, x_2]$ is a *nonparabolic region* for g. A weakly nonparabolic region is similarly defined. Using Lemma 6.6 for u and $u(t, -x)$, we obtain:

6.12. THEOREM. *Assume Assumptions* 6.1 *and* 6.2 *on* $\tilde{\gamma}$ *and* β. *Suppose that* u_0 *is globally Lipschitz and convex. Let* $[x_1(t), x_2(t)]$ *is a nonparabolic region for* $v(t, \cdot)$ *defined by* (5.1) *from* u *solving* (2.4) *with initial data* u_0. *Suppose that* $-\infty < x_1(0) < x_2(0) < \infty$. *Then* $x_1(t)$ (*resp.* $x_2(t)$) *is a continuous, increasing* (*resp. decreasing*) *function for* $0 \leq t < T$. *Here* T *is either infinity or the time when* $x_1(T) = x_2(T)$, *where* $x_i(T) =$

$\lim_{t \uparrow T} x_i(t)$. *Moreover,* $[x_1(t), x_2(t)]$ *agrees with the weakly nonparabolic region of* $v(t, \cdot)$ *for* $t>0$.

6.13. COROLLARY. *Assume the same hypotheses of Theorem* 6.12. *Suppose that* $c=0$ *or that* $c>0$ *and* $\beta(m(\theta))=\rho(\theta)$ *for* θ, $\theta_1 \leq \theta \leq \theta_2$ *with* $\rho(\theta)=\alpha_1 \cos\theta+\alpha_2 \sin\theta+\alpha_3$, $\alpha_i \in \boldsymbol{R}$. *Suppose that* $\beta(\vec{m}(\theta)) \geq \rho(\theta)$ *for* θ *near* θ_1 *and* θ_2. *Then* $T<\infty$.

PROOF. It is not difficult to see that assumption $b \geq 0$ in Lemma 6.6 (vi) is fulfilled under the assumption of Corollary 6.13. The conclusion $T<\infty$ easily follows from Lemma 6.6(vi). □

6.14. REMARK. Note that Lemma 6.6 applies to the case when Lipschitz initial data u_0 is convex on $(-\infty, x_1(0))$ and concave in $(x_2(0), \infty)$, where $p_1 < u_{0x}(x) < p_2$ if and only if $x_1(0) < x < x_2(0)$. We can define nonparabolic region and prove that it decreases in time. We do not state results explicitly. We note that such a configuration of initial data appeared in a material science literature of Mullins [Mu].

6.15. THEOREM ON INSTANT LOSS OF SMOOTHNESS. *Suppose that* u_0 *is convex and smooth with bounded first derivative. Suppose that* u_{0x} *is increasing and* $-\infty < x_1(0) < x_2(0) < \infty$. *Then* v *(in Theorem* 6.12) *becomes nondifferentiable in* x *for* $t>0$ *close to zero.*

PROOF. We apply Lemma 6.6 to get

$$v(t, x)=u_0(x) \text{ for } x_1(t) \leq x \leq x_2(t)$$

and $v(t, x_1(t)-0) \leq p_1$. Since $u_{0x}(x)$ and $x_1(t)$ are increasing, $v(t, x_1(t)+0) > p_1$ for $0<t<T$. Thus v is not differnetiable in x at $x=x_1(t)$. □

References

[A] S. B. ANGENENT, Parabolic equations for curves on surfaces — part 2, Ann. of Math. 133, 171-215, (1991).

[AG1] S. B. ANGENENT and M. E. GURTIN, Multiphase thermomechanics with interfacial structure 2, Evolution of an isothermal interface, Arch. Rational Mech. Anal. 108, 323-391, (1989).

[AG2] ———, Anisotropic motion of a phase interface, preprint.

[CGG1] Y.-G. CHEN, Y. GIGA and S. GOTO, Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations, J. Differential Geometry 33, 749-786, (1991).

[CGG2] ———, Analysis toward snow crystal growth " Proc. of International symposium on Functional analysis and related topics ", (ed.S. Koshi) pp. 43-57, World Scientific, Singapore (1991).

[CGG3] ———, Remarks on viscosity solutions for evolution equations, Proc. Japan Acad. Ser. A 67, 323-328, (1991).

[CI] M. G. CRANDALL and H. ISHII, The maximum principle for semicontinuous functions, Diff. Int. Equations 3, 1001-1014, (1990).

[CIL] M. G. CRANDALL, H. ISHII and P. L. LIONS, User's guide to viscosity solutions of second order partial differential equations, Bull. Amer. Math. Soc. 27, 1-67, (1992).

[ES] L. C. EVANS and J. SPRUCK, Motion of level sets by mean curvature I, J. Differential Geometry 33, 635-681, (1991).

[GG1] Y. GIGA and S. GOTO, Motion of hypersurfaces and geometric equations, J. Math. Soc. Japan 44, 99-111, (1992).

[GG2] ————, "On the evolution of phase boundaries", (eds. M. E. Gurtin, G. B. McFadden), IMA volumes in mathematics and its applications 43 (1992), pp. 51-65, Springer-Verlag, New York.

[GGIS] Y. GIGA, S. GOTO, H. ISHII and M.-H. SATO, Comparison principle and convexity preserving properties for singular degenerate parabolic equations on unbounded domains, Indiana Univ. Math. J. 40, 443-470, (1991).

[GSS] M. E. GURTIN, H. M. SONER and P. E. SOUGANIDIS, Anisotropic motion of an interface relaxed by formation of infinitesimal wrinkles, J. Differential Equations (to appear).

[I] H. ISHII, Perron's method for Hamilton-Jacobi equations, Duke Math. J. 55, 369-384, (1987).

[LSU] O. A. LADYZHENSKAYA, V. A. SOLONNIKOV and N. N. URAL'CEVA, Linear and Quasilinear Equations of Parabolic Type, Amer. Math. Soc., Providence, 1968.

[M] A. M. MEIRMANOV, The Stefan Problem, Walter de Gruyter, Berlin-New York (1992). (English translation).

[Mu] W. W. MULLINS, Theory of linear facet growth during thermal etching, Philosophical Magazine 6 (1961), 1313-1341.

[N] L. NIRENBERG, A strong maximum principle for parabolic equations, Comm. Pure and Appl. Math. 6 (1953), 167-177.

[OhS] M. OHNUMA and M.-H. SATO, Singular degenerate parabolic equations with applications to geometric evolutions, Diff. Int. Equations, 6, 1265-1280, (1993).

[PW] M. H. PROTTER and H. F. WEINBERGER, Maximum principles in Differential Equations, Englewood Cliffs, N. E., Prentice-Hall, 1967.

[S] H. M. SONER, Motion of a set by the curvature of its boundary, J. Differential Equations 101 (1993), 313-372.

[TCH] J. E. TAYLOR, J. W. CAHN and C. A. HANDWERKER, Geometric models of crystal growth, Acta Metallurgica et Materialia 40 (1992), 1443-1474.

Department of Mathematics
Hokkaido University
Sapporo 060, Japan