Hokkaido Mathematical Journal Vol. 24 (1995) p. 567–573

Exotic circles of $PL_{+}(S^{1})$

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(Received January 6, 1995; Revised March 3, 1995)

Abstract. Let G be a subgroup of $\mathrm{Homeo}_{+}(S^{1})$. An exotic circle of G is a subgroup of G which is topologically conjugate to $SO(2)$ but not conjugate to $SO(2)$ in G. This shows us the subgroup G is far from being a Lie group. In this paper, we prove that $PL_{+}(S^{1})$ has exotic circles.

Key words: exotic circle, $PL_{+}(S^{1})$, topologically conjugate, PL conjugate, bendeng point.

Introduction

Let G be a Lie group and M an oriented manifold of class $C^{k}(1\leq k\leq\infty)$. Let $\mathrm{Diff}_{+}^{k}(M)$ denote the group of all C^{k} diffeomorphisms of M. A topological action is a continuous map $\varphi: G\times M\rightarrow M$ such that
1) $\varphi_{e}(x)=x$,
2) $\varphi_{gh}(x)=\varphi_{g}(\varphi_{h}(x))$.
where e is the unit of G and $\varphi_{g}(x)=\varphi(g, x)$. D. Montgomery and L. Zippin proved the following theorem ([4]).

Theorem 0.1 *Let φ be a topological action. If every φ_{g} belongs to $\mathrm{Diff}_{+}^{k}(M)$ then φ is a map of class C^{k}.*

In the case where $G=M=S^{1}$, this theorem implies the following corollary.

Corollary 0.2 *If every $h\circ R_{x}\circ h^{-1}$ is contained in $\mathrm{Diff}_{+}^{k}(S^{1})$, then h belongs to $\mathrm{Diff}_{+}^{k}(S^{1})$. Here, $R_{x}: S^{1}\rightarrow S^{1}$ is the rotation of S^{1}, i.e., $R_{x}(y)=x+y$.*

Indeed, for $\varphi(x, y)=h\circ R_{x}\circ h^{-1}(y)$. $\varphi: S^{1}\times S^{1}\rightarrow S^{1}$ is a topological action with $\varphi_{x}\in \mathrm{Diff}_{+}^{k}(S^{1})$. Then φ is of class C^{k} by Theorem 0.1. Fix a point y_{0} and define the C^{k} diffeomorphism ϕ of S^{1} by $\phi(x)=\varphi(x, y_{0})$. Then we can see easily $\phi^{-1}\circ\varphi_{x}\circ\phi=R_{x}$. So $\phi^{-1}\circ h=R_{z}$ for some $z\in S^{1}$. This implies h belongs to $\mathrm{Diff}_{+}^{k}(S^{1})$.

Let $SO(2)=\{R_{x}|x\in S^{1}\}$ be the group of all rotations of S^{1}. Corollary

1991 Mathematics Subject Classification : 58E40, 58F03.

0.2 says that $\mathrm{Diff}_+^k(S^1)$ has no exotic circle in the following sense.

Let G be a subgroup of $\mathrm{Homeo}_+(S^1)$.

Definition 0.3 1) A subgroup $S \subset \mathrm{Homeo}_+(S^1)$ is called a *topological circle* if $S = h \circ SO(2) \circ h^{-1}$ for some $h \in \mathrm{Homeo}_+(S^1)$.

2) A topological circle $S \subset G$ is an *exotic circle* of G if h does not belong to G.

The aim of this paper is to show the existence of an exotic circle of $PL_+(S^1)$.

The author would like to thank the referee for his useful comments.

1. Piecewise linear homeomorphisms with two bending points

Let $\mathrm{Homeo}_+^\sim(S^1)$ be the group of all orientation preserving homeomorphisms of R which commutes with the translation T_1. Here $T_b(x) = x + b$ $(x, b \in R)$ is the translation by b. Every $F \in \mathrm{Homeo}_+^\sim(S^1)$ induces a homeomorphism $f : S^1 \to S^1$ $(S^1 = R/Z)$. So we define

$$p : \mathrm{Homeo}_+^\sim(S^1) \to \mathrm{Homeo}_+(S^1)$$

by $p(F) = f$. Conversely for any $f \in \mathrm{Homeo}_+(S^1)$, there exists a $\tilde{f} \in \mathrm{Homeo}_+^\sim(S^1)$ such that $p(\tilde{f}) = f$. Such $\tilde{f}$ is called a *lift* of f. We can easily check that

$$p^{-1}(f) = \{T_n \circ \tilde{f} \mid n \in Z\}.$$

Let $PL_+^\sim(S^1)$ be the group of $\mathrm{Homeo}_+^\sim(S^1)$ defined as follows. $F \in \mathrm{Homeo}_+^\sim(S^1)$ belongs to $PL_+^\sim(S^1)$ if F is piecewise linear and bending points of F have no accumulation points in R. Then we define $PL_+(S^1) = p(PL_+^\sim(S^1))$.

Let $\pi : R \to S^1 = R/Z$ denote the quotient map. A point $\tilde{x} \in R$ with $\pi(\tilde{x}) = x$ is called a *lift* of x. We may use the notation $\pi(\tilde{x}) = [\tilde{x}]$.

Definition 1.1 Let f be an element of $PL_+(S^1)$.

$$d_R f(x) = \lim_{\varepsilon \to 0, \varepsilon > 0} \frac{\tilde{f}(\tilde{x} + \varepsilon) - \tilde{f}(\tilde{x})}{\varepsilon}$$

$$d_L f(x) = \lim_{\varepsilon \to 0, \varepsilon > 0} \frac{\tilde{f}(\tilde{x}) - \tilde{f}(\tilde{x} - \varepsilon)}{\varepsilon}$$

This definition is well defined, because each right-hand side does not depend on the choices of lifts $\tilde{f}, \tilde{x}$.

Definition 1.2 $BP(f) = \{x \in S^1 \mid d_R f(x) \neq d_L f(x)\}$.

$BP(f)$ is a finite set, since the bending points of $\tilde{f}$ do not accumulate in R.

Lemma 1.3 *For any* $x \in S^1$, $\lambda (\lambda > 0,\ \lambda \neq 1)$ *and* $\beta > 0$, *there exists the unique* $f \in PL_+(S^1)$ *such that*
1) $BP(f) = \{x, f(x)\}$,
2) $d_R f(x) = d_L(f(x)) = \lambda, d_R f(f(x)) = d_L f(x) = \lambda^{-\beta}$.

Proof. By taking a conjugation by a rotation, we may assume $x = [0]$. We define $F_{\lambda,\beta} \in PL_+^{\sim}(S^1)$ by

$$F_{\lambda,\beta}|_{[0,1]} = \begin{cases} \lambda \tilde{y} & 0 \leq \tilde{y} \leq \tilde{a}_{\lambda,\beta} \\ \lambda^{-\beta}(\tilde{y} - 1) + 1 & \tilde{a}_{\lambda,\beta} \leq \tilde{y} \leq 1 \end{cases}$$

where $\tilde{a}_{\lambda,\beta} = (\lambda^{\beta} - 1)/(\lambda^{1+\beta} - 1)$. $f = p(T_{\tilde{a}_{\lambda,\beta}} \circ F_{\lambda,\beta})$ is the required PL homeomorphism.

Suppose $g \in PL_+(S^1)$ satisfies 1),2). $g' = R_{-g(x)} \circ g$ fixes x. Then g' must be equal to $R_x \circ p(F_{\lambda,\beta}) \circ R_x^{-1}$. This implies the uniqueness of such g. □

Let $f_{x,\lambda,\beta}$ denote the unique element of $PL_+(S^1)$ which satisfies 1), 2) of Lemma 1.3.

Lemma 1.4 *Let* x, λ, β *be as above.* $f_{x,\lambda,\beta}$ *has no fixed points.*

Proof. Since $f_{[0],\lambda,\beta} = R_x^{-1} \circ f_{x,\lambda,\beta} \circ R_x$, it suffices to prove the lemma for $f = f_{[0],\lambda,\beta}$. We show that the graph of the lift $T_{\tilde{a}_{\lambda,\beta}} \circ F_{\lambda,\beta}$ of f lies in the area $\{(s,t) \in R^2 | s < t < s+1\}$. In fact,

$$0 < \tilde{a}_{\lambda,\beta} \leq T_{\tilde{a}_{\lambda,\beta}} \circ F_{\lambda,\beta}(\tilde{y}) - \tilde{y} \leq \lambda \tilde{a}_{\lambda,\beta} < 1,$$

if $\lambda > 1$ and

$$0 < \lambda \tilde{a}_{\lambda,\beta} \leq T_{\tilde{a}_{\lambda,\beta}} \circ F_{\lambda,\beta}(\tilde{y}) - \tilde{y} \leq \tilde{a}_{\lambda,\beta} < 1$$

if $0 < \lambda < 1$. □

Lemma 1.5 *For any $n \in N$,*

$$BP(f^n_{x,\lambda,\beta}) \subset \{f_{x,\lambda,\beta}(x), f^{-n+1}_{x,\lambda,\beta}(x)\}.$$

Especially, if $f^n_{x,\lambda,\beta}(x) = x$, then $f_{x,\lambda,\beta}(x) = id$.

Proof. Let $f = f_{x,\lambda,\beta}$. Since $d_R f^2(x) = \lambda^{1-\beta} = d_L f^2(x)$, then we can see that f^n is differentiable at each point of $S^1 - \{f(x), f^{-n+1}(x)\}$. That is,

$$BP(f^n) \subset \{f(x), f^{-n+1}(x)\}.$$

If $f^n(x) = x$, then $f^{-n+1}(x) = f(x)$. So $\sharp BP(f^n) \leq 1$. Each $g \in PL_+(S^1)$ must have at least two bending points whenever $g \notin SO(2)$. Therefore f^n has no bending points and fixes x. Thus f^n must be identity. □

The rotation number $\rho : \mathrm{Homeo}_+(S^1) \to S^1$ is a well-known semi-conjugacy invariant which has the following properties ([1], [5], [6]);
1) $\rho(R_\alpha) = \alpha$.
2) $\rho(f \circ g) = \rho(f) + \rho(g)$ if $f \circ g = g \circ f$.
3) If $\rho(f) = \alpha$, then $R_\alpha^{-1} \circ f$ has an fixed point.
4) If f^n has no fixed points for any $n \in Z$, then $\rho(f)$ is irrational.

Corollary 1.6 *If β is irrational. then the rotation number of $f_{x,\lambda,\beta}$ is irrational.*

Proof. Suppose $f = f_{x,\lambda,\beta}$. For any $n \geq 1$, there exist $n_1, n_2 \in N$ with $n_1 + n_2 = n$ such that

$$d_R f^n(x) = \lambda^{n_1 - n_2\beta}.$$

Since β is irrational, $d_R f^n(x) \neq 1$. Therefore f^n does not belong to $SO(2)$. This means $\sharp BP(f^n) \geq 2$. Then $BP(f^n) = \{f(x), f^{-n+1}(x)\}$ and $f^n(x) \neq x$ by Lemma 1.5. Since $f^n(f^{-n+1}(x)) = f(x)$, Lemma 1.4 implies that f^n has no fixed points. Then f has an irrational rotation number. □

Definition 1.7 Let $a^i, b^i (i = 1, 2)$ be points of S^1 such that $a^1 \neq a^2$, $b^2 \neq b^2$. Let $pl(a^1, a^2 : b^1, b^2)$ denote the element of $PL_+(S^1)$ such that
1) $BP(pl(a^1, a^2 : b^1, b^2)) = \{a^1, a^2\}$,
2) $pl(a^1, a^2 : b^1, b^2)(a^i) = b^i (i = 1, 2)$.

We give the set of periodic functions on R the maximum norm. Then it induces the topologies of $\mathrm{Homeo}^{\sim}_+(S^1)$ and $\mathrm{Homeo}_+(S^1)$. We use this topology in the rest of this paper.

Lemma 1.8 *Let* $\{a^i_k\}_{k\in N}$, $\{b^i_k\}_{k\in N} \subset S^1$ $(i = 1, 2)$ *be two sequeces with* $a^1_k \neq a^2_k$, $b^1_k \neq b^2_k$ *and* a^i, $b^i (i = 1, 2)$ *two points of* S^1 *with* $a^1 \neq a^2$, $b^1 \neq b^2$. *If* $a^i_k \to a^i$, $b^i_k \to b^i$ *as* $k \to \infty$, *then* $pl(a^1_k, a^2_k : b^1_k, b^2_k) \to pl(a^1, a^2 : b^1, b^2)$ *as* $k \to \infty$.

Proof. Let $f_k = pl(a^1_k, a^2_k : b^1_k, b^2_k)$ and $f = pl(a^1, a^2 : b^1, b^2)$. We take lifts $\tilde{f}, \tilde{a^1}, \tilde{a^2}$ such that $\tilde{a^1} < \tilde{a^2} < \tilde{a^1} + 1$. Since $a^i_k \to a^i, b^i_k \to b^i$ as $k \to \infty$, then there exist lifts $\tilde{a^i}_k, \tilde{b^i}_k$ such that $\tilde{a^i}_k \to \tilde{a^i}, \tilde{b^i}_k \to \tilde{b^i}$ as $k \to \infty$. Then we can take lifts $\tilde{f}_k$ such that $\tilde{f}_k(\tilde{a^i}_k) = \tilde{b^i}_k$. For any $\epsilon > 0$, we define

$$U_\epsilon = \{(s,t) \in R^2 \mid \tilde{f}(s) - \epsilon < t < \tilde{f}(s) + \epsilon\}.$$

Then U_ϵ is an open set of R^2 which contains the graph Γ of $\tilde{f}$. It suffices to show that for any U_ϵ there exists a positive integer N such that the graph Γ_k of $\tilde{f}_k$ lies in U_ϵ if $k \geq N$. Now suppose U_ϵ is given. We take open convex sets O_1, O_2 such that

$$K_1 = \Gamma|_{[\tilde{a^1}, \tilde{a^2}]} \subset O_1 \subset U_\epsilon$$

and

$$K_2 = \Gamma|_{[\tilde{a^2}, \tilde{a^1}+1]} \subset O_2 \subset U_\epsilon$$

where $\Gamma|[s,t] = \Gamma \cap [s,t] \times R$. This is possible, because K_i is a compact segment. Since $(\tilde{a^i}, \tilde{b^i}) \in O^1$ $(i = 1, 2)$, there exists a positive intger N_1 such that $(\tilde{a^i}_k, \tilde{b^i}_k) \in O^1$ for any $k \geq N_1$. That is, $\Gamma_k|_{[\tilde{a^i}_k, \tilde{b^i}_k]} \subset O^1 \subset U_\epsilon$, because O^1 is convex and $\Gamma_k|_{[\tilde{a^i}_k, \tilde{b^i}_k]}$ is the segments with endpoints $(\tilde{a^i}_k, \tilde{b^i}_k), (\tilde{a^i}_k, \tilde{b^i}_k)$. We can show that there exists a positive integer N_2 such that $\Gamma_k|_{[\tilde{b^i}_k, \tilde{a^i}_k+1]} \subset O^2 \subset U_\epsilon$ for any $k \geq N_k$ in the same way. Since U_ϵ, Γ and $\Gamma_k (k = 1, 2)$ are all periodic, then this implies $\Gamma_k \subset U_\epsilon$ for any $k \geq \max\{N_1, N_2\}$. This completes the proof. □

It is the time to show the existence of exotic circles of $PL_+(S^1)$.

2. Existence of exotic circles

Let $x \in S^1$, $\lambda > 1$ and β a positive irrational number. Then $f = f_{x,\lambda,\beta}$ has an irrational rotation number $\alpha \in S^1$ by Cor 1.6. So there exists $h \in \mathrm{Homeo}_+(S^1)$ such that

$$h \circ f \circ h^{-1} = R_\alpha.$$

We define

$$S = h^{-1} \circ SO(2) \circ h.$$

Lemma 2.1 *$S \subset PL_+(S^1)$ and $\sharp BP(g) \leq 2$ for any $g \in S$.*

Proof. Any $g \in S$ is written as $g = h^{-1} \circ R_{\alpha_g} \circ h$. We can take a sequece $\{n_k\}_{k \in N} \subset N$ such that $R_\alpha^{n_k} \to R_{\alpha_g}$ and $R_\alpha^{-n_k} \to R_{\alpha_g}^{-1}$ as $k \to \infty$. Then $f^{n_k} \to g$ and $f^{-n_k} \to g^{-1}$ as $k \to \infty$. Since $BP(f^n) = \{f(x), f^{-n+1}(x)\}$, then

$$f^n = pl(f^{-n+1}(x), f(x) : f(x), f^{n+1}(x)).$$

By Lemma 1.8,

$$\begin{aligned} f_{n_k} &= pl(f^{-n_k+1}(x), f(x) : f(x), f^{n_k+1}(x)) \\ &\longrightarrow f^\infty = pl(g^{-1}(f(x)), f(x) : f(x), g(f(x))). \end{aligned}$$

Thus $g = f^\infty \in PL_+(S^1)$. □

Lemma 2.2 *h does not belong to $PL_+(S^1)$.*

Proof. We assume $h \in PL_+(S^1)$ and deduce a contradiction. Since $BP(f)$ is not empty, h must bend and $\sharp BP(h) \geq 2$. Since $BP(h)$ is a finite set, there exists a rotation R_{α_1} such that

$$R_{\alpha_1}(h(BP(h))) \cap h(BP(h)) = \emptyset.$$

Then we have

$$BP(h^{-1} \circ R_{\alpha_1} \circ h) = BP(h) \cup h^{-1} \circ R_{\alpha_1}^{-1} \circ h(BP(h)).$$

The right hand side is the disjoint union by the choice of α_1. So $BP(h^{-1} \circ R_{\alpha_1} \circ h)$ contains at least four points. On the other hand, since $h^{-1} \circ R_{\alpha_1} \circ h$ belongs to S, it has at most two bending points. This is the contradiction. □

Thus we have shown the existence of exotic circles of $PL_+(S^1)$. For every element $f \in \mathrm{Homeo}_+(S^1)$, let the symbol α_f donote the rotation number.

Proposition 2.3 *Let $S = h \circ SO(2) \circ h^{-1}$ be an exotic circle of $PL_+(S^1)$. Any element $f \in S$ is not PL conjugate to the rotation R_{α_f}, if α_f is irrational.*

Proof. Let $f \in PL_+(S^1)$ be with the irrational rotation number. Suppose $k \circ f \circ k^{-1} = R_{\alpha_f}$. Then we have $(k \circ h) \circ R_{\alpha_f} \circ (k \circ h)^{-1} = R_{\alpha_f}$. Since α_f is irrational, $k \circ h$ must belong to $SO(2)$. By the fact that $SO(2)$ is a subgroup of $PL_+(S^1)$ and h does not belong to $PL_+(S^1)$, we can see that k does not belong to $PL_+(S^1)$. This completes the proof. □

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